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PROBABILITY DENSITY FUNCTIONS FOR HOMOGENEOUS ISOTROPIC TURBULENCE

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ABSTRACT

We investigate homogeneous isotropic turbulence using the Lundgren–Monin–Novikov hierarchy of multi-point probability density functions. To use the symmetries of homogeneity and isotropy, we introduce scalar invariants constructed from the sample space velocities and spatial increment vectors in a suitably adapted coordinate system. In doing so, the set of independent variables, is reduced by six.

For the two-point case, we show that the invariant PDF representation is not only consistent with the classical von Kármán–Howarth framework but is a significant extension of the latter. In particular, the longitudinal and transversal velocity correlation functions are derived as exact second-order moments of the reduced two-point PDF. Moreover, the incompressibility constraint relating these correlations is recovered directly from the continuity equations of the PDF formulation, thus establishing a further link between the LMN hierarchy and classical von Kármán-Howarth moment-based turbulence theory.

Finally, we transform the one-point LMN transport equation into the invariant coordinate system, yielding a exact dimensionally reduced representation based on the symmetry invariants of HIT, which may enable future analyses based on the LMN hierarchy. In particular, the isotropic PDF naturally induces not only the second moment, but all higher moments.

Introduction

Homogeneous isotropic turbulence (HIT) represents one of the fundamental idealizations in turbulence theory, providing a framework in which the statistical properties of turbulent flows can be investigated independently of boundary effects and large-scale anisotropy. The idea of isotropic turbulence was introduced by G.I. Taylor (1935), who established the use of statistical averages and correlation functions for describing turbulent motion. The assumption of isotropy simplifies the statistical description by reducing the dependence of turbulent quantities to rotationally invariant functions of spatial increments.

A major development of the statistical theory of HIT was achieved by von Kármán & Howarth (1938), who derived the evolution equation for the two-point velocity correlation tensor. Their approach provides a description of the dynamics of second-order statistics through the longitudinal and transversal correlation functions. Although the von Kármán–Howarth equation remains a cornerstone of turbulence theory, it is limited to second order moments and does not directly describe the complete probability distribution of turbulent velocity fluctuations.

The concept of local isotropy was introduced by Kolmogorov, who proposed that at sufficiently large Reynolds numbers, small scales of turbulent fluctuations become statistically independent of the anisotropic large-scale motions and exhibit universal properties (Kolmogorov, 1941, 1991). Based on this assumption, Kolmogorov developed the classical K41 theory, predicting self-similar power-law scalings of velocity increment statistics in the inertial range for moments of order two and three. Extending it to arbitrary higher increment moments, however, experimental and numerical observations revealed deviations from these predictions, known as anomalous scaling, demonstrating that turbulent fluctuations are not fully self-similar and that intermittency effects play an important role (Anselmet *et al.*, 1984). To describe these deviations, refined theories such as Kolmogorov's refined similarity hypothesis (Kolmogorov, 1962) or the intermittency model of She & Lévêque (1994) introduced corrections based on the non-uniform distribution of energy dissipation.

A rather complete statistical description of turbulence is provided by the Lundgren–Monin–Novikov (LMN) hierarchy, which describes the evolution of turbulence directly in terms of multi-point probability density functions (PDFs) (Lundgren, 1967). In contrast to moment-based approaches, the LMN hierarchy contains "vertical" statistical information of the turbulent velocity field and, provided the PDF is known, provides access to arbitrary higher-order moment statistics(Monin & Yaglom, 1975).

The connection between the PDF formulation and the classical von Kármán–Howarth description follows naturally from the two-point PDF. The velocity correlation tensor is obtained as a second-order moment of the two-point PDF, meaning that the longitudinal and transversal correlations are projections of the underlying PDF rather than independent statistical quantities (Pope, 2000). Furthermore, the incompressibility constraint relating these correlations can be recovered from the continuity equations of the PDF formulation (Lundgren, 1967; Monin & Yaglom, 1975).

In this work, and in some analogy to the von Kármán-Howarth equation, the symmetries of HIT are used to reduce the dimensionality of the LMN equations through a formulation based on scalar invariants. According to the results given by Görtz *et al.* (2026), this approach provides a direct link between the dimension-reduced two-point PDF and the classical correlation functions of isotropic turbulence by von Kármán-Howarth. Furthermore, the one-point LMN transport equation

is transformed into the corresponding adapted coordinate system, yielding a reduced representation of the PDF dynamics that preserves the statistical content of the original hierarchy while reducing the number of variables by six.

Basic definitions and notation

We consider multi-point PDFs for homogeneous isotropic turbulence. For this, we introduce the instantaneous velocity field as a random variable at an arbitrary point $\boldsymbol{x}_k$, i.e.

$$\boldsymbol{u}_k = \boldsymbol{u}(\boldsymbol{x}_k, t). \tag{1}$$

The multi-point velocity PDF f_n is weakly defined by the expectation of any function of the random velocity field, given by

$$\langle Q(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)\rangle \stackrel{!}{=} \int_{\mathbb{R}^{3n}} Q(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)\, f_n(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, t)\, \mathrm{d}^3\boldsymbol{v}_1 \cdots \mathrm{d}^3\boldsymbol{v}_n. \tag{2}$$

Here, $\boldsymbol{v}_k$, with $k = 1 \ldots n$, are the velocity sample space variables. Accordingly, the PDF depends on $6n+1$ coordinates. We will later reduce this dependency for the case of HIT (Görtz *et al.*, 2026).

