14th International Symposium on Turbulence and Shear Flow Phenomena (TSFP14)
Heidelberg, Germany, July 28–31, 2026

AN ITERATIVE MODEL FOR MEAN VELOCITY AND TOTAL SHEAR STRESS PROFILES IN TURBULENT BOUNDARY LAYERS WITH ADVERSE PRESSURE GRADIENTS

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ABSTRACT

Pressure gradients distort the universal velocity distributions observed in canonical zero-pressure-gradient turbulent boundary layers (ZPG TBLs), thereby hindering the accurate modelling of their mean streamwise profiles. This study presents an iterative model for predicting the mean velocity and total shear stress profiles in turbulent boundary layers with adverse pressure gradients (APGs). The present model does not rely on any case-by-case parameter recalibration. Comparison with multiple datasets indicates that the present model can accurately predict the profiles of mean streamwise velocity, wall-normal velocity, and total shear stress.

INTRODUCTION

Turbulent boundary layers have been extensively studied for their relevance to engineering applications. The simplest case for this is the flat-plate ZPG TBL, for which several well-acknowledged models for the mean streamwise velocity profiles have been proposed, such as the van Driest mixing length model (Van Driest, 1951) and Spalding's formula (Spalding *et al.*, 1961) for the near-wall region and Coles' wake function (Coles, 1956) for the outer region. These models have been widely used in turbulence modeling, such as Reynolds-averaged Navier-Stokes models and wall-modelled large-eddy simulation. However, for APG TBLs, these classical models are not accurate (Monty *et al.*, 2011; Devenport & Lowe, 2022).

Recently, Subrahmanyam *et al.* (2022) proposed a more comprehensive mixing-length model. This model can successfully predict the profiles of mean streamwise velocity, wall-normal velocity, and total shear stress for pipe turbulence, channel turbulence, and ZPG TBLs. However, for APG TBLs, some model parameters are no longer universal and instead require case-by-case recalibration, undermining the universality of the model. This model is fundamentally derived from internal flows, where a linear total shear stress distribution is guaranteed. On the other hand, for external flows like boundary layers, the total shear stress profile is not linear and cannot be determined *a priori*. Subrahmanyam *et al.* (2022) continued to employ the linear approximation of the total shear stress to develop the model for boundary layers. While this approximation may be reasonable for ZPG cases, it breaks down for APG cases where the total shear stress profile deviates significantly from linear profiles. To address this issue, Shu & Xu (2025) proposed a one-step iterative model for APG TBLs, which successfully captures the outer peak in the total shear stress profile. Nevertheless, due to its inherent mathematical complexity, performing multiple iterations is challenging, and the convergence of results is not guaranteed. Furthermore, this model still relies on the initial guess of the mean total shear stress profile.

In this work, we propose a new iterative model, which can iterate multiple times until convergence. This model is asymptotically consistent with the boundary layer equations and requires no parameter recalibration. It accurately predicts the mean velocity and total shear stress profiles in APG turbulent boundary layers and is independent of the initial guess for the total shear stress.

METHODOLOGY

The following Reynolds-averaged equations for two-dimensional incompressible turbulent boundary layers are considered:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{1}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}P_e}{\mathrm{d}x} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}\overline{u'v'}, \tag{2}$$

where P_e is the free-stream pressure, U and V are the mean streamwise and wall-normal velocities, $\overline{u'v'}$ is the Reynolds shear stress, ρ and ν are the density and kinematic viscosity, respectively. Owing to the additional Reynolds stress term

$\overline{u'v'}$, these equations are not closed. Subrahmanyam *et al.* (2022) proposed the following mixing length model:

$$\frac{\mathrm{d}U^+}{\mathrm{d}y^+} = \frac{-1+\sqrt{1+4\left(\lambda^+(y^+)\right)^2\tau_t^+}}{2\left(\lambda^+(y^+)\right)^2}, \tag{3}$$

$$\lambda^+(y^+) = ky^+\frac{1-e^{-(y^+/a)^m}}{\left(1+\left(\frac{y^+}{bRe_\tau}\right)^n\right)^{1/n}}, \tag{4}$$

