SCALE-BY-SCALE TURBULENT ENERGY TRANSFERS IN VERY STEEPLY NON-HOMOGENEOUS TURBULENCE WITHOUT MEAN-FLOW GRADIENTS

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ABSTRACT

Direct numerical simulations (DNS) of non-homogeneous turbulence are investigated in the framework of the two-point Karman-Howarth-Monin-Hill (KHMH) equation. The domain is a triply periodic box with one long (z) and two short (x,y) directions and the flow is forced locally at its centre $z=0$ with zero mean momentum injection rate. It is found that the magnitudes of the non-linear energy transport processes in both physical space and scale space are proportional to the local dissipation rate $\varepsilon(z)$ even though $\varepsilon(z)$ varies by an order of magnitude as the distance z from the forcing varies. The dependence of the inter-space non-linear turbulence transport rate on two-point separation scale r appears to be approximately constant with r over a range bounded from below by the Taylor length λ and from above by the large scale $2\pi H/4$ of the system. Work is under way to test this result and investigate the behaviour of the inter-scale turbulence transport rate and the pressure related transport rate at higher Reynolds numbers.

INTRODUCTION

For the most part, turbulence cascades have been studied in the context of homogeneous turbulence where there is no average spatial transport of turbulent energy. In this case the cascade of energy to smaller scales balances dissipation on average over the inertial range of scales (see Frisch (1995)). This is the Kolmogorov scale-by-scale equilibrium. However, inhomogeneous turbulence, where spatial turbulent energy transport matters, is the rule rather then the exception in nature and engineering, e.g. turbulent wakes, jets, boundary layers etc. Here we examine a turbulent flow introduced by Alexakis (2023) where the forcing acts in a small sub-volume of the domain and is zero outside this sub-volume and look for scaling laws that govern two-point energy transfer processes in both physical and scale spaces. Our goal is to determine whether laws and scalings of these transfers exist in non-homogeneous turbulence where scale-by-scale Kolmogorov equilibrium is absent.

PHYSICAL AND SCALE-SPACE ENERGY TRANSFERS AND THE KHMH EQUATION

The framework of the two-point Karman-Howarth-Monin-Hill (KHMH) equation Hill (2002) is often used to access the magnitudes of the inter-space and inter-scale transfers of energy, as well as the transfer of energy by pressure-velocity correlations in inhomogeneous flows. We consider two points in the flow, ξ_1 and ξ_2. We take $\mathbf{X}$ to be their centroid and their separation vector to be $2\mathbf{r}$. The KHMH equation (1) is derived from the Navier-Stokes equations at the two points with $\delta\mathbf{u}(\mathbf{X},\mathbf{r}) = \frac{\mathbf{u}(\mathbf{X}+\mathbf{r})-\mathbf{u}(\mathbf{X}-\mathbf{r})}{2}$, $\mathbf{u}_X(\mathbf{X},\mathbf{r}) = \frac{\mathbf{u}(\mathbf{X}+\mathbf{r})+\mathbf{u}(\mathbf{X}-\mathbf{r})}{2}$ where $\mathbf{u}$ is the velocity field, $\delta p(\mathbf{X},\mathbf{r}) = \frac{p(\mathbf{X}+\mathbf{r})-p(\mathbf{X}-\mathbf{r})}{2}$ where p is the pressure field (implicitly normalised by mass density) and $\delta\mathbf{f}(\mathbf{X},\mathbf{r}) = \mathbf{f}(\mathbf{X}+\mathbf{r}) - \mathbf{f}(\mathbf{X}-\mathbf{r})$ where $\mathbf{f}$ is the external forcing applied.

$$\begin{aligned}\frac{\partial|\delta\mathbf{u}|^2}{\partial t} + \nabla_{\mathbf{X}}\cdot(\mathbf{u_x}|\delta\mathbf{u}|^2) + \nabla_{\mathbf{r}}\cdot(\delta\mathbf{u}|\delta\mathbf{u}|^{\mathbf{2}}) &= -2\nabla_{\mathbf{X}}\cdot(\delta\mathbf{u}\delta p) \\ &+ \frac{\nu}{2}\nabla^{\mathbf{2}}_{\mathbf{X}}|\delta\mathbf{u}|^{\mathbf{2}} + \frac{\nu}{2}\nabla^{\mathbf{2}}_{\mathbf{r}}|\delta\mathbf{u}|^{\mathbf{2}} - \frac{1}{2}\varepsilon^{+} - \frac{1}{2}\varepsilon^{-} + \delta\mathbf{f}\cdot\delta\mathbf{u}.\end{aligned} \tag{1}$$

