14th International Symposium on Turbulence and Shear Flow Phenomena (TSFP14)
Heidelberg, Germany, July 28–31, 2026

ANISTROPIC INCREMENTS IN 2D DECAYING STRATIFIED TURBULENCE

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ABSTRACT

Stratified turbulence, whether driven by temperature or salinity, represents a multiscale phenomenon where waves and vortices interact across a wide range of scales. To explore this complex interplay, we take advantage of the computational capabilities of GPUs to develop a code that computes increments over a complete 2D field. This method is applied to two simulations of the Boussinesq equations, with or without stratification. By calculating the statistics of moments of increments across the entire domain, we first validate the evolution equation for second-order structure functions, ensuring its validity . We then analyze each term of the equation in detail, focusing on scale-dependent behaviors and deviations from isotropy. Through this work, we seek to highlight the significant potential of full-field increment calculations in deepening our understanding of stratified turbulence dynamics.

1 Introduction

Internal gravity waves, observable in fluids with stable density stratification such as the oceans and parts of the atmosphere, enhance the spatial propagation of energy, in addition to turbulent transport. Moreover they can interact with turbulent eddies and large-scale motions over a wide range of scales (Gregg, 1977), as observed experimentally (Savaro *et al.*, 2020) and numerically (Clark Di Leoni & Mininni, 2015). The stratified turbulence framework represents a simplified approach, compared to actual geophysical flows, but it retains three essential ingredients, namely waves, eddies and large-scale motion also called shear mode (Zoppi *et al.*, 2025). In this context, the evolution of fluctuations of velocity $\mathbf{u}$, and of buoyancy b around the mean constant gradient, is given by the Navier-Stokes equations under the Boussinesq approximation:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla).\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + b\mathbf{y} \tag{1}$$

$$\partial_t b + \mathbf{u}\cdot\nabla b = -N^2 u_y + \chi\nabla^2 b \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{3}$$

where p is the pressure field, ν the kinematic viscosity, χ the thermal/solutal diffusivity. $N = \sqrt{(g/\rho_0)\partial_y\langle\rho(y)\rangle}$ is the Brunt-Väisälä frequency with a constant mean gradient of density $\partial_y\langle\rho(y)\rangle$, g the gravity, and ρ_0 the reference density. Without stratification, at $N = 0$, equation (2) reduces to an advection-diffusion equation, and b evolves as a passive scalar. From the Boussinesq equations, relevant dimensionless numbers in the context of stratified turbulence can be derived: the Froude number $Fr = \varepsilon_u/Nu_h^2$ and the buoyancy Reynolds number $Re_b = \varepsilon_u/\nu N^2$ (Maffioli *et al.*, 2016). ε_u is the kinetic energy dissipation. Here, in a first approach, we focus on two-dimensional stratified flows with velocity $\mathbf{u} = (u_x, u_y)$, thus reducing the flow complexity in comparison with the 3D case.

According to Brethouwer *et al.* (2007), different regimes are possible depending on the values of Fr and Re_b. For a strong stratification $Fr \ll 1$, we distinguish between the regime of a viscosity-affected stratified flow (VASF) where $Re_b \ll 1$ with large horizontal layers and a regime of strongly stratified turbulence (SST) with $Fr \gg 1$, where large vertically sheared horizontal flow (VSHF, or shear modes) observed numerically (Fitzgerald & Farrell, 2018) and theoretically (Godeferd & Cambon, 1994), and three-dimensional overturning structures are observed. In addition, it is possible to add the wave turbulence regime (WTR) with $Re_b < 1$ and $Fr \ll 1$, where non linear interactions occur mainly between internal gravity waves.

The typical turbulent length in turbulence is the Kolmogorov $L_\eta = (\nu^3/\varepsilon_u)^{1/4}$ at which most of the dissipation occurs. In stratified turbulence, the Ozmidov length scale $L_O = \sqrt{(\varepsilon_u/N^3)}$ represents the scale above which internal gravity waves dominate the flow and below which the classical turbulent cascade dominates the flow. The buoyancy Reynolds number $Re_b = (L_0/L_\eta)^{3/4}$ also serves as a measure of the scale separation between L_O and L_η (Davidson, 2013). Consequently, the anisotropy induced by the stratification manifests only at scales larger than the Ozmidov scale. When a clear scale separation exists ($L_O \gg L_\eta$), the flow recovers isotropy at small scales.