Statistical description of turbulence and the infinite Lundgren–Monin–Novikov hierarchy

The starting point of the Lundgren–Monin–Novikov (LMN) hierarchy are the incompressible Navier–Stokes equations, written in the pressure-eliminated form using Green's function as a solution of the Poisson equation on an unbounded domain ($\mathbb{R}^3$)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = -\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{4\pi}\int_{\boldsymbol{R}^3} \frac{\partial/\partial \boldsymbol{x}' \cdot \{\boldsymbol{u}\cdot(\partial \boldsymbol{u}/\partial \boldsymbol{x}')\}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3\boldsymbol{x}' + \nu\frac{\partial}{\partial \boldsymbol{x}}\cdot\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}, \tag{3}$$

which governs each realization of the turbulent velocity field. The velocity PDFs are defined as

$$f_1(\boldsymbol{v}_1, \boldsymbol{x}_1, t) = \langle \delta(\boldsymbol{u}(\boldsymbol{x}_1, t) - \boldsymbol{v}_1)\rangle, \tag{4}$$

$$f_2(\boldsymbol{v}_1, \boldsymbol{x}_1, \boldsymbol{v}_2, \boldsymbol{x}_2, t) = \langle \delta(\boldsymbol{u}(\boldsymbol{x}_1, t) - \boldsymbol{v}_1)\delta(\boldsymbol{u}(\boldsymbol{x}_2, t) - \boldsymbol{v}_2)\rangle, \tag{5}$$

for the one-point and two-point case. For higher order PDFs an analog definition is used. To obtain an evolution equation for f_1, Eq. (4) is differentiated with respect to time. Applying the chain rule yields a term proportional to $(\partial_t \boldsymbol{u}(\boldsymbol{x}_1, t))$, which is subsequently replaced using Eq. (3). The resulting expression contains contributions from advection, the non-local pressure term and viscous diffusion.

The advection term can be transformed by integration by parts in velocity space. Using the defining property of the Dirac delta distribution, the velocity field, multiplied by the delta function is replaced by the sample-space variable $(\boldsymbol{v}_1)$, giving the convective term

$$\left\langle \boldsymbol{u}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1}\delta(\boldsymbol{u}_1 - \boldsymbol{v}_1)\right\rangle = \boldsymbol{v}_1 \cdot \nabla_{\boldsymbol{x}_1} f. \tag{6}$$

The pressure and viscous terms involve products of velocity fields evaluated at different spatial positions. These cannot be expressed solely in terms of the one-point PDF and therefore introduce the two-point distribution function f_2. In particular, inserting the identity

$$1 = \int \delta(\boldsymbol{u}_2 - \boldsymbol{v}_2)\,\mathrm{d}^3\boldsymbol{v}_2 \tag{7}$$

allows ensemble averages containing $(\boldsymbol{u}_2)$ to be rewritten in terms of f_2 as follows

$$\int \langle \boldsymbol{u}_2 \delta(\boldsymbol{u}_2 - \boldsymbol{v}_2)\delta(\boldsymbol{u}_1 - \boldsymbol{v}_1)\rangle\,\mathrm{d}^3\boldsymbol{v}_2 = \int \boldsymbol{v}_2 f_2\,\mathrm{d}^3\boldsymbol{v}_2. \tag{8}$$

From now on, integrals without concrete bounds refer to integration over the full $\mathbb{R}^3$ or the full space of the respective variable. Consequently, the evolution equation for the one-point PDF is not closed: its dynamics depend on the two-point statistics.

Collecting the transformed advection, pressure and viscous contributions yields the first order of the LMN hierarchy,

$$\begin{aligned}\frac{\partial f_1}{\partial t} = &-\boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}_1} \\ &- \frac{\partial}{\partial \boldsymbol{v}_1}\cdot\Bigg\{ + \lim_{\boldsymbol{x}_2 \to \boldsymbol{x}_1} \nu \frac{\partial}{\partial \boldsymbol{x}_2}\cdot\frac{\partial}{\partial \boldsymbol{x}_2}\int \boldsymbol{v}_2 f_2\,\mathrm{d}^3\boldsymbol{v}_2 \\ &- \frac{1}{4\pi}\int \left(\frac{\partial}{\partial \boldsymbol{x}_1}\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\right)\cdot\left(\boldsymbol{v}_2\cdot\frac{\partial}{\partial \boldsymbol{x}_2}\right)^2 f_2\,\mathrm{d}^3\boldsymbol{r}_2\,\mathrm{d}^3\boldsymbol{v}_2\Bigg\},\end{aligned} \tag{9}$$

which relates the evolution of the one-point velocity PDF to the two-point PDF. Equation (9) is the first equation of an infinite hierarchy in which the (n)-order distribution function is coupled to the (n+1)-order distribution function.