where $\tau_t = \nu\partial U/\partial y - \overline{u'v'}$ is the total shear stress, Re_τ is the friction Reynolds number, and k, a, b, m, n are model parameters to be calibrated. The superscript $+$ denotes quantities normalized by wall units, i.e., the friction velocity $u_\tau = \sqrt{\tau_w/\rho}$ and the viscous length $\delta_\nu = \nu/u_\tau$, where τ_w is the wall shear stress. By assuming $\tau_t^+ = 1-y^+/Re_\tau$, Eq. (3) can be readily integrated to obtain the mean velocity profile $U^+(y^+)$, which is the foundation of the model by Subrahmanyam *et al.* (2022). Due to the artificial assumption of τ_t^+, this model is inconsistent with the boundary layer equations. On the other hand, it is worth noting that Eqs. (1)–(4) already form a closed system. Therefore, no explicit specification of τ_t^+ is required in principle. Consequently, the only remaining challenge is solving these equations, which is complicated by their nonlinear nature. Moreover, the applicability of Eqs. (3)–(4) to APG TBLs is also questionable. To tackle these issues, we propose the following procedure.

First, we note that Eqs. (3)–(4) are normalized by wall units, which is a natural choice for ZPG TBLs due to the constant shear stress assumption. However, for APG TBLs, τ_t^+ is no longer constant in the near-wall region, but instead varies with the wall distance due to PG effects. Thus, it is necessary to consider a new normalization procedure for this. Here, we follow Shu & Xu (2025) in using the following scales to replace wall units in Eq. (3)–(4):

$$u_\beta = u_\tau(1+\beta), \quad \delta_\beta = \frac{\nu}{u_\beta}, \tag{5}$$

where $\beta = \frac{\delta_d}{\tau_w}\frac{\mathrm{d}Pe}{\mathrm{d}x}$ is the Clauser pressure gradient parameter (Clauser, 1954) and δ_d is the displacement thickness. By replacing the wall units with the new scales defined in Eq. (5), Eqs. (3)–(4) can be rewritten as:

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\tilde{y}} = \frac{-1+\sqrt{1+4\left(\tilde{\lambda}(\tilde{y})\right)^2\tilde{\tau}_t}}{2\left(\tilde{\lambda}(\tilde{y})\right)^2}, \tag{6}$$

$$\tilde{\lambda}(\tilde{y}) = k\tilde{y}\frac{1-e^{-(\tilde{y}/a)^m}}{\left(1+\left(\frac{111\tilde{y}}{bRe_\tau}\right)^n\right)^{1/n}}, \tag{7}$$

where the tilde above the variables denotes quantities normalized by the new scales defined in Eq. (5).

To solve Eqs. (1)–(2) and Eqs. (6)–(7), we propose an iterative procedure as follows:

1. Set an initial guess for τ_t.
2. Integrate Eq. (6) to compute the streamwise velocity U.
3. Integrate Eq. (1) to compute the wall-normal velocity V.
4. Integrate Eq. (2) to update the total shear stress τ_t.
5. Repeat steps 2–4 until convergence is achieved.

The key step of this method is the calculation of the streamwise derivative in Eqs. (1)–(2). Since the profile obtained from step 2 is only a function of y, the streamwise derivative cannot be directly calculated. The previous studies of Subrahmanyam *et al.* (2022) and Shu & Xu (2025) theoretically derive the streamwise derivative from the Kármán momentum equation. However, this method cannot be applied to multiple iterations due to the mathematical complexity. Here, we propose a method to calculate the streamwise derivative numerically. We consider two streamwise-adjacent velocity profiles $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$. $U_1^+(y_1^+)$ is the profile at the given Reynolds number $Re_{\tau 1} = Re_\tau$ and Clauser parameter β, while $U_2^+(y_2^+)$ is the profile at an infinitesimal downstream distance, which has the Reynolds number of $Re_{\tau,2} = Re_\tau + \Delta Re_\tau$. ΔRe_τ needs to be relatively small to ensure that the streamwise distance between $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$ is small enough so that the streamwise derivative can be approximated accurately. Here, we choose $\Delta Re_\tau/Re_\tau = 0.05$. Strictly speaking, the Clauser parameter β may also vary with the streamwise location. However, for the cases considered in this study, this effect is relatively small and is therefore neglected for simplicity. Before calculating the streamwise derivative through the difference between $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$, two issues are still to be addressed. First, $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$ are both normalized by the local wall scales u_τ and δ_ν, which also vary with the streamwise location. Thus, it is necessary to renormalize $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$ using the same scales. Second, the streamwise distance Δx between $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$ is also unknown. To address the first problem, we convert the normalization of wall scales into the corresponding outer scales (i.e., free-stream velocity U_e and boundary layer thickness δ) and define $U_1^*(y_1^*) = U_1/U_{e1} = U_1^+/U_{e1}^+$, $U_2^*(y_2^*) = U_2/U_{e2} = U_2^+/U_{e2}^+$, where the superscript $*$ denotes quantities normalized by the local outer scales U_e and δ. Therefore, once the ratios of outer scales U_{e2}/U_{e1} and δ_2/δ_1 are determined, these two profiles can be converted to the same normalization. U_{e2}/U_{e1} and δ_2/δ_1 can be derived from the Kármán momentum integral equation and the Bernoulli equation with the following results:

