14th International Symposium on Turbulence and Shear Flow Phenomena (TSFP14)
Heidelberg, Germany, July 28–31, 2026

HELICOCLINICITY: HELICITY EFFECTS ON PASSIVE SCALAR TRANSPORT IN INHOMOGENEOUS TURBULENCE

Nobumitsu Yokoi
Department of Earth and Planetary Science
University of Tokyo
7-3-1 Hongo, Bunkyo, Tokyo 113-0003, Japan
nobyokoi@gmail.com

ABSTRACT

The effects of helicity (velocity–vorticity correlation) on scalar transport in inhomogeneous turbulence is investigated with the aid of multiple-scale renormalised perturbation theory. Helicity is ubiquitously present in geophysical, astrophysical, and engineering flows where vorticity, rotation and density stratification play key roles. It is suggested that the inhomogeneity of turbulent helicity represented by $\nabla\langle\mathbf{u}'\cdot\boldsymbol{\omega}'\rangle$ [$\mathbf{u}'$: velocity fluctuation, $\boldsymbol{\omega}'(=\nabla\times\mathbf{u}')$: vorticity fluctuation] coupled with the non-uniform scalar $\nabla\Theta$ gives a contribution to the turbulent scalar flux $\langle\mathbf{u}'\theta'\rangle$ (Θ: mean scalar, θ': fluctuating scalar). In the inhomogeneous turbulence, this helicity effect alters the turbulent scalar flux as compared with the non-helical turbulence. It is also pointed out that non-similar spatial distributions between the turbulent energy and turbulent helicity should be important for this helicity effect.

INTRODUCTION

Helicity, an inviscid invariant of fluid equation, is a measure of non-reflectional symmetry of fluid system. Helicity contributes in a crucial way to the dynamic properties of turbulence and its statistical behaviours (Pouquet & Yokoi, 2022). As for its roles on the effective momentum transport, the inhomogeneous turbulent helicity density $H \equiv \langle\mathbf{u}'\cdot\boldsymbol{\omega}'\rangle$ [$\mathbf{u}'$: velocity fluctuation, $\boldsymbol{\omega}'(=\nabla\times\mathbf{u}')$: vorticity fluctuation] enters the Reynolds stress as the coupling coefficient of the mean absolute vorticity, and counter-balances the eddy viscosity, which coupled with the mean velocity strain (Yoko & Yoshizawa, 1993; Yokoi & Brandenburg, 2016). This helicity effect on the turbulent momentum transport has been recently implemented into the subgrid-scale (SGS) modelling and the validity of the helicity SGS model has been confirmed with a series of high-resolution direct numerical simulations (DNSs) (Yokoi, et al., 2026). This effect was applied to the stellar convection problem to elucidate the angular-momentum transport in a star (Yokoi & Miesch, 2025).

On the other hand, the helicity effects in the scalar transport have not been fully explored. There are only a few investigations on this problem (Brandenburg, et al. 2025, and works cited therein). In particular, the helicity effects on turbulent scalar flux in inhomogeneous turbulence is still open problem. In this work, we address this problem with the aid of the multiple-scale renormalised perturbation theory: a closure theory of inhomogeneous and anisotropic turbulence.

TURBULENT SCALAR DIFFUSION

A mean scalar Θ obeys the equation

$$\frac{\partial\Theta}{\partial t} + (\mathbf{U}\cdot\nabla)\Theta = -\nabla\cdot\mathbf{H}_\theta + \kappa\nabla^2\Theta \tag{1}$$

where κ is the molecular diffusivity and $\mathbf{H}_\theta$ is the turbulent scalar flux define by

$$\mathbf{H}_\theta = \langle\mathbf{u}'\theta'\rangle \tag{2}$$

In the following, with the aid of multiple-scale renormalised perturbation expansion theory (Yoshizawa 1984; Yokoi, 2020; Yokoi 2024, and references cited therein), we explore the turbulent helicity effect on the turbulent scalar flux $\mathbf{H}_\theta$.

MULTIPLE-SCALE ANALYSIS

In the multiple-scale formulation, we introduce the fast variables $(\xi;\tau)$ and the slow variables $(\mathbf{X};T)$ with the definitions of

$$\xi = \mathbf{x},\ \mathbf{X} = \delta\mathbf{x};\ \ \tau = t,\ T = \delta t \tag{3}$$

The scale parameter δ is linked to the large-scale spatial and temporal derivatives as

$$\nabla_\mathbf{x} = \nabla_\xi + \delta\nabla_\mathbf{X},\ \ \frac{\partial}{\partial t} = \frac{\partial}{\partial\tau} + \delta\frac{\partial}{\partial T} \tag{4}$$

A field quantity f is decomposed into the mean and fluctuations as

$$f(\xi,\mathbf{X};\tau,T) = F(\mathbf{X};T) + f'(\xi,\mathbf{X};\tau,T) \tag{5}$$

The mean field F depends on the slow variables, but the fluctuating field f' depends on both the fast and slow variables. We assume that the fluctuating field is homogeneous with respect to the fast space variable ξ. Then the Fourier transform of the fluctuating field is written as

$$f'(\xi,\mathbf{X};\tau,T) = \int f'(\mathbf{k},\mathbf{X};\tau,T)\exp[-i\mathbf{k}\cdot(\xi-\mathbf{U}\tau)]d\mathbf{k} \tag{6}$$

where the factor $\mathbf{k}\cdot(\xi-\mathbf{U}\tau)$ means that the transform is taken in the frame moving with the mean velocity $\mathbf{U}$.