Isotropy and the idea of scalar invariants

In this and the following chapters, we closely follow the derivation of the correlation functions for homogeneous isotropic PDF introduced and derived in Görtz *et al.* (2026).

To obtain a reduced PDF for the case of HIT, we recall that HIT is understood as independence of the turbulent multi-point statistics of both a given base point and the orientation of the coordinate system. To incorporate this property into the PDF, we consider spatial increments, meaning that the points in space are from now on described by

$$\boldsymbol{x}_1 = \boldsymbol{x}\,,\ \boldsymbol{x}_2 = \boldsymbol{x} + \boldsymbol{r}\,,\ \boldsymbol{x}_3 = \boldsymbol{x} + \boldsymbol{r}_3\,,\ \ldots\,,\ \boldsymbol{x}_n = \boldsymbol{x} + \boldsymbol{r}_n \tag{10}$$

where $\boldsymbol{x}$ is the base point and $\boldsymbol{r}, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_n$ are the spatial increments. Since we focus on the two-point PDF later, we introduce $\boldsymbol{r} = \boldsymbol{r}_2$ for the sake of brevity. With this, we are now in position to describe a dimensionally reduced, homogeneous, isotropic multi-point PDF. First, homogeneity is included, which may also be understood as invariance of the PDF under translations of the base point, by dropping the dependency of the base point $\boldsymbol{x}$. The homogeneous n-point PDF is accordingly written as

$$f_n = f_n(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n; \boldsymbol{r}, \ldots, \boldsymbol{r}_n, t), \tag{11}$$

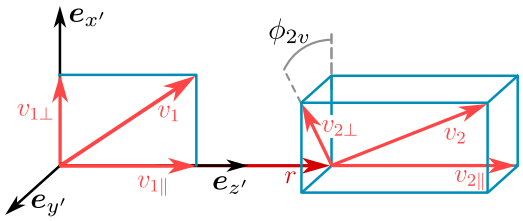

Figure 1. Adapted coordinate system used for the determination of scalar invariants.

For the following reduction of the former to isotropy, we recall that isotropy means invariance under all rotations of the system, i.e. passive rotations of the coordinate system. Specifically, the PDF is invariant w.r.t. rotations of the arguments by the orthogonal linear map $\boldsymbol{A} \in \mathrm{O}(3)$, i.e. $\det\boldsymbol{A} = \pm 1$ and $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$, according to

$$\boldsymbol{r}_k^* = \boldsymbol{A}(\boldsymbol{r}_k), \qquad \boldsymbol{v}_k^* = \boldsymbol{A}(\boldsymbol{v}_k). \tag{12}$$

This implies that we have to demand in the case of HIT that

$$f_n(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;\boldsymbol{r},\ldots,\boldsymbol{r}_n,t) \overset{!}{=} f_n(\boldsymbol{v}_1^*,\ldots,\boldsymbol{v}_n^*;\boldsymbol{r}^*,\ldots,\boldsymbol{r}_n^*,t). \tag{13}$$

However, the PDF is a scalar density and depends on invariant scalars I_k constructed from the arguments in (11). Therefore, we have for the general homogeneous isotropic PDF a dependency of type

$$f_n = f_n(I_1,\ldots,I_{6n-6},t), \quad \text{with} \quad I_k(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;\boldsymbol{r},\ldots,\boldsymbol{r}_n) \in \mathbb{R}. \tag{14}$$

To obtain invariance under rotations as defined in (13), an isotropic PDF must only depend on the scalar invariants I_k under the mapping $\boldsymbol{A}$, which are characterized by

$$I_k(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;\boldsymbol{r},\ldots,\boldsymbol{r}_n) = I_k(\boldsymbol{v}_1^*,\ldots,\boldsymbol{v}_n^*;\boldsymbol{r}^*,\ldots,\boldsymbol{r}_n^*), \tag{15}$$

with the rotated vectors given by (12). To obtain further insights into the structure of the homogeneous isotropic PDF, we will in the following derive a suitable choice of scalar invariants. This choice is not unique, and a new set of scalar invariants can always be derived via recombining a known set of such scalar invariants. For practical reasons, we will find a set of scalar invariants which is close to the formulation of von Kármán & Howarth (1938).