Since turbulence is a multiscale phenomenon, what happens when waves, large-scale structures, and eddies are present? To investigate interactions between scales, the most relevant statistical quantity is the moment of buoyancy field increment:

$$\delta b^p(\mathbf{r},t) = \langle (b(\mathbf{x}+\mathbf{r},t) - b(\mathbf{x},t))^p \rangle_{x,y} \tag{4}$$

Since the field is homogeneous, we use the ergodic hypothesis to replace the spatial average, denoted as $\langle \cdot \rangle_{x,y}$, with an ensemble average. Consequently, the increment depends only on the separation distance r and not on the spatial coordinates. Furthermore, if statistical stationarity is achieved, a time average can also be employed.

In the present work, we focus on the increment of buoyancy, δb, which is a scalar quantity, in contrast with previous works dedicated to statistics of velocity increment $\delta \mathbf{u}$. Following the approach of the Kármán-Howarth-Monin equation used to obtain the evolution equation for the velocity structure function δu^2 in homogeneous, unstratified turbulence (Frisch *et al.*, 1995; Monin & Yaglom, 1975), we write an analogous evolution equation for δb^2 (see Augier *et al.* 2012 or in a more general case Danaila, L.).

$$\partial_t \delta b^2 + \partial_{r_j}(\delta u_j \delta b^2) = -2N^2 \delta u_y \delta b + 2\chi \partial_{r_j r_j} \delta b^2 - 4\varepsilon_b \tag{5}$$

$\varepsilon_b = (\nabla b)^2$ is the dissipation rate of available potential energy. The evolution equation reveals that the non-linear term $\partial_{r_j} \delta u_j \delta b^2$ is key to studying nonlinear scale interactions and energy cascades. In the absence of stratification, b behaves as a passive scalar, exhibiting only a direct cascade toward smaller scales.

When stratification is introduced, the dynamics at horizontal scales smaller than the Ozmidov scale resemble that of isotropic 2D turbulence. In this regime, in addition to the direct enstrophy cascade, an inverse energy cascade emerges with a power-law spectrum $\propto k^{-5/3}$ (Kraichnan, 1967), analogous to the direct cascade observed in 3D isotropic turbulence. However, in actual stratified atmospheric or oceanic flows, the dynamics is never completely 2D. This raises a fundamental question: What is the origin of the observed -5/3 horizontal spectra in natural stratified flows? To address this question, Cho & Lindborg (2001) demonstrated that increment-based analysis provides a way to address this question. For a comprehensive review of this topic, see Deusebio *et al.* (2014).

This example illustrates the complementary and essential insights provided by increment analysis in understanding turbulence dynamics. In this context, we perform numerical simulation, as detailed in section 2.1, in which we compute the increments across the entire domain with a methodology described in Section 2.2. In section 3, we analyze the various terms of equation (5) for two distinct simulations, with or without stratification. This final section will highlight the added value of the full domain increment calculations by examining the contributions of each terms of the evolution equation (5).

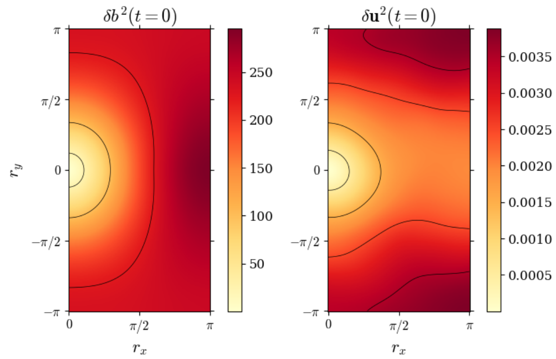

Figure 1. Second-order structure functions of the fields b (left) and $\mathbf{u}$ (right) at $t = 0$.

2 Numerical methods

2.1 Direct numerical simulation

We use a standard pseudo-spectral Fourier algorithm in a 2D-periodic spatial domain with phase shifting to treat aliasing in the nonlinear term (see Lam *et al.* 2020, for details) and with a spatial resolution of 512^2 points. The Prandtl number is $Pr = \nu/\chi = 1$. We consider decaying turbulence initialized with a velocity field obtained from a previous 2D isotropic simulation. The buoyancy field is initialized using a power-law spectrum. From these initial fields, two simulations are performed: one for homogeneous isotropic turbulence (at least at small scales) and a second with stratification. The parameters for both simulations are summarized in Table 1.

