LIE-SYMMETRY BASED SCALING LAWS AND NON-EQUILIBRIUM DISSIPATION SCALING IN TURBULENT PLANE JETS

Nils Benedikt, Paul Braun, Martin Oberlack
Chair of Fluid Dynamics
Technische Universität Darmstadt
Otto-Berndt-Straße 2, 64287 Darmstadt, Germany
benedikt@fdy.tu-darmstadt.de,
paul_konrad.braun@stud.tu-darmstadt.de,
oberlack@fdy.tu-darmstadt.de

ABSTRACT

We derive scaling laws for turbulent plane jets from the Lie-symmetries of the multi-point-moment-equations. Therein, we omit the Reynolds decomposition and the boundary layer approximation. The scaling laws contain an additional free parameter compared to the classical scaling laws. We find that this parameter encodes non-equilibrium dissipation scaling, which can be connected to viscosity induced symmetry breaking and the strong dissipative anomaly. However, in contrast to previous approaches, the scaling laws cannot be validated experimentally.

INTRODUCTION

Turbulent jets are canonical flows which allow insight into the fundamental mechanisms of turbulence. One powerful tool to study such mechanisms is Lie-symmetry theory, whose application to fluid dynamics goes back to Oberlack (2001). Initially focusing on the symmetries of the Navier-Stokes equations, Oberlack & Rosteck (2010) later discovered that there are additional symmetries, which only appear in statistical descriptions of turbulence, while they are absent in the Navier-Stokes equations themselves. The statistical symmetries were used to derive scaling laws for Poiseuille flow with wall transpiration (Avsarkisov *et al.* 2014), temporally evolving plane jets (Sadeghi *et al.* 2018) and spacially evolving round jets (Nguyen *et al.* 2025) and the corresponding transport of passive scalars (Sadeghi *et al.* 2021, Nguyen & Oberlack 2025), as well as for arbitrary velocity moments beyond the log law in wall-bounded shear flows (Oberlack *et al.* 2022). Furthermore, the implications of statistical symmetries for RANS turbulence modelling were investigated by Klingenberg *et al.* (2020) for Reynolds-stress transport models and Klingenberg & Oberlack (2022) for eddy-viscosity models. While a study on the infinite Lundgren-Monin-Novikov-hierarchy for the probability density functions of velocity provided first insights into the physical interpretation of statistical symmetries, revealing a connection to intermittency and non-Gaussianity (Wacławczyk *et al.* 2014), the physical interpretation of the statistical symmetries is still an open question. We aim to investigate this in the present study by connecting the statistical scaling symmetry to non-equilibrium dissipation.

One of the pillars of our understanding of the physics of turbulence is that the rate of dissipation of turbulent kinetic energy is related to the largest scales of turbulence as

$$\varepsilon \sim \frac{\mathscr{U}^3}{\mathscr{L}}, \tag{1}$$

where $\mathscr{U}$ is the characteristic velocity and $\mathscr{L}$ is the integral length scale. The reasoning behind (1) lies in the equilibrium cascade of turbulence. If energy is passed through all scales by the same rate, then the rate of dissipation of said energy must be described by the scales at which the energy is inserted (i.e. the large scales $\mathscr{U}$ and $\mathscr{L}$). Therefore, (1) is referred to as equilibrium dissipation scaling. Many commonly accepted practices in turbulence research and application rely on equilibrium dissipation scaling, for example the estimation of the Kolmogorov scale when setting up direct numerical simulation, or the estimation of the eddy viscosity in RANS modelling (Vassilicos 2015). However, equilibrium dissipation scaling was challenged as early as George (1995), stating that (1) is only true in the infinite Reynolds number limit. More recently, a growing number of publications (Vassilicos 2015, Vassilicos & Laval 2025) suggest that (1) does not necessarily hold. The theoretical basis for this was laid out by Vassilicos (2015), finding that at the large scales, the equilibrium cascade might not accurately describe reality. The work of Yasuda & Vassilicos (2018) supported this by finding that the equilibrium cascade does not capture the strong intermittency in interscale energy transfers. A deviation from the classical cascade seems especially likely for decaying turbulence, i.e. decaying homogeneous turbulence or convected decay as it can be observed in free shear flows, where production and dissipation of turbulent kinetic energy do not happen at the same place and time. As a consequence, a non-equilibrium dissipation scaling is required to describe the behavior of such flows. Taking the square root of the turbulent kinetic energy k as the characteristic velocity $\mathscr{U}$ and the half-width δ of the shear layer as a proxy for the integral length scale $\mathscr{L}$, the non-equilibrium dissipation scaling takes the form