Adapted coordinates

To find such a set of scalar invariants, we use the principle of adapted coordinates. The idea is as follows: First, we use homogeneity and rotational freedom to position the adapted Cartesian coordinate system, $\boldsymbol{e}_{x'}$-$\boldsymbol{e}_{y'}$-$\boldsymbol{e}_{z'}$. The scalar invariants under shift and rotation of the coordinate system are given by the minimal set of scalars which is needed to describe the vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n;\boldsymbol{r},\ldots,\boldsymbol{r}_n$ in the adapted system. The chosen adapted system is displayed in Figure 1. The adapted coordinate system is chosen as follows: First, a translation is applied to place the origin of the adapted system in the first point, accounting for homogeneity. Consequently, in the adapted system, we always have $\boldsymbol{x}_1 = 0$. Next the system is rotated around the origin such that the $\boldsymbol{e}'_z$-axis is parallel to the vector $\boldsymbol{r}$. With that the position of the second point is described in the adapted system with $\boldsymbol{r}' = |\boldsymbol{r}|\,\boldsymbol{e}'_z = r\,\boldsymbol{e}'_z$. Finally, the remaining rotational degree of freedom around the $\boldsymbol{e}_{z'}$-axis is fixed by rotating the system such that the vector $\boldsymbol{v}_1$ lies in the $\boldsymbol{e}_{x'}$–$\boldsymbol{e}_{z'}$ plane. The sample velocity $\boldsymbol{v}_1$ can then be described using only two components such that $\boldsymbol{v}'_1 = v_{1\perp}\,\boldsymbol{e}'_x + v_{1\parallel}\,\boldsymbol{e}'_z$. The first component $v_{1\perp}$ is the part of $\boldsymbol{v}_1$ perpendicular to $\boldsymbol{r}$ and the second component $v_{1\parallel}$ is the part collinear to $\boldsymbol{r}$. For the description of any further sample space velocities a third component is needed. Therefore, the angle ϕ_{kv} is introduced as the angle between $v_{k\perp}$ and $v_{1\perp}$. This is shown in Figure 1 for the example of the sample-space velocity at the second point. Since the adapted system is now fixed, the description of any further point or sample space velocity vector requires the full set of three coordinates.

In view of this discussion, we choose the scalar invariant dependency of homogeneous isotropic PDFs as follows:

$$f_1(\boldsymbol{v}_1,t) = f_1(v_{1\parallel}^2 + v_{1\perp}^2, t), \tag{16}$$

$$f_2(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t) = f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\phi_{2v},r,t). \tag{17}$$

The scalar invariants introduced above follow as

$$v_{1/2\parallel} = \frac{\boldsymbol{v}_{1/2}\cdot\boldsymbol{r}}{r}, \quad v_{1/2\perp} = \left|\boldsymbol{v}_{1/2} - v_{1/2\parallel}\frac{\boldsymbol{r}}{r}\right|, \tag{18}$$

$$\cos(\phi_{2v}) = \frac{\left(\boldsymbol{v}_1 - v_{1\parallel}\frac{\boldsymbol{r}}{r}\right)\cdot\left(\boldsymbol{v}_2 - v_{2\parallel}\frac{\boldsymbol{r}}{r}\right)}{\left|\boldsymbol{v}_1 - v_{1\parallel}\frac{\boldsymbol{r}}{r}\right|\left|\boldsymbol{v}_2 - v_{2\parallel}\frac{\boldsymbol{r}}{r}\right|}, \quad r = |\boldsymbol{r}|.$$

For PDFs of order higher than two, no additional symmetry-based reduction beyond the six removed degrees of freedom is obtained. For the description of position vectors $\boldsymbol{r}_n$ and sample-space velocities $\boldsymbol{v}_n$ with $n \geq 3$, three components are needed, which must also be included in the respective PDF. This observation is consistent with the fact that both homogeneity and isotropy are three-dimensional symmetry groups. Accordingly, a dimensional reduction of up to six variables is possible, which is already achieved for the homogeneous isotropic two-point PDF. We will therefore focus in the following on the two-point PDF (17).

From the usual understanding of isotropy in the two-point case, also symmetry with respect to reflections is included. We may therefore demand that the two-point PDF is invariant under reflections of the vector $\boldsymbol{v}_2$ on the $\boldsymbol{e}_{x'}$-$\boldsymbol{e}_{z'}$ - plane, which allows to further simplify (17), giving

$$f_2(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{r},t) = f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\cos(\phi_{2v}),r,t). \tag{19}$$

Relation between the PDF Description and the von Kármán-Howarth Framework (Görtz *et al.*, 2026)

To prove the consistency of the scalar invariant approach to homogeneous isotropic PDFs with the usual understanding of HIT, we show that the present approach allows the derivation of the von Kármán-Howarth equation (von Kármán & Howarth, 1938). An isotropic two-point moment follows from the PDF by integration, i.e. we define the two-point velocity moment

$$\mathbf{R} = \langle \boldsymbol{u}_1(\boldsymbol{x},t)\otimes\boldsymbol{u}_2(\boldsymbol{x}+\boldsymbol{r},t)\rangle = \iint \boldsymbol{v}_1\otimes\boldsymbol{v}_2 f_2\,\mathrm{d}^3\boldsymbol{v}_1\mathrm{d}^3\boldsymbol{v}_2, \tag{20}$$

where the PDF in the two-point case is written depending on the scalar invariants. From (18), we have $r \in [0,\infty)$, $\boldsymbol{v}_{1/2\parallel} \in (-\infty,\infty)$, $\boldsymbol{v}_{1/2\perp} \in [0,\infty)$, $\cos(\phi_{2v}) \in [0,2\pi)$. We will now use the dimensionally reduced two-point PDF (19) in (20) to show consistency with the approach made by von Kármán & Howarth (1938). In the chosen framework, the sample space velocity differential is given by

$$\mathrm{d}^3\boldsymbol{v}_k = v_{k\perp}\mathrm{d}\phi_{kv}\mathrm{d}v_{k\perp}\mathrm{d}v_{k\parallel}. \tag{21}$$