Here, we just show the fluctuating scalar θ' equation. Hereafter, the dependence of f' on the slow variables $\mathbf{X}$ and T may not be explicitly written.

In this two-scale formulation, the scalar field $\theta'(\xi;\tau)$ equation in the wave-number space is written as

$$\begin{aligned}\mathscr{L}[\theta',\mathbf{u}']&\left[\equiv\frac{\partial\theta'(\mathbf{k};\tau)}{\partial\tau}+\lambda k^2\theta'(\mathbf{k};\tau)\right.\\&\left.\quad-ik_\ell\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,u'_\ell(\mathbf{p};\tau)\theta'(\mathbf{q};\tau)\right]\\&=\delta\left[-u'_\ell(\mathbf{k};\tau)\frac{\partial\Theta}{\partial X_\ell}-\frac{D\theta'(\mathbf{k};\tau)}{DT}-\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\right.\\&\left.\quad\times\frac{\partial}{\partial X_\ell}\left(u'_\ell(\mathbf{p};\tau)\theta'(\mathbf{q};\tau)\right)-\delta(\mathbf{k})\frac{\partial H_{\theta\ell}}{\partial X_\ell}\right]\end{aligned}\tag{7}$$

($\mathscr{L}$ is the operator defined by the first equality).

A field quantity f' is further expanded as

$$f'(\mathbf{k};\tau)=\sum_{n=0}^{\infty}\delta^n f'_n(\mathbf{k};\tau)\tag{8}$$

The zeroth-order field f'_0 corresponds to the homogeneous isotropic turbulence, and the large-scale inhomogeneities in space and time enter through the higher-order terms in δ.

With this derivative expansion, the equations of the first- and second-order scalar fields, $\theta'_1(\mathbf{k};\tau)$ and $\theta'_2(\mathbf{k};\tau)$, are written, respectively, as

$$\begin{aligned}\mathscr{L}[\theta'_1,\mathbf{u}'_0]=&\ \frac{\partial\theta'_1(\mathbf{k};\tau)}{\partial\tau}+\lambda k^2\theta'_1(\mathbf{k};\tau)\\&-ik_\ell\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,u'_{0\ell}(\mathbf{p};\tau)\theta'_1(\mathbf{q};\tau)\\&=-u'_{0\ell}(\mathbf{k};\tau)\frac{\partial\Theta}{\partial X_\ell}-\frac{D\theta'_0(\mathbf{k};\tau)}{DT}\\&-\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,\frac{\partial}{\partial X_\ell}\left(u'_{0\ell}(\mathbf{p};\tau)\theta'_0(\mathbf{q};\tau)\right)\\&+ik_\ell\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,u'_{1\ell}(\mathbf{p};\tau)\theta'_0(\mathbf{q};\tau)\end{aligned}\tag{9}$$

$$\begin{aligned}\mathscr{L}[\theta'_2,\mathbf{u}'_0]=&\ \frac{\partial\theta'_2(\mathbf{k};\tau)}{\partial\tau}+\lambda k^2\theta'_2(\mathbf{k};\tau)\\&-ik_\ell\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,u'_{0\ell}(\mathbf{p};\tau)\theta'_2(\mathbf{q};\tau)\\&=-\frac{D\theta'_1(\mathbf{k};\tau)}{DT}-\frac{\partial\Theta}{\partial X_\ell}u'_{\mathrm{S}1\ell}(\mathbf{k};\tau)+i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\frac{\partial u'_{0m}(\mathbf{k};\tau)}{\partial X_m}\\&+\mathrm{N.T.}\end{aligned}\tag{10}$$

where some higher-order terms are dropped, and $\mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau)$ is the solenoidal velocity defined by

$$u'_{\mathrm{S}1i}(\mathbf{k};\tau)=u'_{1i}(\mathbf{k};\tau)+i\frac{k_i}{k^2}\frac{\partial u'_{0m}(\mathbf{k};\tau)}{\partial X_m}\tag{11}$$

which automatically satisfies the solenoidal condition as

$$\mathbf{k}\cdot\mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau)=0\tag{12}$$

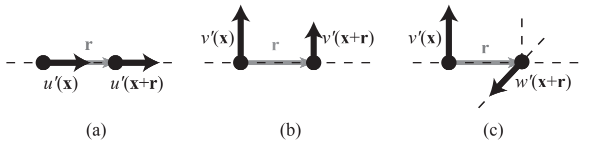


Figure 1. Two-point and two-time velocity correlation $\langle\mathbf{u}'(\mathbf{x};t)\mathbf{u}'(\mathbf{x}';t')\rangle$ in homogeneous and isotropic turbulence. The displacement vector of two points is defined by $\mathbf{r}=\mathbf{x}'-\mathbf{x}$, and $r=|\mathbf{r}|$. (a) Longitudinal correlation, (b) transverse correlation, and (c) cross correlation.

Note that $\partial u'_{0m}/\partial X_m$ does not vanish in general.