This two-point energy balance involves $\langle\nabla_{\mathbf{X}}\cdot(\mathbf{u_x}|\delta\mathbf{u}|^2)\rangle \equiv T_x(\mathbf{X},\mathbf{r})$ which represents inter-space transfer of energy of two-point velocity differences, i.e. to neighbouring points in space, $\langle\nabla_{\mathbf{r}}\cdot(\delta\mathbf{u}|\delta\mathbf{u}|^{\mathbf{2}})\rangle \equiv \Pi_r(\mathbf{X},\mathbf{r})$ which represents inter-scale transfer of such energy and is associated with the turbulence cascade, and $\langle -2\nabla_{\mathbf{X}}\cdot(\delta\mathbf{u}\delta p)\rangle \equiv T_p(\mathbf{X},\mathbf{r})$ which represents transport of energy by pressure-velocity correlations. Angular brackets denote an appropriate averaging over realisations which we defined further down in this paper for this specific flow. The turbulence dissipation rates at ξ^+ and ξ^- are denoted by ε^+ and ε^- respectively. The two-point dissipation often has a weak dependence on two-point separation. In the particular turbulent flow considered here and described in the following section, the turbulence dissipation rate depends only on one spatial direction and is independent of the other two. Chosing the separation vector $\mathbf{r}$ to be in the sub-space of these other two directions and performing an average over time and over the space of these two other directions, $\langle \frac{1}{2}\varepsilon^{+} + \frac{1}{2}\varepsilon^{-}\rangle$ is independent of $\mathbf{r}$ and can be denoted as $\varepsilon(\mathbf{X})$.

The terms governing viscous diffusion of two-point turbulent kinetic energy, $\langle \frac{\nu}{2}\nabla^2_{\mathbf{X}}|\delta \mathbf{u}|^2\rangle \equiv \mathscr{D}_c(\mathbf{X},\mathbf{r})$ and $\langle \frac{\nu}{2}\nabla^2_{\mathbf{r}}|\delta \mathbf{u}|^2\rangle \equiv \mathscr{D}_r(\mathbf{X},\mathbf{r})$ have been proven under mild generally valid kinematic assumptions to be negligible if the components of $\mathbf{r}$ are larger than the local Taylor length $\lambda(\mathbf{X})$ (see Valente & Vassilicos (2015)). Therefore the energy balance reduces to

$$T_x(\mathbf{X},\mathbf{r}) + \Pi_r(\mathbf{X},\mathbf{r}) = T_p(\mathbf{X},\mathbf{r}) - \varepsilon(\mathbf{X}) \qquad (2)$$

for two-point separations scales larger than $\lambda(\mathbf{X})$ and regions in space where the external forcing's power is negligible compared to the local turbulence dissipation rate.

In homogeneous turbulence, $T_x(\mathbf{X},\mathbf{r}) = T_p(\mathbf{X},\mathbf{r}) = 0$ and the average inter-scale transfer rate of turbulent energy is constant in the inertial range of scales and equal the average turbulence dissipation rate in magnitude. In non-homogeneous turbulent flows, however, the inter-space transport of turbulent energy and the transport of energy by pressure-velocity correlations can have non-zero average values. Here we investigate how they and the inter-scale turbulence transfer rate vary in physical and scale spaces, i.e. how the three terms on the left hand side of equation (2) vary in physical and scale spaces, when the turbulence is not statistically homogeneous. We do this in the non-homogeneous turbulent flow recently introduced by Alexakis (2023).

DETAILS OF THE FLOW AND DNS

The flow introduced in Alexakis (2023) was introduced for the purpose of computationally investigating how locally injected turbulence spreads in space. The flow is in a cuboidal periodic box of size $2\pi H$ along the x and y directions and of size $2\pi L$ along the z direction, with $H = 1$ and $L = 8H$. The forcing of the flow has the functional form

$$\mathbf{f}(t,\mathbf{x}) = \begin{bmatrix} \partial_y[\psi(t,\mathbf{x}/\ell) - \psi(t,-\mathbf{x}/\ell)] \\ \partial_x[\psi(t,-\mathbf{x}/\ell) - \psi(t,\mathbf{x}/\ell)] \\ 0 \end{bmatrix} \exp\left[\frac{L^2}{\ell^2}\left(\cos\left(\frac{z}{L}\right) - 1\right)\right]$$

where $\psi(t,\mathbf{x}/\ell)$ is a random function including only Fourier modes with wavevectors k satisfying $0 < |k\ell| \le 1$.