Inserting (21) and (19) into (20) and carrying out the integration over the angles ϕ_{1v} and ϕ_{2v} gives

$$\mathbf{R}(r,t) = \mathrm{g}(r,t)\,\mathbf{I} + (\mathrm{f}(r,t)-\mathrm{g}(r,t))\,\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{r^2}, \tag{22}$$

where $\mathbf{I}$ refers to the unit tensor. The correlation functions $f(r,t)$ and $g(r,t)$ are given by

$$\begin{aligned}\mathrm{f}(r,t) =& \pi \iint v_{1\perp}v_{1\parallel} \iint v_{2\perp}v_{2\parallel} \int_{-1}^{1} \frac{1}{\sqrt{1-\xi^2}} \\ & f_2\left(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r,t\right)\mathrm{d}\xi\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel}\mathrm{d}v_{1\perp}\mathrm{d}v_{1\parallel},\end{aligned} \tag{23}$$

$$\begin{aligned}\mathrm{g}(r,t) =& 2\pi \iint v_{1\perp}^2 \iint v_{2\perp}^2 \int_{-1}^{1} \frac{\xi}{\sqrt{1-\xi^2}} \\ & f_2\left(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r,t\right)\mathrm{d}\xi\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel}\mathrm{d}v_{1\perp}\mathrm{d}v_{1\parallel},\end{aligned} \tag{24}$$

with $\xi = \cos(\phi_{2v})$ (Görtz *et al.*, 2026). We observe that equation (22) is exactly the approach made by von Kármán & Howarth (1938). With that, it is proven that the ideas based on scalar-invariants for a dimensional reduction of PDFs, as given by equation (19), is in full accordance with the classical understanding of isotropic turbulence. Furthermore with (23) and (24) a connection between the well-known longitudinal and transversal correlations of von Kármán & Howarth (1938) and the isotropic two-point PDF is derived.

To deepen the link between PDFs and the classical, moment-based picture of isotropic turbulence, we investigate the continuity equation. Similar to the derivation of the von Kármán-Howarth equation, the continuity equations for the PDFs also introduce an additional constraint. We start with the two continuity equations or divergence conditions of the general two-point PDF (Lundgren, 1967), given in absolute coordinates by

$$\frac{\partial}{\partial\boldsymbol{x}_1}\cdot\int \boldsymbol{v}_1 f_2(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{x}_1,\boldsymbol{x}_2,t)\,\mathrm{d}\boldsymbol{v}_1 = 0\,, \tag{25}$$

$$\frac{\partial}{\partial\boldsymbol{x}_2}\cdot\int \boldsymbol{v}_2 f_2(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{x}_1,\boldsymbol{x}_2,t)\,\mathrm{d}\boldsymbol{v}_2 = 0\,. \tag{26}$$

With the transformations to relative coordinates (10) and introducing the homogeneous isotropic PDF, i.e. $\partial/\partial\boldsymbol{x} = 0$ and (19), the two continuity equations transform to

$$\frac{\partial}{\partial\boldsymbol{r}}\cdot\int \boldsymbol{v}_1 f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\cos(\phi_{2v}),r,t)\,\mathrm{d}\boldsymbol{v}_1 = 0, \tag{27a}$$

$$\frac{\partial}{\partial\boldsymbol{r}}\cdot\int \boldsymbol{v}_2 f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\cos(\phi_{2v}),r,t)\,\mathrm{d}\boldsymbol{v}_2 = 0. \tag{27b}$$

If we now introduce the invariants in (18) and the differential in (21), we obtain the reduced forms of (27)

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\Bigg(r^2 \iint & v_{1\perp}v_{1\parallel} \\ & f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\cos(\phi_{2v}),r,t)\,\mathrm{d}v_{1\perp}\mathrm{d}v_{1\parallel}\Bigg) = 0,\end{aligned} \tag{28a}$$

and

$$\begin{aligned}&-\frac{v_{1\parallel}}{v_{1\perp}r}\iint v_{2\perp}^2\int_{-1}^{1}\frac{\xi f_2(\ldots,\xi,r,t)}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel} \\ &+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\iint v_{2\perp}v_{2\parallel}\int_{-1}^{1}\frac{f_2(\ldots,\xi,r,t)}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel}\right) = 0.\end{aligned} \tag{28b}$$

A more detailed derivation of the reduced continuity equations can be found in Görtz *et al.* (2026).