Associated with (9) and (10), the Green's function of the scalar equations, $G'_\theta(\mathbf{k},\mathbf{X};\tau,\tau',T)$, is introduced as

$$\mathscr{L}[G'_\theta(\mathbf{k};\tau,\tau'),\mathbf{u}'_0]=\delta(\tau-\tau')\tag{13}$$

TURBULENT SCALAR FLUX

The turbulent scalar flux is calculated as

$$\begin{aligned}\mathbf{H}_\theta&=\langle\mathbf{u}'(\xi,\tau)\theta'(\xi;\tau)\rangle=\int d\mathbf{k}\frac{\langle\mathbf{u}'(\mathbf{k};\tau)\theta'(\mathbf{k}';\tau)\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\&=\int d\mathbf{k}\,\frac{\left\langle\mathbf{u}'_0(\mathbf{k};\tau)\theta'_0(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\&+\delta\int d\mathbf{k}\left(\frac{\left\langle\mathbf{u}'_0(\mathbf{k};\tau)\theta'_1(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}+\frac{\left\langle\mathbf{u}'_1(\mathbf{k};\tau)\theta'_0(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\right)\\&+\delta^2\int d\mathbf{k}\left(\frac{\left\langle\mathbf{u}'_0(\mathbf{k};\tau)\theta'_2(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}+\frac{\left\langle\mathbf{u}'_1(\mathbf{k};\tau)\theta'_1(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\right.\\&\left.\quad+\frac{\left\langle\mathbf{u}'_2(\mathbf{k};\tau)\theta'_0(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\right)+O\left(\delta^3\right)\end{aligned}\tag{14}$$

As for the lowest-order velocity field, we assume it the homogeneous isotropic. In homogeneous and isotropic turbulence, the two-point and two-time velocity correlation $\langle\mathbf{u}'(\mathbf{x};t)\mathbf{u}'(\mathbf{x}';t')\rangle$ depends only on the distance $r=|\mathbf{r}|$ of the two points [$\mathbf{r}(=\mathbf{x}'-\mathbf{x})$: displacement vector], and is expressed as

$$\begin{aligned}\langle u'_i(\mathbf{x};t)u'_j(\mathbf{x}';t')\rangle&=\langle u'_i(\mathbf{0};t)u'_j(\mathbf{r};t')\rangle\\&=g\frac{r_ir_j}{r^2}+f\left(\delta_{ij}-\frac{r_ir_j}{r^2}\right)+\varepsilon_{ij\ell}h\frac{r_\ell}{r}\end{aligned}\tag{15}$$

where g, f, and h are the coefficients linked to the dilatational energy, solenoidal energy, and helicity, respectively (Figure 1). Equation (15) is the mathematically generic expression of the velocity correlation of homogeneous and isotropic turbulence in the configuration space. Note that in mirror-symmetric system h vanishes. In the case of incompressible fluids, g associated with the dilatational energy vanishes.

Corresponding to Eq. (15), the expressions for the two-point and two-time velocity correlation function Q_{ij} and the Green's function G_{ij} in incompressible turbulence in the wave-number space are expressed as

$$\begin{aligned}Q_{ij}(\mathbf{k},\mathbf{X};\tau,\tau',T)&=\frac{\langle u'_{0i}(\mathbf{k},\mathbf{X};\tau,T)u'_{0j}(\mathbf{k}',\mathbf{X};\tau',T)\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\&=D_{ij}(\mathbf{k})Q(k,\mathbf{X};\tau,\tau',T)+\frac{i}{2}\frac{k_\ell}{k^2}\varepsilon_{ij\ell}H(k,\mathbf{X};\tau,\tau',T)\end{aligned}\tag{16}$$

$$G_{ij}(k,\mathbf{X};\tau,\tau',T) = \left\langle G'_{ij}(\mathbf{k},\mathbf{X};\tau,\tau',T)\right\rangle = D_{ij}(\mathbf{k})G(k,\mathbf{X};\tau,\tau',T) \tag{17}$$

where $D_{ij}(\mathbf{k})(=\delta_{ij}-k_ik_j/k^2)$ is the solenoidal projection operator (δ_{ij}: Kronecker delta, $\varepsilon_{ij\ell}$: alternating tensor) (Figure 1). For the scalar counterpart, we assume

$$Q_\theta(\mathbf{k},\mathbf{X};\tau,\tau',T) = \frac{\left\langle \theta'_0(\mathbf{k},\mathbf{X};\tau,T)\theta'_0(\mathbf{k}',\mathbf{X};\tau',T)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')} \tag{18}$$

$$G_\theta(k,\mathbf{X};\tau,\tau',T) = \left\langle G'_\theta(\mathbf{k},\mathbf{X};\tau,\tau',T)\right\rangle \tag{19}$$

Hereafter, for the sake of simplicity of notation, we suppress the explicit writing of slow variables $\mathbf{X}$ and T dependence.