$$\psi(t,\mathbf{x}/\ell) = \sum_{0<|k\ell|\le 2,\, k_x\neq 0} a_k \exp(\mathrm{i}[\mathbf{k}\cdot\mathbf{x} + \phi_k(t)]). \qquad (3)$$

The phases ϕ_k of these modes are delta-correlated in time (they change randomly at every time step Δt of the simulation) while the amplitudes $a_k \propto 1/\sqrt{\Delta t}$ are fixed in time and independent of k. The forcing injects no linear or angular momentum. For $|z| << L$, the exponential factor in the forcing scales as $\exp -\left(\frac{z^2}{\ell^2}\right)$, so the forcing is limited to the range $|z| \propto l$. Here, we choose $l = H$. The flow starts from an initial condition of zero velocity.

If the turbulence in the flow remains localised, defining Reynolds numbers based on the kinetic energy or the turbulence dissipation averaged over the entire box of the flow (as typically done in periodic boxes with equal periods in all directions and spatially distributed forcing) would lead to the Reynolds numbers going to zero in the limit $L >> H$, which is the limit that we are interested in. In order to compensate for this, Alexakis (2023) defined a typical turbulence dissipation rate $\varepsilon = \frac{2\mathscr{D}L}{H}$ where $\mathscr{D}$ is the turbulence dissipation rate averaged over the entire computational box and over time once the flow attains a steady state. This is done so as to ensure that the Reynolds numbers defined on the basis of the typical velocity and typical dissipation remain finite in the $L >> H$ limit. In this flow, we control the energy injection rate which is equal, on average to the turbulence dissipation rate $\mathscr{D}$. Therefore, Alexakis (2023) introduced the Reynolds number $Re_\varepsilon = \frac{\varepsilon^{\frac{1}{3}} H^{\frac{4}{3}}}{\nu}$.

The DNS are performed using the pseudospectral code GHOST (Mininni *et al.*, 2011) which uses a 2/3 dealiasing rule and a second-order Runge–Kutta method for time advancement. A uniform grid was used with grid spacings $\Delta x = 2\pi H/N_x$, $\Delta y = 2\pi H/N_y$ and $\Delta z = 2\pi L/N_z$, with $N_z = 8N_y = 8N_x$. It was verified that $k_{max}\eta > 1$, where $k_{max} = N_y/3$ is the highest wavenumber and $\eta = HRe_\varepsilon^{\frac{-3}{4}}$ is the Kolmogorov lengthscale.

It was found in Alexakis (2023) that, at high enough Reynolds numbers and long enough times, $E(z)$, the one point turbulent kinetic energy averaged over the $x-y$ plane and averaged over time (plotted in figure 1 for two Reynolds numbers given in Table 1), has a Reynolds number-independent profile that decays away from the forcing. It is clear from figure 1 that the energy profile $E(z)$ is very steep and that the turbulence is therefore very steeply non-homogeneous. We also see that the lower of our two Reynolds numbers is not high enough to reach the Reynolds number-independent energy profile observed in Alexakis (2023). However, we checked that the $E(z)$ profile that we obtain for the higher of our two Reynolds number does not reach it either. As mentioned in the results section of this short paper, higher Reynolds number simulations are currently under way to complement our analysis.

Table 1. The grid-sizes of the run, its Re_εs and maximum resolved wave-number in terms of the Kolmogorov Scale η

$N_x \times N_y \times N_z$	Re_ε	$k_{max}\eta$
$256 \times 256 \times 2048$	180	1.75
$128 \times 128 \times 1024$	100	1.32

Our aim is to study equation (2) in this very steeply non-homogeneous turbulence. The terms in this equation have been obtained by averaging over the $x-y$ plane and over the time after the flow reaches statistical stationarity. We set r_z (the component of $\mathbf{r}$ in the long direction z) to 0 and average each term over circles of radius $r = \sqrt{r_x^2 + r_y^2}$ in the $r_x - r_y$ plane. After this averaging, each term is a function of z and r except $\varepsilon(z)$ which is a function of z but not of r. We study the the r-dependent terms in (2) as functions of normalised two-point