$$\frac{\delta_2}{\delta_1} = \frac{-(1-\theta_1^*C)+\sqrt{(1-\theta_1^*C)^2+4\theta_2^*CD}}{2\theta_2^*C}, \tag{8}$$

$$\frac{U_{e2}}{U_{e1}} = D\frac{Re_{\tau 1}}{Re_{\tau 2}}, \tag{9}$$

where θ_1, θ_2 are the momentum thicknesses, C and D are quantities that can be determined from the known profiles $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$:

$$C = -\frac{\beta\delta_1/\delta_{d1}}{1+\beta+2\beta\theta_1/\delta_{d1}}, \tag{10}$$

$$D = \frac{Re_{\tau 2}U_{e2}^+}{Re_{\tau 1}U_{e1}^+}. \tag{11}$$

In the limit of $\beta \to 0$, Eqs. (8) and (9) reduce to $\delta_2/\delta_1 = D$ and $U_{e2}/U_{e1} = 1$, which is consistent with the results of ZPG TBLs. The streamwise distance Δx can also be determined by the Kármán momentum integral equation:

$$\frac{\Delta x}{\Delta\theta} = \frac{U_e^{+2}}{1+\beta+2\beta\theta/\delta_d}. \tag{12}$$

With the above equations, profiles $U_1^+(y_1^+)$ and $U_2^+(y_2^+)$ can be converted into the same normalization scales, and the streamwise distance can also be calculated. Thus, the streamwise derivative in Eqs. (1)–(2) can be calculated as:

$$\frac{\partial U}{\partial x} \approx \frac{\Delta U}{\Delta x}. \tag{13}$$

RESULTS AND DISCUSSION

Before predicting the profiles, the five parameters in Eq. (4), namely, k, a, m, b, and n, must be calibrated. These parameters are calibrated using large-eddy simulation data of a ZPG turbulent boundary layer from Eitel-Amor *et al.* (2014). The calibrated values are summarized in Table 1. As shown later, although the parameters are calibrated with ZPG TBL data, they can be directly applied to APG TBLs without any modification, which indicates the universality of the present model.

Table 1. Parameters calibrated with the ZPG TBL dataset.

k	a	m	b	n
0.4002	21.4204	1.2540	0.2048	2.6062

Table 2. Summary of the incompressible APG TBL datasets used in the present study.

Case	Re_τ	β	References
b10B	600	1.0	Bobke *et al.* (2017)
b21B	600	2.1	Bobke *et al.* (2017)
m13B	600	1.1	Bobke *et al.* (2017)
m16B	600	2.4	Bobke *et al.* (2017)
m18B	600	4.0	Bobke *et al.* (2017)
b13P	1746	1.3	Pozuelo *et al.* (2022)
b14Y	803	1.4	Yoon *et al.* (2018)
b07L	376	0.73	Lee (2017)
b22L	364	2.2	Lee (2017)

To validate the accuracy of the present model, we compare the predicted profiles with reference data from multiple datasets of incompressible APG TBLs, as summarized in Table 2. The results are presented in Fig. 1. Salient features of APG TBLs include the enhanced wake of the streamwise velocity profile and the emergence of an outer peak in the total shear stress profile. These features are all well captured by the present model. In general, for all APG TBL cases, the model is able to reproduce the profiles of U, V, and τ_t with good accuracy.

CONCLUSION

In summary, we have proposed an iterative model for turbulent boundary layers, capable of accurately predicting the mean velocity and total shear stress profiles for both ZPG and APG cases. This model is asymptotically consistent with the boundary layer equations, and the model parameters are universal.

The remaining challenge lies in incorporating history effects and extending the model to more complex scenarios, such as compressible turbulent boundary layers. Furthermore, under strong pressure gradients, the neglected Reynolds stress components may become significant, and the applicability of Eqs. (1)–(2) and (6)–(7) needs to be revalidated. These issues will be addressed in future work.