FIRST-ORDER ANALYSIS

With the aid of G'_θ [Eq. (13)], Eq. (9) can be formally solved as

$$\begin{aligned}\theta'_1(\mathbf{k};\tau) = &-\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)u'_{0\ell}(\mathbf{k};\tau_1)\\ &-\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)\frac{D\theta'_0(\mathbf{k};\tau_1)}{DT} - \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\\ &\quad\times\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)\frac{\partial}{\partial X_\ell}\left(u'_{0\ell}(\mathbf{p};\tau_1)\theta'_0(\mathbf{q};\tau_1)\right)\\ &+ik_\ell\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\\ &\quad\times\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)u'_{1\ell}(\mathbf{p};\tau_1)\theta'_0(\mathbf{q};\tau_1)\end{aligned} \tag{20}$$

Multiplying $\mathbf{u}'_0(\mathbf{k};\tau)$ and taking an ensemble average, from the first term in Eq. (16), we have

$$\begin{aligned}&\frac{\left\langle u'_{0i}(\mathbf{k};\tau)\theta'_1(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &= -\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ \frac{\left\langle G'_\theta(\mathbf{k};\tau,\tau_1)u'_{0i}(\mathbf{k};\tau_1)\theta'_{0\ell}(\mathbf{k}';\tau_1)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &= -\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ \frac{\left\langle G'_\theta(\mathbf{k};\tau,\tau_1)\right\rangle\left\langle u'_{0i}(\mathbf{k};\tau_1)\theta'_{0\ell}(\mathbf{k}';\tau_1)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &= -\frac{\partial\Theta}{\partial X_i}\frac{2}{3}\int_{-\infty}^{\tau} d\tau_1\ G_\theta(k;\tau,\tau_1)Q(k;\tau,\tau_1)\end{aligned} \tag{21}$$

which leads to

$$\mathbf{H}_\theta^{(1)} = \int d\mathbf{k}\ \frac{\left\langle \mathbf{u}'_0(\mathbf{k};\tau)\theta'_1(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = -\kappa_{\mathrm{T}}\nabla\Theta \tag{22}$$

where κ_{T} is the turbulent scalar diffusivity given by

$$\kappa_{\mathrm{T}} = \frac{2}{3}\int d\mathbf{k}\ \int_{-\infty}^{\tau} d\tau_1\ G_\theta(k;\tau,\tau_1)Q(k;\tau,\tau_1) \tag{23}$$

At this level of expansion, we have no contribution of helicity represented by the second or H-related term in Eq. (16).

SECOND-ORDER ANALYSIS

With the aid of G'_θ [Eq. (13)], we formally solve Eq. (10) as

$$\begin{aligned}\theta'_2(\mathbf{k};\tau) = &-\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)\frac{D\theta'_1(\mathbf{k};\tau_1)}{DT}\\ &-\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)u'_{\mathrm{S}1\ell}(\mathbf{k};\tau_1)\\ &+i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ G'_\theta(\mathbf{k};\tau,\tau_1)\frac{\partial u'_{0m}(\mathbf{k};\tau_1)}{\partial X_m} + \mathrm{R.T.}\end{aligned} \tag{24}$$

(R.T. denotes the residual terms). Then, the turbulent scalar flux due to θ'_2 is given by

$$\begin{aligned}&\frac{\left\langle u'_{0i}(\mathbf{k};\tau)\theta'_2(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &= -\int_{-\infty}^{\tau} d\tau_1\ \frac{1}{\delta(\mathbf{k}+\mathbf{k}')}\left\langle G_\theta(k;\tau,\tau_1)u'_{0i}(\mathbf{k};\tau)\frac{D\theta'_1(\mathbf{k}';\tau_1)}{DT}\right\rangle\\ &-\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ \frac{\left\langle G_\theta(k;\tau,\tau_1)u'_{0i}(\mathbf{k};\tau)u'_{\mathrm{S}1\ell}(\mathbf{k}';\tau_1)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &+i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ \frac{1}{\delta(\mathbf{k}+\mathbf{k}')}\times\\ &\quad\left\langle G_\theta(k;\tau,\tau_1)u'_{0i}(\mathbf{k};\tau)\frac{\partial u'_{0m}(\mathbf{k}';\tau_1)}{\partial X_m}\right\rangle + \mathrm{N.T.}\end{aligned} \tag{25}$$

The third term of Eq. (25) is from the second term of the solenoidal velocity fluctuation [Eq. (11)], which is related to the pressure interaction effect. This term can be rewritten as

$$\begin{aligned}&i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau}d\tau_1\frac{1}{\delta(\mathbf{k}+\mathbf{k}')}\left\langle G'_\theta(\mathbf{k};\tau,\tau_1)u'_{0i}(\mathbf{k};\tau)\frac{\partial u'_{0m}(\mathbf{k};\tau_1)}{\partial X_m}\right\rangle\\ &= i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ G_\theta(k;\tau,\tau_1)\frac{\partial}{\partial X_m}\frac{\left\langle u'_{0i}(\mathbf{k};\tau)u'_{0m}(\mathbf{k};\tau_1)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &-i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ \frac{1}{\delta(\mathbf{k}+\mathbf{k}')}G_\theta(k;\tau,\tau_1)\times\\ &\qquad\left\langle\frac{\partial u'_{0i}(\mathbf{k};\tau)}{\partial X_m}u'_{0m}(\mathbf{k};\tau_1)\right\rangle\end{aligned} \tag{26}$$