Those equations can now be further simplified by integrating them with respect to r. With that, (28a) simplifies to

$$\begin{aligned}\iint v_{1\perp}v_{1\parallel}\,f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\cos(\phi_{2v}),r,t)\,\mathrm{d}v_{1\perp}\mathrm{d}v_{1\parallel} \\ = \frac{C_1(v_{2\parallel},v_{2\perp},\phi_{2v})}{r^2}.\end{aligned} \tag{29}$$

If we assume that the integral is bounded in limit $r\to 0$, the only possible choice is $C_1 = 0$, leading to

$$\iint v_{1\parallel}v_{1\perp}f_2(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\cos(\phi_{2v}),r,t)\,\mathrm{d}v_{1\perp}\mathrm{d}v_{1\parallel} = 0. \tag{30}$$

The second continuity equation (28b) is reduced to

$$\begin{aligned}& r^2\iint v_{2\parallel}v_{2\perp}\int_{-1}^{1}\frac{1}{\sqrt{1-\xi^2}} \\ & \quad f_2\left(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r,t\right)\mathrm{d}\xi\,\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel} \\ &= \int_0^r\frac{\tilde{r}v_{1\parallel}}{v_{1\perp}}\iint v_{2\perp}^2\int_{-1}^{1}\frac{\xi}{\sqrt{1-\xi^2}} \\ & \quad f_2\left(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,\tilde{r},t\right)\mathrm{d}\xi\,\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel}\mathrm{d}\tilde{r}.\end{aligned} \tag{31}$$

With that result, we recall to the definition of the longitudinal correlation function f in (23) and obtain

$$\begin{aligned}\mathrm{f}(r,t) =& 4\pi\frac{1}{r^2}\int_0^r\tilde{r}\iint v_{1\perp}^2\iint v_{2\perp}^2\int_{-1}^{1}\frac{\xi}{\sqrt{1-\xi^2}} \\ & f_2\left(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,\tilde{r},t\right)\mathrm{d}\xi\,\mathrm{d}v_{2\perp}\mathrm{d}v_{2\parallel}\,\mathrm{d}v_{1\perp}\mathrm{d}v_{1\parallel}\mathrm{d}\tilde{r}.\end{aligned} \tag{32}$$

Comparing with the definition of the transversal correlation function g, given by (24), we can identify g in the above equation, leading to the well known relation between longitudinal and transversal correlation in the form

$$\mathrm{f}(r,t) = \frac{2}{r^2}\int_0^r\tilde{r}\mathrm{g}(\tilde{r},t)\,\mathrm{d}\tilde{r}. \tag{33}$$

First derived by von Kármán & Howarth (1938) based on the continuity equation. It nicely interlinks the isotropic correlation functions and the isotropic PDF, as given by equations (23) and (24) and is a new result first derived by Görtz *et al.* (2026).

Dimensionally reduced one point transport equation

To obtain the dynamical equation governing HIT within the reduced PDF framework, we transform the one-point equation of the LMN hierarchy into the invariant coordinate system (18). Starting from the one-point transport equation (9), homogeneity is first applied by introducing relative coordinates (10). Consequently, the first position vector vanishes and the second position vector is replaced by the separation vector $\boldsymbol{r}$.
A direct transformation of the complete equation into adapted coordinates is cumbersome. Therefore, the isotropic representation of the one- and two-point PDFs is first introduced in the Cartesian formulation as given in the equations (16) and (19). With this representation, the convective contribution vanishes because the one-point PDF only depends on the magnitude of the velocity,

$$-\boldsymbol{v}_1 \cdot \frac{\partial f_1(\boldsymbol{v}_1,\boldsymbol{r}_1,t)}{\partial \boldsymbol{r}_1} = \boldsymbol{v}_1 \cdot \frac{\partial f_1(v_{1\parallel}^2+v_{1\perp}^2,t)}{\partial \boldsymbol{r}} = 0. \qquad (34)$$

The remaining terms are transformed by applying the chain rule to replace Cartesian derivatives by derivatives with respect to the scalar invariants (18). All scalar products between vectors are subsequently rewritten using the adapted variables. The resulting equation is therefore fully rotational invariant and can be transformed into the adapted coordinate system without further assumptions.
After transforming the differentials,

$$\mathrm{d}^3\boldsymbol{r} = r^2 \sin\theta \,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\theta\,\mathrm{d}\phi, \qquad \mathrm{d}^3\boldsymbol{v}_2 = v_{2\perp}\,\mathrm{d}v_{2\perp}\,\mathrm{d}v_{2\parallel}\,\mathrm{d}\phi_{2v}, \quad (35)$$

and performing the angular integrations for θ and ϕ as well as substituting $\cos(\phi_{2v}) = \xi$, the reduced one-point transport equation reads