N	ν	Re_λ	Fr	Re_b	L_η	L_O
0	10^{-3}	1	-	-	0.1	-
10	10^{-3}	1	10^{-3}	0.02	0.02	10^{-3}

Table 1. Simulations parameters

Note that, since we simulate decaying turbulence, vertically sheared horizontal flow does not have time to develop significantly. With the chosen parameters, we simulate a regime intermediate between wave turbulence ($Fr \ll 1$ and $Re_b < 1$) and a viscosity-affected stratified flow regime ($Fr \ll 1$ and $Re_b \ll 1$). The characteristic scales, L_O and L_η, indicate that the resolution ought to be larger to achieve scale separation. Viscous dissipation acts at scales larger than those at which isotropic 2D turbulence would typically emerge.

The second-order structure functions of the initial fields are presented on Fig. 1. The analysis reveals a pronounced anisotropy in the velocity field, starting at intermediate scales. This comes from the initialization field in which some condensation is observed at large scales. In contrast, the density field exhibits significantly greater isotropy by construction at initial time.

2.2 Computation of increments

Calculating increments is computationally expensive. For 2D simulations with a resolution of $N_x = N_y = 512$ and $N_t = 1024$ time steps, the total number of increment computations amounts to $N_x \times N_y \times N_t \times N_{r_x} \times N_{r_y} \sim 7 \times 10^{13}$ subtractions, in addition to the operations required for averaging. Given the algorithmic simplicity and independence of these operations, GPU code is highly suitable.

To address this challenge, we developed a GPU code using C++ and CUDA. To optimize performance, it is critical to maximize the reuse of data loaded into the fastest memory layers closest to the processors (e.g. L1 cache and registers). Otherwise, the memory bandwidth between RAM and the processors becomes a significant bottleneck, severely limiting computational speed. A further optimization involves the averaging process in the x and y directions. To minimize memory bandwidth saturation, partial averages are computed within individual warps after calculating a subset of terms, followed by a final reduction within each block.

Using the optimized GPU code, the computation of $\delta b^2(\mathbf{r},t)$ takes approximately 1 min on Nvidia H100 GPUs. For comparison, the same calculation would require roughly 500 hours on a CPU with python and specialized libraries.

3 Results

This section is dedicated to the analysis of the two simulations and is structured around two main objectives. First we are going to validate the evolution equation (5), and second we will analyze the contributions of its individual terms. We begin by examining the case of homogeneous turbulence without stratification, followed by an investigation of the stratified case.

3.1 Unstratified 2D turbulence

For the simulation of unstratified turbulence, the time-integrated terms of equation (5) are presented in Fig. 3. The relative error of the integrated equation is computed as

$$Err = \left\langle \frac{\int_0^T \left(\partial_t \delta b^2 + \partial_{r_j}(\delta u_j \delta b^2) + 2N^2 \delta u_y \delta b - 2\chi \partial_{r_j r_j} \delta b^2 + 4\varepsilon_b\right) \mathrm{d}t}{\int_0^T \partial_t \delta b^2 \mathrm{d}t} \right\rangle_{r_x, r_y} \tag{6}$$

For the unstratified case we find a relative error of the order of $Err = 1\%$, confirming its validity. As expected, the δb^2 term exhibits isotropy at small scales r. However, the initial velocity field is not isotropic across all scales, which explains the anisotropy and the two preferred directions visible on Figs. 3b and c for the two redistribution terms. Furthermore, in the absence of coupling with the buoyancy field which remains isotropic the initial anisotropy is preserved throughout the simulation.

The temporal variation term shown on Fig. 3a is negative because it is a simulation of decaying turbulence. Here, δb^2 represents the cumulative quantity of b^2 between 0 and $\mathbf{r}$, averaged over the entire domain (Campagne *et al.*, 2014). Since $b^2 \geq 0$, δb^2 increases with r, which explains the observed increase in $\delta b^2(t=T) - \delta b^2(t=0)$ with r even though energy is dissipated at small scales.

The diffusion and nonlinear terms can be expressed as divergences of a field, and thus represent interscale transfer terms. Consequently, the total dissipation observed in the temporal term is fully accounted by the dissipation of buoyancy: $\int_0^T \varepsilon_b(t)dt = 29.4$.

The nonlinear term provides insight into the direction of the cascade: positive values indicate energy gain, while negative values signify loss. In Fig. 3c, we observe a gain in b^2 at small r and a loss at large r, consistent with a forward cascade as expected for a passive scalar.