Substituting Eq. (16), from the first term of Eq. (26), we have

$$\begin{aligned}&\int d\mathbf{k}\ i\frac{k_\ell}{k^2}\frac{\partial\Theta}{\partial X_\ell}\int_{-\infty}^{\tau} d\tau_1\ G_\theta(k;\tau,\tau_1)\frac{\partial}{\partial X_m}\frac{\left\langle u'_{0i}(\mathbf{k};\tau)u'_{0m}(\mathbf{k};\tau_1)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\\ &= \frac{1}{6}\varepsilon_{im\ell}\left(\int d\mathbf{k}\ \frac{1}{k^2}\int_{-\infty}^{\tau} d\tau_1\ G_\theta(k;\tau,\tau_1)\times\right.\\ &\qquad\left.\frac{\partial H(k,\mathbf{X};\tau,\tau_1,T)}{\partial X_m}\right)\frac{\partial\Theta}{\partial X_\ell}\end{aligned} \tag{27}$$

Then, we have a contribution of the turbulent helicity H to the turbulent scalar flux $\mathbf{H}_\theta$ as

$$\mathbf{H}_\theta^{(2)} = \int d\mathbf{k}\frac{\langle\mathbf{u}'_0(\mathbf{k};\tau)\theta'_2(\mathbf{k}';\tau)\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = \Gamma\times(\nabla\Theta) \tag{28}$$

with

$$\Gamma = \frac{1}{6}\int d\mathbf{k}\frac{1}{k^2}\int_{-\infty}^{\tau} d\tau_1\ G_\theta(k;\tau,\tau_1)\nabla H(k;\tau,\tau_1) \tag{29}$$

Equation (28) with Eq. (29) indicates that the turbulent scalar flux $\langle \mathbf{u}'\theta' \rangle$ can be induced in the direction perpendicular both to the turbulent helicity gradient ∇H and the gradient of the mean scalar gradient $\nabla\Theta$.

It follows from Eqs. (22) and (34) that the turbulent scalar flux $\mathbf{H}_\theta = \langle \mathbf{u}'\theta' \rangle$ is expressed as

$$\mathbf{H}_\theta = \mathbf{H}_\theta^{(1)} + \mathbf{H}_\theta^{(2)} = -\kappa_{\mathrm{T}} \nabla\Theta + \eta_H (\nabla H) \times (\nabla\Theta) \tag{30}$$

This theoretical result shows that, in addition to the usual eddy-diffusivity effect $-\kappa_{\mathrm{T}}\nabla\Theta$ [Eq. (22)], which gives a turbulent flux anti-parallel to the mean scalar gradient $\nabla\Theta$, we have a turbulent scalar flux in the direction perpendicular both to the mean scalar gradient, $(\nabla H) \times (\nabla\Theta)$. In the following, this turbulent helicity effect in scalar diffusion will be discussed further.

ONE-POINT TURBULENCE MODELLING

The analytical expressions of the transport coefficients in Eq. (29), κ_{T} [Eq. (23)] and η_{T} [Eq. (29)], contain the spectral functions of turbulent energy and turbulent helicity, $Q(k)$ and $H(k)$, and the response function of the scalar equation, $G_\theta(k)$. With the aid of Eqs. (23) and (29), we construct a model for the turbulent scalar flux $\mathbf{H}_\theta = \langle \mathbf{u}'\theta' \rangle$ using appropriate one-point turbulent statistical quantities. According to Eqs. (23) and (29), the primary one-point quantities are the local density of turbulent energy K and the local density of turbulent helicity H defined by

$$K = \langle \mathbf{u}'^2 \rangle / 2 \tag{31}$$

$$H = \langle \mathbf{u}' \cdot \omega' \rangle \tag{32}$$

In addition, we consider the timescale of turbulent scalar, τ_θ, reflecting the scalar response function G_θ. The timescale τ_θ may be expressed in terms of the turbulent scalar variance $K_\theta (= \langle \theta'^2 \rangle)$ and its dissipation rate ε_θ.

In the simplest possible case that the spectral function Q can be evaluated independently to the time integral, Eq. (23) provides a simple mixing-length expression of the turbulent diffusivity κ_{T} as

$$\kappa_{\mathrm{T}} = C_{\theta 1} \tau_\theta K = C_{\theta 1} \frac{K_\theta}{\varepsilon_\theta} K \tag{33}$$

where $C_{\theta 1}$ is the model constant and $\tau_\theta (= K_\theta / \varepsilon_\theta)$ is the timescale of scalar variance. Equation (22) with Eq. (33) corresponds to the conventional simplest gradient diffusion model.

On the other hand, with H [Eq. (32)], the turbulent scalar flux can be modelled as

$$\mathbf{H}_\theta^{(2)} = \eta_H (\nabla H) \times (\nabla\Theta) \tag{34}$$

with

$$\eta_H = C_{\theta 2} \ell^2 \tau_\theta = C_{\theta 2} K \frac{K^2}{\varepsilon} \frac{K_\theta}{\varepsilon_\theta} \tag{35}$$

where $C_{\theta 2}$ is the model constant, and $\ell (= K^{3/2}/\varepsilon)$ is the turbulence length scale. This η_H is the transport coefficient for the turbulent scalar flux due to the helicity effect except for the helicity gradient ∇H. We should note that η_H is not uniform but varies in space and time in inhomogeneous turbulence ($\nabla K \neq 0$).