$$\begin{aligned}
\frac{\partial f_1}{\partial t} = &-\int \frac{6}{r^2} \iiint_{-1}^{1} \left(2v_{2\parallel}v_{2\perp}^2\,\xi \left(\frac{1}{v_{1\perp}} + \frac{\partial}{\partial v_{1\perp}} \right) \right. \\
&\left. + v_{2\perp}\left(v_{2\perp}^2 - 2v_{2\parallel}^2\right) \frac{\partial}{\partial v_{1\parallel}} \right) \frac{f_2}{\sqrt{1-\xi^2}}\, d\xi\, dv_{2\perp}\, dv_{2\parallel}\, dr \\
&- \lim_{r\to 0} \nu \iiint_{-1}^{1} \Big[a(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r) \\
&+ b(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r)\,\partial_{v_{1\perp}} \\
&+ \frac{2v_{2\perp}^2\,\xi}{r v_{1\perp}}\,\partial_r + \frac{(v_{1\perp}v_{2\perp}\xi - 2v_{1\parallel}v_{2\parallel})v_{2\perp}}{r^2}\,\partial^2_{v_{1\parallel}^2} \\
&+ c(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r)\,\partial^2_{v_{1\parallel}v_{1\perp}} \\
&+ \frac{2v_{2\parallel}v_{2\perp}}{r}\,\partial^2_{v_{1\parallel}r} + \frac{(2v_{1\parallel}^2 - v_{1\perp}^2)v_{2\perp}^2\,\xi}{r^2 v_{1\perp}}\,\partial^2_{v_{1\perp}^2} \\
&+ \frac{2v_{2\perp}^2\,\xi}{r}\,\partial^2_{v_{1\perp}r} + \frac{v_{2\perp}^2\,\xi}{v_{1\perp}}\,\partial^2_{r^2} + \frac{v_{1\perp}^2 v_{2\parallel}v_{2\perp}}{r^2}\,\partial^3_{v_{1\parallel}^2} \\
&+ d(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r)\,\partial^3_{v_{1\parallel}^2 v_{1\perp}} \\
&+ e(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r)\,\partial^3_{v_{1\parallel}v_{1\perp}^2} \\
&+ v_{2\parallel}v_{2\perp}\,\partial^3_{v_{1\parallel}r^2} + \frac{v_{2\perp}^2\,\xi\,v_{1\parallel}^2}{r^2}\,\partial^3_{v_{1\perp}^3} \\
&+ v_{2\perp}^2\,\xi\,\partial^3_{v_{1\perp}r^2} \Big] \frac{f_2}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi\,\mathrm{d}v_{2\perp}\,\mathrm{d}v_{2\parallel}
\end{aligned}$$

where the left-hand side contains the usual transient term, the first integral on the right-hand side represents the pressure integral, and this is followed by the viscous term.
The coefficients a,b,c,d,e were introduced here for sake of brevity. Those coefficients are partly long nonlinear rational functions of the invariants and are the result of the scalar product of the existing vectors in the original equation and the terms generated in the application of the chain rule. The definitions of the coefficients are given in the Appendix (37).
The same method can be used to transform the continuity equations given in equations (27) into the adapted system. Using the same substitution as before the continuity equations become

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{r}} \cdot \int \boldsymbol{v}_1 f_2\,\mathrm{d}\boldsymbol{v}_1 = \qquad (36a) \\
&2\pi \iint \left\{ \frac{4v_{1\parallel}v_{1\perp}}{r} f_2 + \frac{v_{1\perp}^2 v_{2\perp}}{r}\,\xi\,\frac{\partial f_2}{\partial v_{2\parallel}} \right. \\
&\quad - \frac{v_{2\parallel}v_{1\perp}}{v_{2\perp}r}\left(2v_{1\perp}v_{2\perp}\xi - v_{1\parallel}v_{2\parallel}\right)\frac{\partial f_2}{\partial v_{2\perp}} \\
&\quad + v_{1\perp}v_{1\parallel}\frac{\partial f_2}{\partial r} + \frac{v_{1\perp}\xi}{r}\left[\frac{v_{1\parallel}}{v_{1\perp}^2}\left(2v_{1\perp}^2 - v_{1\parallel}^2\right) \right. \\
&\quad \left. \left. + \frac{v_{2\parallel}}{v_{2\perp}^2}\left(2v_{1\perp}v_{2\perp}\xi - v_{1\parallel}v_{2\parallel}\right) \right] \frac{\partial f_2}{\partial \xi} \right\} \mathrm{d}v_{1\perp}\,\mathrm{d}v_{1\parallel} = 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{r}} \cdot \int \boldsymbol{v}_2 f_2\, d\boldsymbol{v}_2 = \qquad (36b) \\
&\iint_{-1}^{1} \left\{ v_{2\parallel}v_{2\perp}\frac{\partial}{\partial r} - \frac{v_{1\parallel}v_{2\perp}}{v_{1\perp}r}\left(2v_{1\perp}v_{2\perp}\xi - v_{1\parallel}v_{2\parallel}\right)\frac{\partial}{\partial v_{1\perp}} \right. \\
&\quad + \frac{v_{1\perp}v_{2\perp}^2}{r}\,\xi\,\frac{\partial}{\partial v_{1\parallel}} - \frac{v_{2\perp}}{r}\left[\frac{2v_{1\parallel}v_{2\perp}}{v_{1\perp}}\,\xi\,\frac{2-\xi^2}{1-\xi^2} \right. \\
&\quad \left. \left. + \frac{1}{1-\xi^2}\left(-\frac{v_{1\parallel}^2 v_{2\parallel}}{v_{1\perp}^2} + \frac{v_{2\parallel}}{v_{2\perp}^2}\left(2v_{2\perp}^2 - v_{2\parallel}^2\right) \right) + 4v_{2\parallel} \right] \right\} \\
&\quad \frac{2f_2}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi\,\mathrm{d}v_{2\perp}\,\mathrm{d}v_{2\parallel} = 0
\end{aligned}$$