$$\varepsilon \sim \frac{k^{3/2}}{\delta}\left(\frac{\mathrm{Re}_G}{\mathrm{Re}_\delta}\right), \tag{2}$$

where Re_G is the global Reynolds number and $\mathrm{Re}_\delta = (\sqrt{k}\delta)/\nu$ is the local Reynolds number. If the ratio between local and global Reynolds number is not a constant for a given flow configuration, (2) is fundamentally different from (1). The scaling (2) has been observed in multiple flow configurations such as decaying turbulence behind fractal grids (Seoud & Vassilicos 2007), round wakes (Dairay *et al.* 2015), and also in plane jets (Cafiero & Vassilicos 2019).

This work aims to establish a connection between the additional parameter a_{Ss}^* in Lie-symmetry based scaling laws discovered by Nguyen *et al.* (2025) and non-equilibrium dissipation scaling. As turbulent round jets, for which Nguyen *et al.* (2025) developed the theory, do not have a variable local Reynolds number, we choose to study plane jets instead. A similar attempt was made by Layek & Sunita (2018) with the important difference that the boundary layer approximation was employed, which we omit. Furthermore, Layek & Sunita (2018) did not use the statistical scaling symmetry, which generates the additional parameter a_{Ss}^*.

First, we will introduce the scaling laws of turbulent plane jets by applying Lie-symmetry theory, where we closely follow the methodology of Nguyen *et al.* (2025). Subsequently, we establish the connection to non-equilibrium dissipation scaling. We then conclude and give an outlook of potential future research.

LIE-SYMMETRY BASED SCALING LAWS IN TURBULENT PLANE JETS

One of the complete statistical descriptions of turbulence are the multi-point moment equations

$$\begin{aligned}\frac{\partial H_{i_{\{n\}}}}{\partial t}+\sum_{l=1}^{n}\Bigg[&\frac{\partial H_{i_{\{n+1\}}[i_{(n)}\mapsto k_{(l)}]}\left[\mathbf{x}_{(n)}\mapsto\mathbf{x}_{(l)}\right]}{\partial x_{k_{(l)}}}\\&+\frac{\partial I_{i_{\{n-1\}}[l]}}{\partial x_{i_{(l)}}}-\frac{1}{Re}\frac{\partial^2 H_{i_{\{n\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\Bigg]=0.\end{aligned} \tag{3}$$

Here, the multi-point velocity moments are defined as

$$H_{i_{\{n\}}}=\overline{U_{i_{(1)}}(\mathbf{x}_{(1)})\cdot\ldots\cdot U_{i_{(n)}}(\mathbf{x}_{(n)})}, \tag{4}$$

where $\mathbf{x}_{(j)}$ denotes the j-th point in space, $i_{(j)}$ denotes the velocity component sampled at that point and $\overline{(\cdot)}$ is an appropriate averaging operator. The further terms in (3) are the convective term

$$\begin{aligned}&H_{i_{\{n+1\}}[i_{(n)}\mapsto k_{(l)}]}\left[\mathbf{x}_{(n)}\mapsto\mathbf{x}_{(l)}\right]=\\&\overline{U_{i_{(1)}}\left(\mathbf{x}_{(1)}\right)\cdot\ldots\cdot U_{k_{(l)}}\left(\mathbf{x}_{(l)}\right)U_{i_{(n+1)}}\left(\mathbf{x}_{(n+1)}\right)}\end{aligned} \tag{5}$$

and the velocity-pressure correlation

$$I_{i_{\{n-1\}}[l]}=\overline{P(\mathbf{x}_{(l)})\prod_{j=1,j\neq l}^{n}U_{i_{(j)}}(\mathbf{x}_{(j)})}. \tag{6}$$

Note that we did not apply the Reynolds-decomposition $U_i=\overline{U_i}+u_i$ here. However, the more familiar Reynolds averaged quantities can be recovered via $H_{ij}=\overline{U}_i\overline{U}_j+\overline{u_iu_j}$.