HELICOCLINICITY

As Eq. (1) shows, the turbulent scalar flux $\mathbf{H}_\theta$ enters the mean scalar equation in a divergence form as $\nabla \cdot \mathbf{H}_\theta$. In homogeneous turbulence, where the transport coefficient η_H is uniform, for $\mathbf{H}_\theta^{(2)}$, we have

$$\nabla \cdot \mathbf{H}_\theta^{(2)} = \eta_H \nabla \cdot [(\nabla H) \times (\nabla\Theta)] = 0 \tag{36}$$

since $\nabla \cdot [(\nabla\phi) \times (\nabla\psi)] = 0$ for arbitrary scalars ϕ and ψ. This means that in homogeneous turbulence, the turbulent scalar flux due to the helicity $\mathbf{H}_\theta^{(2)}$ does not contribute to the mean scalar evolution. In other words, at this level of analysis of helicity effect, the kinetic helicity will not change the evolution of the mean scalar Θ even if we locally have a non-zero turbulent scalar flux $\mathbf{H}_\theta = \langle \mathbf{u}'\theta' \rangle$ in the direction perpendicular to the gradient of the scalar, $\nabla\Theta$.

On the other hand, in inhomogeneous turbulence, where η_H varies in space, $\mathbf{H}_\theta^{(2)}$ contributes to the mean scalar transport as

$$\nabla \cdot \mathbf{H}_\theta^{(2)} = (\nabla \eta_H) \cdot [(\nabla H) \times (\nabla\Theta)] \tag{37}$$

This suggests that the turbulent helicity H contributes to the scalar flux perpendicular to the gradient of scalar $\nabla\Theta$. This can be rewritten as

$$\nabla \cdot \mathbf{H}_\theta^{(2)} \propto \nabla\Theta \cdot [(\nabla K) \times (\nabla H)] \tag{38}$$

This gives an effective scalar transport if the helicity gradient ∇H is oblique to the gradient of turbulent energy, ∇K. This may be called "*helicoclinic*" effect on the analogy of the baroclinic effect in the vorticity generation arising from the misalignment between the pressure and density gradients $\nabla \times [(-\nabla p)/\rho] = (\nabla\rho) \times (\nabla p)/\rho^2$ in a variable density fluid. Equation (38) indicates that if

$$-\nabla \times \left(\frac{\nabla H}{K} \right) \propto \frac{(\nabla K) \times (\nabla H)}{K^2} \neq 0 \tag{39}$$

is non-zero in the direction of the mean scalar gradient $\nabla\Theta$, we have a non-zero scalar transport due to turbulent helicity.

It is worth while noting that Eq. (38) is written in the form of the triple product of the gradients of functions Θ, K, and H (Nambu bracket). It is well known that, in the velocity space, the velocity evolves on the intersection of the Hamiltonian = const. and the Casimir = const. in the Euler flow (Fukumoto & Zou, 2024). If we remember that the Casimir for the Euler flow is the helicity, the expression (38) is suggestive for expressing the local mean scalar evolution of Θ associated with the energy gradient ∇K and the helicity gradient ∇H. A further investigation is required in this subject.

NUMERICAL SET-UP

In order to confirm the helicity effect in the turbulent scalar diffusion, some numerical simulations are to be performed.

One of the possible numerical set-ups is the one with an inhomogeneous turbulent energy K and an inhomogeneous helicity H are externally imposed by forcing in the x-y plane (Figure 2). The mean scalar gradient $\nabla\Theta$, which is homogeneous in the x-y plane, is externally imposed with a constant $\partial\Theta/\partial z$ in the z direction. The diffusivity η_H is determined by the turbulent energy K ($\eta_H \propto K$). In the presence of the modulation of turbulent energy, it leads to $\nabla\eta_H \neq 0$. Without helicity ($H = 0$), we have no scalar transport in the x-y plane. Even with non-uniform helicity ($\nabla H \neq 0$), if the spatial pattern of the H contour is similar to the K counterpart ($\nabla H \parallel \nabla K$: "*helicotropic*" state), we have no net Θ transport since $\nabla \cdot \mathbf{H}^{(2)} = \nabla \cdot [\eta_H \{(\nabla H) \times (\nabla\Theta)\}] \propto [(\nabla K) \times (\nabla H)] \cdot \nabla\Theta = 0$ (Left). In this case, spatial distribution of Θ in the x-y plane must be uniform. On the other hand, in the case of non-similar distributions of K and H with each other [$(\nabla K) \times (\nabla H) \neq 0$], it is expected that a non-trivial granular spatial pattern of the Θ contour should be observed in the x-y plane (Right). Such a non-trivial Θ pattern is purely caused by the helicity effect (the "*helicoclinic*" effect), since the anisotropy effects due to the mean velocity shear are avoided in this numerical set-up.

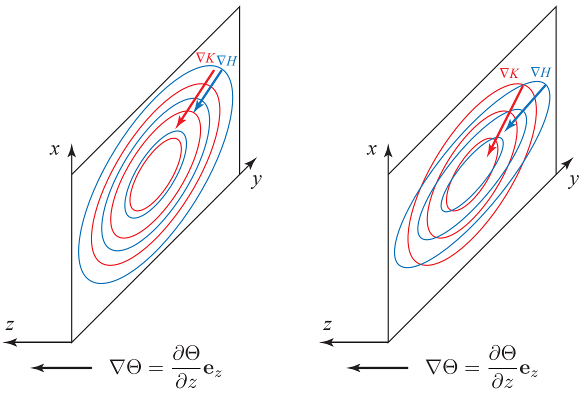


Figure 2. A possible DNS set-up for examining the helicity effects on scalar diffusion.