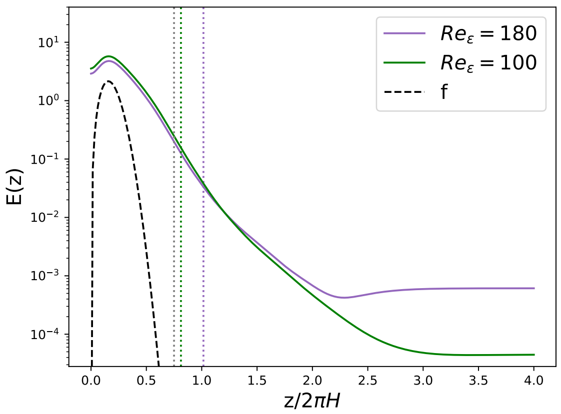

Figure 1. E(z), the one-point turbulent kinetic energy averaged over the two short directions of the box and over time, once the flow has attained a statistically steady state. The black dashed line denotes the forcing. KHMH statistics of each simulation are studied in the region of space from the grey dotted line to the dotted line in their respective colour in the legend.

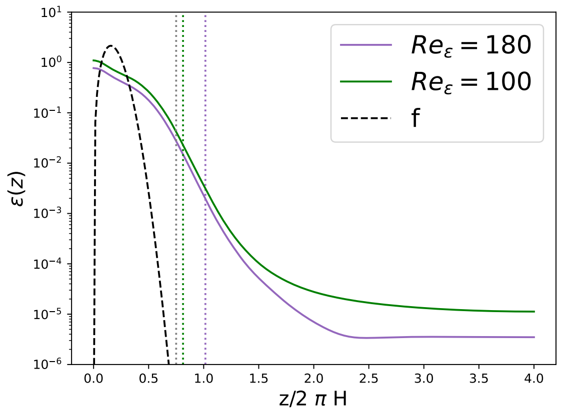

Figure 2. $\varepsilon(z)$, the turbulence dissipation averaged over the two short directions of the box and over time, once the flow has attained a statistically steady state. The black dashed line denotes the forcing. KHMH statistics of each simulation are studied in the region of space from the grey dotted line to the dotted line in their respective colour in the legend.

separation distance $\frac{r}{\lambda(z)}$, where $\lambda(z) = \sqrt{10\nu E(z)/\varepsilon(z)}$ is the local Taylor length. In figure 2 we plot the profile along z of the turbulence dissipation rate $\varepsilon(z)$ which appears in equation (2), and in figure 3 we plot the Taylor length's evolution with z obtained from $E(z)$ and $\varepsilon(z)$. The very strong non-homogeneity evidenced in $E(z)$ is also present in $\varepsilon(z)$. Normalising two-point distance by $\lambda(z)$ allows to separate the range below $\lambda(z)$ where viscous diffusion matters from the range above $\lambda(z)$ where it does not. Figure 3 shows how the boundary between these two r-ranges grows with z.

REGION OF SPACE CONSIDERED FOR OUR ANALYSIS OF EQUATION (2)

The turbulence remaining localised near the forcing region, coupled with the truncation of the Navier-Stokes equation in Fourier space causes modes with wavenumbers near the truncation wavenumber k_{max} to be in a thermal equilibrium

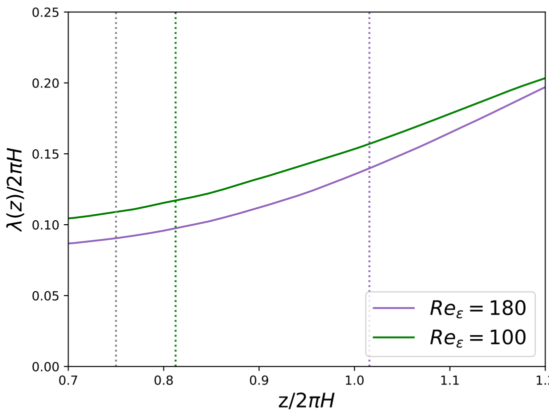

Figure 3. Local Taylor lengthscale for the runs defined as $\lambda(z) = \sqrt{10\nu E(z)/\varepsilon(z)}$. Vertical dotted lines represent the extent of space where KHMH statistics are be studied.