While the reduced continuity equations (36) appear different from the continuity equations (29) and (31), they are merely different representations of the same equations. The difference between the two sets of equations lies in the treatment of the derivative with respect to $\boldsymbol{r}$. In the former case, the derivative is transformed by evaluating the total derivative of f_2 with respect to $\boldsymbol{r}$, whereas in equations (29) and (31), the derivative is eliminated through integration of the equations.

The resulting equations are a fully dimension-reduced representation of the one-point hierarchy equation and the continuity equations. They contain only scalar variables associated with the isotropic geometry and therefore provides a reduced framework for studying the dynamics of homogeneous isotropic turbulence in terms of reduced probability density functions.

Conclusion

In this work, we investigated the statistical description of homogeneous isotropic turbulence within the framework of the LMN hierarchy. By using the symmetries of homogeneity and isotropy, the multi-point probability density functions were reformulated in terms of scalar invariants, providing a dimensional reduction of the statistical description.

The consistency of this invariant formulation with the classical description of homogeneous isotropic turbulence was demonstrated by relating the isotropic two-point PDF to the von Kármán-Howarth correlation functions. The longitudinal and transversal velocity correlations were obtained as exact moments of the reduced two-point PDF, showing that the classical correlation tensor arises naturally from the probabilistic description. Furthermore, the continuity equations of the PDF formulation were shown to recover the known incompressibility constraint between longitudinal and transversal correlations.

Beyond the kinematic description, the one-point LMN equation was transformed into the adapted coordinate system introduced for the isotropic reduction. This formulation avoids the explicit dependence on the orientation of the coordinate system and provides a dimensionally reduced representation of the PDF dynamics in homogeneous isotropic turbulence. Through the use of scalar invariants, the number of independent variables is reduced from twelve spatial and velocity variables to six scalar invariants, while preserving the complete statistical information contained in the original formulation.

This framework establishes a direct connection between the LMN hierarchy and classical moment-based turbulence statistics while simultaneously providing a reduced representation of the LMN hierarchy. Future work could leverage this dimensional reduction of the PDF hierarchy to develop tractable closure models at the level of the two-point statistics and to explore approximate numerical solution strategies for the resulting reduced equations, thereby using the invariant formulation to simplify the treatment of homogeneous isotropic turbulence.

Appendix

The coefficients used in equation (36a) are defined as follows:

$$a(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r) = \frac{v_{2\perp}^2\xi(\xi-1)^2(\xi+1)^2v_{1\parallel}^2 - 2v_{2\parallel}v_{2\perp}\left(v_{2\perp}\xi^3 - 4v_{2\perp}\xi + 3v_{1\perp}\right)v_{1\parallel} + v_{2\perp}^2\xi(\xi-1)^2(\xi+1)^2v_{1\perp}^2}{(1-\xi^2)^2r^2v_{1\perp}^3} \tag{37a}$$

$$b(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r) = \frac{v_{2\perp}^2\xi(\xi-1)^2(\xi+1)^2v_{1\parallel}^2 - 2v_{2\parallel}v_{2\perp}\left(v_{2\perp}\xi^3 - 4v_{2\perp}\xi + 3v_{1\perp}\right)v_{1\parallel} + v_{2\perp}^2\xi(\xi-1)^2(\xi+1)^2v_{1\perp}^2}{(1-\xi^2)^2r^2v_{1\perp}^2} \tag{37b}$$

$$c(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r) = \frac{(-4v_{1\parallel}v_{1\perp}v_{2\perp}\xi + v_{1\parallel}^2v_{2\parallel} - v_{1\perp}^2v_{2\parallel})v_{2\perp}}{r^2v_{1\perp}} \tag{37c}$$

$$d(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r) = \frac{(v_{1\perp}v_{2\perp}\xi - 2v_{1\parallel}v_{2\parallel})v_{1\perp}v_{2\perp}}{r^2} \tag{37d}$$

$$e(v_{1\parallel},v_{1\perp},v_{2\parallel},v_{2\perp},\xi,r) = \frac{(-2v_{1\perp}v_{2\perp}\xi + v_{1\parallel}v_{2\parallel})v_{1\parallel}v_{2\perp}}{r^2} \tag{37e}$$

(37f)

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