PRELIMINARY NUMERICAL RESULTS

A series of numerical simulations on the turbulent scalar diffusivity are in progress in collaboration with Hamba (2026). Although the initial results are preliminary, some of them will be presented here.

In the line of thoughts described above, the turbulent energy K and the turbulent helicity H are imposed by external forcing. The spatial distribution of the turbulent helicity is different from that of the turbulent energy (Figure 3). The turbulent energy K, which is positive-definite, has two peaks at $(x,y) = (\pi/2, \pi)$ and $(x,y) = (3\pi/2, \pi)$ (Left), while the turbulent helicity H, which is not positive-definite, has a positive peak at $(x,y) = (\pi/2, \pi)$ and a negative peak at $(x,y) = (3\pi/2, \pi)$ (Right).

The contour of the z component of turbulent scalar flux $H_{\theta z}$ (Left) and the vector arrows of the x and y components of turbulent scalar flux, $\mathbf{H}_\theta = (H_{\theta x}, H_{\theta y})$ (Right) are shown in Figure 4. Depending on the spatial distributions of K and H, non-zero spatial distributions of the turbulent scalar flux $\mathbf{H}_\theta$

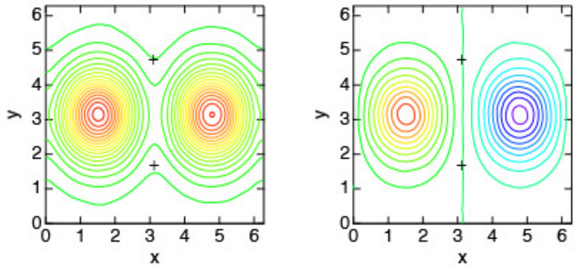


Figure 3. Spatial distributions of the turbulent energy $K(x,y)$ (Left) and the turbulent helicity $H(x,y)$ (Right).

are observed. This non-zero turbulent flux $\mathbf{H}_\theta$ in the x-y plane validates the turbulent scalar flux $\mathbf{H}_\theta^{(2)}$ [Eq. (34)] due to the kinetic helicity.

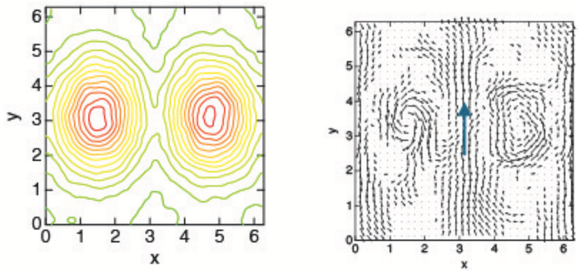


Figure 4. The contour of the z component of turbulent scalar flux, $H_{\theta z}(x,y)$ (Left), and the vector arrows of the x and y components of turbulent scalar flux $\mathbf{H}_\theta = (H_{\theta x}, H_{\theta y})$ (Right).

The spatial distribution of the mean scalar Θ in the x-y plane is shown in Figure 5. We observe a non-trivial modulation of Θ because of the helicity effect in the turbulent scalar diffusion.

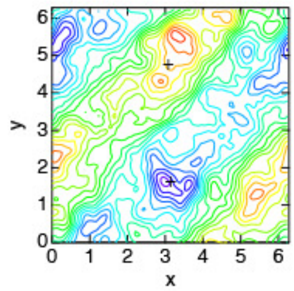
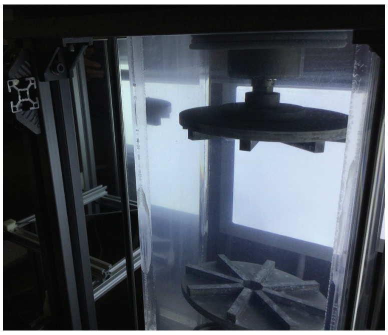


Figure 5. Spatial distribution of the mean scalar $\Theta(x,y)$ in the x-y plane.

A further examination is needed for the detailed understanding the conditions for and the properties of the helicity effect in scalar diffusion.

EXPERIMENTAL SET-UP

As Eq . (34) shows, the turbulent scalar flux due to helicity, $\mathbf{H}_\theta^{(2)}$, is perpendicular to the mean scalar gradient $\nabla\Theta$. It occurs only when the turbulent helicity gradient ∇H is oblique



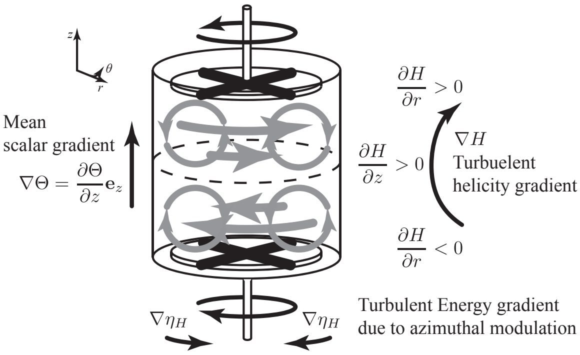


Figure 6. A possible experiment of helicity effect on scalar diffusion in laboratory: von Kàrmàn swirling flow at University of Buenos Aires (Top photo). A schematically depicted set-up (Bottom image).

to the mean scalar gradient $\nabla\Theta$. Such configurations are ubiquitously presented in nature.