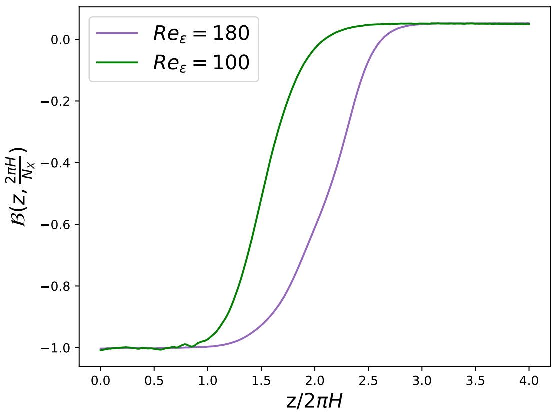

Figure 4. The balance at the smallest non-zero scale for the two runs.

state (Alexakis & Brachet (2020)). This numerical artefact decreases to zero as $k_{max}\eta$ becomes much larger than unity. In physical space, this thermal noise leads to a weak thermal noise that is uniform throughout the domain. This causes the equation 1 to not be satisfied far from the forcing where the mean energy $E(z)$ is so small that is comparable to this thermal noise. Because $E(z)$ drops fast with z it limits the range of z that we can consider.

We define the 'KHMH Balance' as

$$\mathscr{B}(z,r) = \frac{T(z,r) + \Pi(z,r) - (-T_p(z,r) + \mathscr{D}_c(z,r) + \mathscr{D}_r(z,r))}{\varepsilon(z,r)}$$

This quantity should equal -1 in those regions of physical space where the two-point energy injection rate is negligible compared to the local turbulence dissipation rate and the thermal noise is negligible compared to $E(z)$. In figure 4, we plot the KHMH balance as a function of z for the two runs at the smallest non-zero scale ($r = \frac{2\pi H}{N_X}$). We can see that it is close to -1 in regions of high turbulence intensity and becomes 0 far from the forcing owing to the blow-up of the dissipation brought about by the thermal noise. The balance stays closer to -1 in a longer region of space for the run with a higher value of $k_{max}\eta$.

In view of this fact, we consider regions of space that meet the following three criteria:

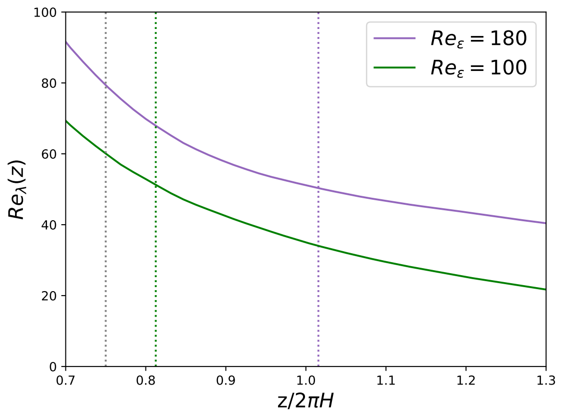

Figure 5. The local Re_λ, defined as $\frac{E(z)}{\sqrt{\nu\varepsilon(z)}}$ for the two simulations. Vertical dotted lines represent the extent of space where KHMH statistics are studied.

1. The region is far enough from the forcing for the two-point energy injection to be negligible compared to turbulence dissipation. This happens at $\frac{z}{2\pi H} = \frac{z_{min}}{2\pi H} = 0.75$ for both runs.
2. The Taylor length is not larger than about half the largest accessible length scale $2\pi H/4$. This is a very minimal requirement for something like a rudimentary inertial range of scales r to exist. From figure 3 and figure 5 where we plot the Taylor length-based Reynolds number Re_λ as a function of z, we see that this minimal rudimentary requirement is that Re_λ should not be smaller than about 50.
3. The KHMH balance for the smallest non-zero scale r is between -0.95 and -1.05.

In Table 2, we give the maximum value z_{max} of z where the last two criteria are met, and the respective factors F_E and F_ε by which $E(z)$ and $\varepsilon(z)$ drop from $z = z_{min}$ to $z = z_{max}$ for our two runs ($Re_\varepsilon = 180$ and $Re_\varepsilon = 100$).