The dissipation term indicates that dissipation occurs predominantly at small scales.

3.2 Stratified 2D turbulence

For the stratified turbulence simulation, the time-integrated terms of equation 5 are presented in Fig. 3. The relative error remains on the order of $Err = 1\%$, confirming the numerical consistency of the results.

Figure 3a shows that anisotropy emerges even at small scales. This anisotropy originates from the stratification term $2N^2\delta u_2 \delta b$ (Fig. 3d), which represents the exchange between the available potential and kinetic energy. This exchange predominantly occurs at large scales, driven by wave motions. A characteristic vertical length scale appears, which may be interpreted as the typical layer thickness. Further investigation is required to establish its relationship with conventional vertical scales in stratified flows.

The dissipation term, which is more significant at small scales, is isotropic (Fig. 3b). However, our simulations do not achieve scale separation, meaning that small scales remain anisotropic as their dynamics is still affected by waves. Nevertheless, the dissipation term exhibits isotropy even at small scales. This may indicate that vortices predominantly dissipate energy. This is closely related to the role of internal gravity waves in the dynamics of small-scale processes. For exemple, several studies have investigated their influence on the forward energy cascade (e.g., Deusebio *et al.* 2013). To address these questions, we would like to decompose the total field into vortical and wave components using the method of Lam *et al.* (2020). We will then compute and analyze the increments of the separated fields. Additionally, in collaboration with L. Danaila, we have derived the evolution equations for the increments of the separated fields (forthcoming publication). These equations provide a complementary framework for analyzing the increments of the decomposed fields.

The nonlinear term reveals a gain of potential energy at both small and large scales, with a loss at intermediate scales. This behavior is indicative of a dual cascade, combining direct and inverse energy transfers. It is worth noting that without computing increments across the entire field, observations would likely have been restricted to the directions $r_x = 0$ or $r_y = 0$, thereby obscuring the detection of the inverse cascade.

The dissipation of buoyancy ε_b is significantly reduced in this case, as the total dissipation is distributed between ε_b and the energy exchange term. Additionally, the two preferred directions observed in the unstratified case (Fig. 3b) disappear due to the coupling between b and u_y.

4 Conclusion

Although this study was limited to the case of decaying 2D turbulence, we have demonstrated that the analysis of the full field increment provides novel insights into the dynamics of turbulent flows. This approach has become possible thanks to the computational capabilities of modern GPUs. The computation of increments across the entire field enabled us to validate the second-order structure function evolution equation for buoyancy (Eq. 5, derived by L.Danaila), confirming its theoretical consistency. Furthermore, the introduction of stratification revealed the emergence of anisotropy, a critical feature that must be incorporated into closure models for stratified turbulence. By accessing domain-wide increments, we obtained quantitative measurements of angular variations, which are essential for the calibration and refinement of these models. Increment analysis also highlighted the directionality of energy