One experimental set-up in laboratory might be a von Kàrmàn swirl flow, in which a global swirl in the azimuthal or θ direction is driven by a pair of disk impellers whose rotation axis is in the vertical or z direction (Figure 3). In von Kàrmàn swirling flows, helicity inhomogeneous in the radial or r direction, $\partial H/\partial r$, and inhomogeneous in the axial direction, $\partial H/\partial z$, is provided by the impellers located at the top and bottom of the cylinder. In addition, it is also possible to impose a scalar quantity (condensation or temperature) gradient in the z direction by the top and bottom boundaries.

With this experimental set-up, the expression of the turbulent scalar flux (30) or its model expression with (33) and (35) should be validated. In case the local turbulent scalar flux $\langle \mathbf{u}'\theta' \rangle$ can be measured in a single point in the von Kàrmàn swirling flow, the measured local flux $\langle \mathbf{u}'\theta' \rangle$ should be compared with the model expression $(\nabla H) \times \nabla\Theta$.

In the laboratory experiments, modulations of turbulent motion in the azimuthal or θ direction is ubiquitously observed in von Karman swirl flows (Berning and Rösgen, 2023). In such a case, a scalar transport in the sense of non-vanishing $\nabla \cdot \mathbf{H}_{\boldsymbol{\theta}}^{(2)} \neq 0$ should be measured depending on the the gradients of turbulent helicity and mean scalar, ∇H and $\nabla\Theta$. An experimental measurement of $\nabla \cdot \mathbf{H}_{\boldsymbol{\theta}}^{(2)}$ might be an interesting subject of study in the future.

CONCLUSION

With the aid of the multiple-scale renormalised perturbation expansion theory, the kinetic helicity effect in the turbulent scalar diffusion was investigated. It was suggested that the inhomogeneous turbulent helicity ∇H coupled with the gradient of mean scalar, $\nabla\Theta$, can contribute to the effective scalar diffusion perpendicular to $\nabla\Theta$. In inhomogeneous turbulence with non-uniform turbulent energy distribution represented by ∇K, this turbulent diffusion effect due to the inhomogeneous helicity can globally contribute to the evolution of the mean scalar distribution. This effect may be called the *helicoclinicity* effect. Numerical and experimental set-ups for the validation of this helicity effect were proposed with preliminary numerical results with the set-up.

ACKNOWLEGMENTS

The author would like to thank Axel Brandenburg, Petri Käpylä, and Igor Rogachevskii for invaluable discussions on the role of helicity in scalar diffusivity. His thanks are also due to Fujihiro Hamba who provided him with some of the initial results of the numerical simulations validating the helicoclinicity effect discussed in this paper.

REFERENCES

Berning, H. and Rösgen, R., 2023, “Suppression of large-scale azimuthal modulations in a von Kàrmàn flow using random forcing”, Phys. Fluids, Vol. 35, 075151, pp. 1-17.

Brandenburg, A., Käpylä, Rogachevskii, I., and Yokoi, N., 2025, “Helicity effect on turbulent passive and active scalar diffusivities”, Astrophys. J., Vol. 984, 88, pp. 1-9.

Fukumoto, Y. and Zou, R., 2024, “Nambu bracket for 3D ideal fluid dynamics and magnetohydrodynamics”, Prog. Theor. Exp. Phys., Vol. 03A101, pp. 1-31.

Hamba, F., 2026, Private communication to NY.

Pouquet, A. and Yokoi, N., 2022, “Helical fluid and (Hall)-MHD turbulence: a brief review”, Phil. Trans. R. Soc. A, Vol. 380, 20210087, pp. 1-18.

Yokoi, N., 2020, “Turbulence, transport and reconnection”, In *Topics in Magnetohydrodynamic Topology, Reconnection and Stability Theory*, (eds. D. MacTaggart, A. Hillier), CISM International Centre for Mechnical Sciences 591, pp. 177-265. Cham, Springer Verlag.

Yokoi, N., 2024, “Transport in helical fluid turbulence”, In *Helicities in Geophysics, Astrophysics, and Beyond*, (eds. K. Kuzanyan, N. Yokoi, M. K. Georgoulis, R. Stepanov) Geophysical Monograph 283, pp. 25-50. Hoboken, John Wiley & Sons, Inc.

Yokoi, N. and Brandenburg, A., 2016, “Large-scale flow generation by inhomogeneous helicity”, Phys. Rev. E, Vol. 93, 033125, pp. 1-14.

Yokoi, N. and Miesch, M. S., 2025, “Kinetic helicity effects in stellar angular-momentum transport”, Geophys. Astrophys. Fluid Dyn., Vol. 119, pp. 1-32.

Yokoi, N. and Yoshizawa, A., 1993, “Statistical analysis of the effects of helicity in inhomogeneous turbulence”, Phys. Fluids A, Vol. 5, pp. 464-477.

Yokoi, N., Mininni, P., Pouquet, A., Rosenberg, D., and Marino, R., 2026, “Helicity subgrid-scale models and their numerical validation”, Phys. Rev. E, (in review), pp. 1-34.

Yoshizawa, A., 1984, “Statistical analysis of the derivation of the Reynolds stress from its eddy-viscosity representation”, Phys. Fluids, Vol. 27, pp. 1377-1387.