Table 2. The maximum values of z (z_{max}) considered and the respective factors F_E and F_ε by which $E(z)$ and $\varepsilon(z)$ drop from $z = z_{min}$ to $z = z_{max}$ in the two runs

Re_ε	$z_{max}/(2\pi H)$	F_E	F_ε
180	1.02	5.95	14.3
100	0.81	1.58	1.83

RESULTS

Chen & Vassilicos (2022) and Beaumard *et al.* (2024) put forward a hypothesis of homogeneous two-point physics in non-homogeneous turbulence which implies $T_x \propto \varepsilon$, $\Pi_r \propto \varepsilon$ and $T_p \propto \varepsilon$ with prefactors independent of both spatial position and two-point separation length in the inertial range when there is negligible turbulence production. Whilst Chen & Vassilicos (2022) and Beaumard *et al.* (2024) report experimental evidence in support of $T_x \propto \varepsilon$ and $\Pi_r \propto \varepsilon$, they do so in regions of turbulent flows where the turbulent kinetic energy and the turbulence dissipation rate are not as steeply non-homogeneous as in the present flows. Do their predictions hold in a turbulence that is as very steeply non-homogeneous as in the present flows?

In figures 6 and 7, we plot $T_x(z,r)/\varepsilon(z)$, $\Pi_r(z,r)/\varepsilon(z)$ and $T_p(z,r)/\varepsilon(z)$ versus $\frac{r}{\lambda(z)}$ for different locations z in the region of space between z_{min} and z_{max} determined in the previous section. The spatial interval between successive locations in z is $\frac{8\pi H}{512}$.

All three normalised transport terms, $T_x(z,r)/\varepsilon(z)$, $\Pi_r(z,r)/\varepsilon(z)$ and $T_p(z,r)/\varepsilon(z)$, are weakly dependent on z even though the turbulent kinetic energy and the turbulence dissipation rate decrease very steeply over the spatial range considered. In support of the hypothesis of homogeneous two-point physics in non-homogeneous turbulence, these three ratios depend very weakly on z, if at all, considering the very steep non-homogeneity of the turbulence.

A second observation is that the inter-space turbulent energy transport rate is approximately independent of the two-point separation r in the range $\lambda(z) \leq r \leq \frac{2\pi H}{4}$. In fact $T_x(z,r) \approx -0.7\varepsilon(z)$ for $Re_\varepsilon = 180$ and $T_x(z,r)/\varepsilon(z) \approx -0.65\varepsilon(z)$ for $Re_\varepsilon = 100$ in that range and in a range of locations z where $E(z)$ and ε decrease very steeply with z. Simulations at higher Reynolds numbers are needed (and are under way) to obtain bigger inertial ranges and verify whether this simple scaling law characterises two-points physics over an increasing range of inertial scales. In particular, the coefficients -0.65 and -0.7 may seem to approach a Reynolds number-independent value with increasing Reynolds number, although the simulations at higher Reynolds numbers are indeed required to ascertain whether this is indeed the case. Interestingly, $\Pi_r(z,r)/\varepsilon(z)$ and $T_p(z,r)/\varepsilon(z)$ are not constant with r over the range of r where $T_x(z,r)/\varepsilon(z)$ is approximately constant. However, their r-dependence is very similar for the two Reynolds numbers tried here. Higher Reynolds number simulations will also shed light on whether the difference between $\Pi_r(z,r)/\varepsilon(z)$ and $T_p(z,r)/\varepsilon(z)$ on the one hand and $T_x(z,r)/\varepsilon(z)$ on the other is or is not a finite Reynolds number effect. Indeed, the global Reynolds numbers in the present study are not high enough to attain the Reynolds number-independent profile observed in Alexakis (2023) and it is possible that global properties of the flow can affect local energy transfer rates.

CONCLUSIONS AND FUTURE PERSPECTIVES

The hypothesis of homogeneous two-point physics in non-homogeneous turbulence seems to hold even in extremely non-homogeneous turbulence such as the one investigated here. However, only one of its three consequences may be verfied in the present flows: the two-point separation length-independence of the inter-space turbulence transfer rate in the inertial range. The absence of such constancy with r of the inter-scale turbulence transfer rate (directly related to the turbulence cascade when the Reynolds number is high enough) and of the pressure-velocity correlation term may, however, be finite Reynolds number effects.

The turbulent flow studied here is an example of a flow where turbulence is allowed to decay in one direction. The closest experimental analogue of this is oscillating grid-generated turbulence Hopfinger & Toly (1976), where, turbulence is allowed to spread in one direction without a mean flow. However,the turbulent kinetic energy does not decay as fast in oscillating grid-generated turbulence than in the present flow.