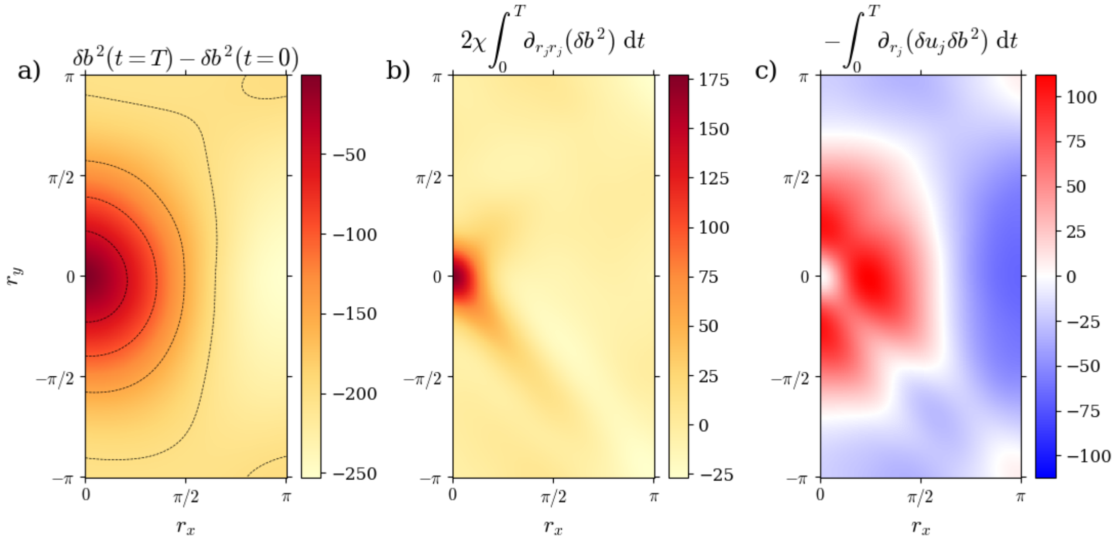

Figure 2. Time-integrated term of the evolution equation δb^2 (Eq. 5) for the homogeneous, nonlinear simulation without stratification. Relative error of the order of 1%. $\int_0^T \varepsilon_b(t)dt = 29.4$, with $T = 204.6$. The periodicity of the domain implies point symmetry of $\delta b^2(\mathbf{r}) = \delta b^2(-\mathbf{r})$ (panels a and b), and antisymmetry of $\delta u_j \delta b^2$ (panel c); hence only $r_x > 0$ is displayed.

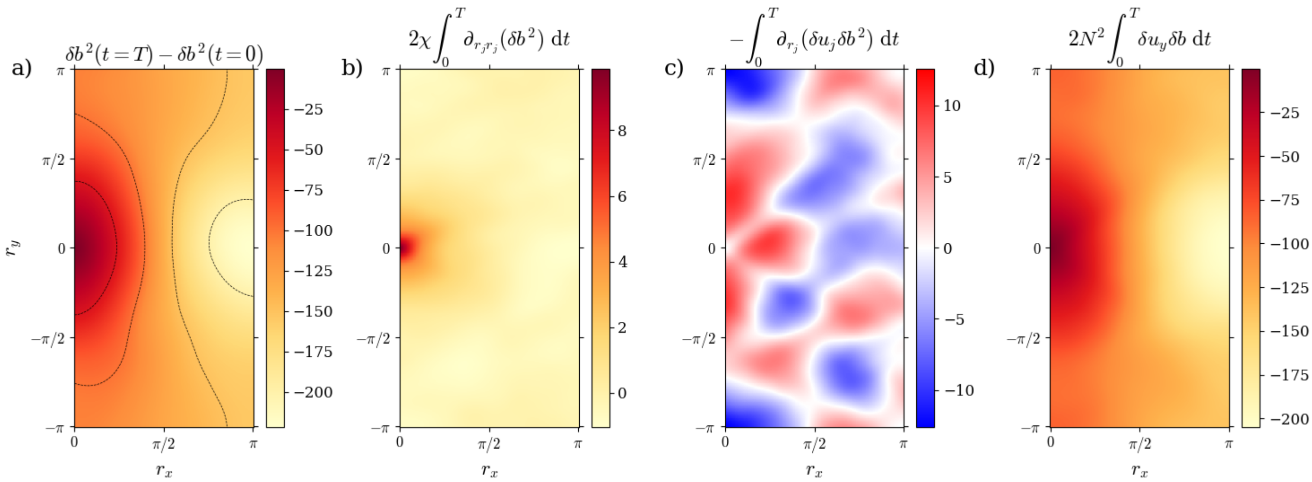

Figure 3. Time-integrated term of the evolution equation δb^2 (Eq. 5) for the homogeneous nonlinear simulation with stratification. Relative error on the order of 1%. $\int_0^T \varepsilon_b(t)dt = 2.5$, with $T = 10.23$. Same remark as on figure regarding the symmetries of the plots.

cascades, confirming the direct cascade for passive scalars and the dual cascade (direct and inverse) in 2D stratified turbulence. These findings emphasize that a comprehensive analysis of the nonlinear term and its associated cascades requires increments computed over the entire domain.

The results presented in this study are preliminary. Our future work will focus on investigating the influence of key dimensionless parameters, such as the Froude number *Fr* and buoyancy Reynolds number Re_b, across both 2D and 3D simulations. This parametric study will allow us to examine differents regime from VASF to wave turbulence regime. Additionally, conducting stationary simulations will provide a more robust framework for analyzing energy cascades and identifying scaling laws. Another promising direction involves decomposing the total flow field into its vortical and wave components, following the approach outlined by (Lam *et al.*, 2021). By computing increments separately for these decomposed fields, we aim to disentangle the distinct contributions of waves and vortices to energy cascades, anisotropy and exchange. This decomposition will enable a detailed investigation of the role of wave in the small scale dynamics, particularly their contribution to the direct cascade (Deusebio *et al.*, 2013).

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