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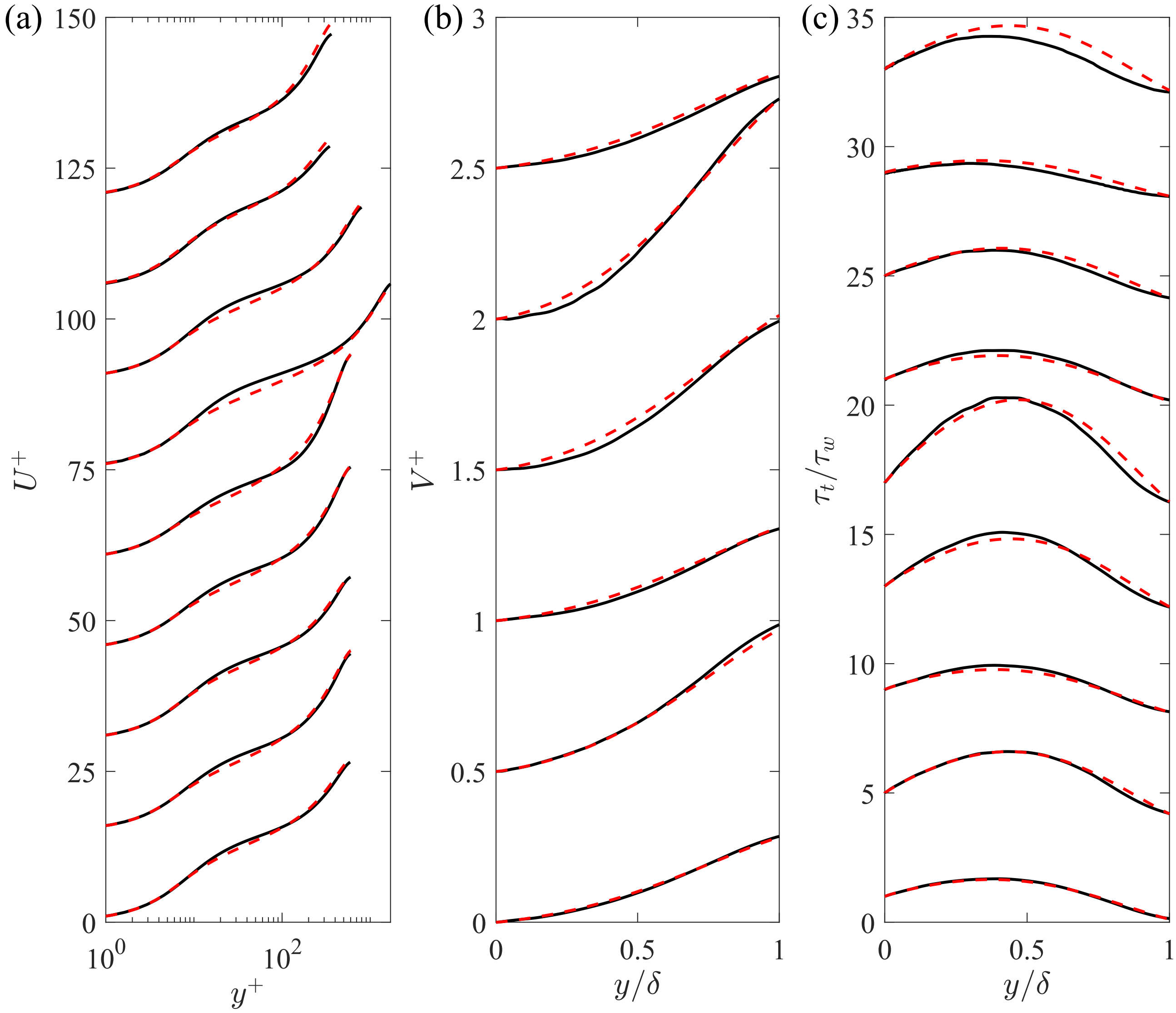

Figure 1. Comparison of the predicted profiles (red dashed lines) from the present framework with the reference data (black solid lines) for cases listed in Table 2. The panels display (a) mean streamwise velocity profiles, (b) mean wall-normal velocity profiles, and (c) total-shear-stress profiles. In each panel, the profiles from bottom to top correspond to cases b10B, b21B, m13B, m16B, m18B, b13P, b14Y, b07L and b22L respectively (cases b14Y, b07L and b22L are omitted in panel (b) due to the unavailability of wall-normal velocity data). To facilitate visualization, consecutive profiles are shifted vertically by increments of $\Delta U^+ = 15$ in panel (a), $\Delta V^+ = 0.5$ in panel (b), and $\Delta\tau_t = 4\tau_w$ in panel (c).