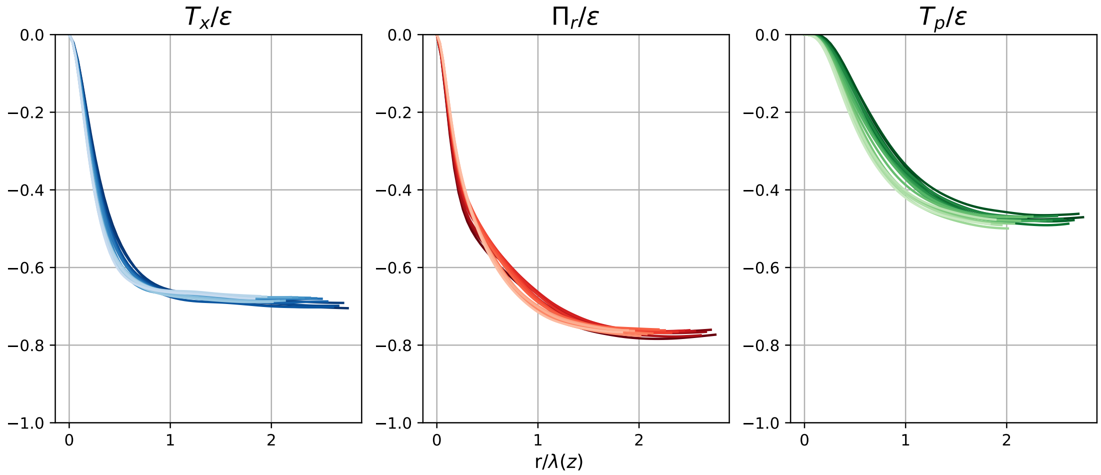

Figure 6. Averaged dissipation-normalised transport terms at 17 equally spaced locations in z in the Re_ε=180 run plotted against $\frac{r}{\lambda(z)}$. Each line represents the transport term averaged as mentioned in the paper's main text at one location in space. Darker colours represent points in z that are closer to the forcing.

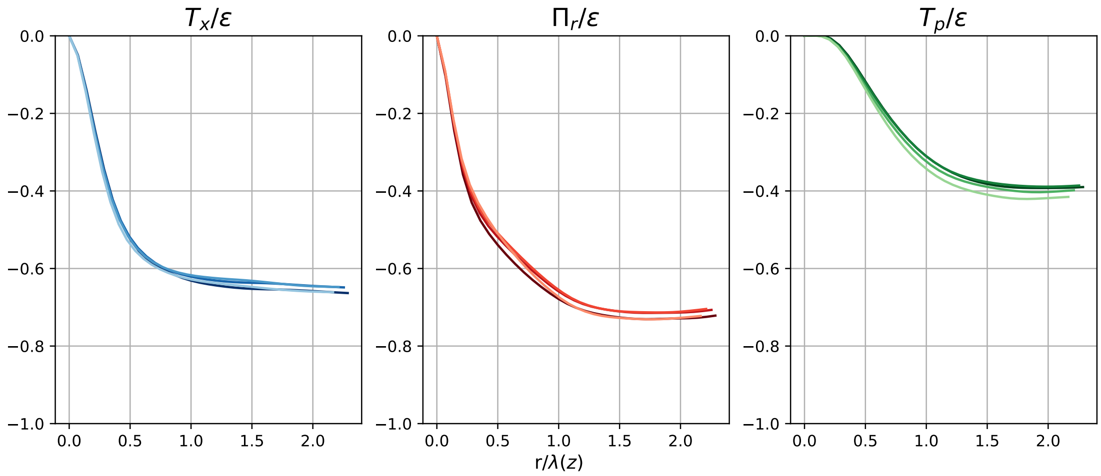

Figure 7. Averaged dissipation-normalised transport terms at 4 equally spaces locations in z in the Re_ε=100 run plotted against $\frac{r}{\lambda(z)}$. Each line represents the transport term averaged as mentioned in the paper's main text at one location in space. Darker colours represent points in z that are closer to the forcing.

Different forcings may therefore generate different variations of turbulent kinetic energy and turbulence dissipation rate in physical space, and this may have an effect on two-point turbulence physics and, in particular, the dimensionless coefficients involved in scalings laws such as $T_x(z,r) \approx -0.7\varepsilon(z)$. We are currently investigating the possibility that the nature of the forcing can have a long-range effect on two-point inter-scale/inter-space physics far downstream where the forcing's power is negligible compared to the local turbulence dissipation rate.

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