Invariant solutions from Lie-symmetry theory

A symmetry in a mathematical sense is a variable transformation

$$T:\boldsymbol{x}^*=\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{u};a),\quad \boldsymbol{u}^*=\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{u};a), \tag{7}$$

which leaves the form of a system of differential equations

$$\boldsymbol{F}\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(1)},...,\boldsymbol{u}^{(p)}\right)=0 \tag{8}$$

unchanged, i.e.

$$\begin{aligned}&\boldsymbol{F}\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}^{(1)},...,\boldsymbol{u}^{(p)}\right)=0\\\iff&\boldsymbol{F}\left(\boldsymbol{x}^*,\boldsymbol{u}^*,\boldsymbol{u}^{*(1)},...,\boldsymbol{u}^{*(p)}\right)=0.\end{aligned} \tag{9}$$

By Taylor expanding (7) in the transformation parameter a, i.e.

$$\begin{aligned}x_i^*&=x_i+\xi_i(\boldsymbol{x},\boldsymbol{u})a+\mathscr{O}(a^2)\\u_j^*&=u_j+\eta_j(\boldsymbol{x},\boldsymbol{u})a+\mathscr{O}(a^2),\end{aligned} \tag{10}$$

the infinitesimals

$$\xi_i(\boldsymbol{x},\boldsymbol{u})=\left.\frac{\partial\phi_i}{\partial a}\right|_{a=0};\quad \eta_j(\boldsymbol{x},\boldsymbol{u})=\left.\frac{\partial\psi_j}{\partial a}\right|_{a=0}. \tag{11}$$

can be computed. According to Lie's first theorem, these infinitesimals already fully describe the symmetries. They furthermore define an infinitesimal generator

$$X=\xi_i\frac{\partial}{\partial x_i}+\eta_k\frac{\partial}{\partial u_k}. \tag{12}$$

Accounting for the infinitesimals of the derivatives up to order p of the dependent variables leads to a prolonged infinitesimal generator $X^{(p)}$. The form invariance condition (9) can then be reformulated by applying the infinitesimal generator to the system of differential equations (8). A transformation (7) is then a symmetry of (8) if and only if

$$X^{(p)}\boldsymbol{F}|_{\boldsymbol{F}=0}=0. \tag{13}$$

The condition (13) can also be used to construct invariant solutions to $\boldsymbol{F}$ by assuming a solution

$$\boldsymbol{u}=\boldsymbol{f}(\boldsymbol{x}) \tag{14}$$

which then has to be invariant under (7), i.e.

$$X^{(p)}\left[\boldsymbol{u}-\boldsymbol{f}(\boldsymbol{x})\right]|_{\boldsymbol{u}=\boldsymbol{f}(\boldsymbol{x})}=0, \tag{15}$$

where (15) is known as invariant surface condition. The invariant surface condition implies a hyperbolic system of PDEs which can be solved for the invariant solutions by the method of characteristics by integrating the characteristic condition

$$\begin{aligned}\frac{\mathrm{d}x_1}{\xi_1(\boldsymbol{x},\boldsymbol{u})} &= \frac{\mathrm{d}x_2}{\xi_2(\boldsymbol{x},\boldsymbol{u})} = \cdots = \frac{\mathrm{d}x_i}{\xi_i(\boldsymbol{x},\boldsymbol{u})}\\ = \frac{\mathrm{d}u_1}{\eta_1(\boldsymbol{x},\boldsymbol{u})} &= \frac{\mathrm{d}u_2}{\eta_2(\boldsymbol{x},\boldsymbol{u})} = \cdots = \frac{\mathrm{d}u_k}{\eta_k(\boldsymbol{x},\boldsymbol{u})}.\end{aligned} \tag{16}$$

If this methodology is applied to scaling symmetries, the invariant solutions are self-similar solutions, the determination of which is the objective of this work.

Governing equations

The equations governing plane turbulent jets are the momentum balances in streamwise direction

$$\frac{\partial \overline{U_x^2}}{\partial x} + \frac{\partial \overline{U_x U_y}}{\partial y} + \frac{\partial \overline{P}}{\partial x} = 0, \tag{17}$$

and cross-stream direction

$$\frac{\partial \overline{U_x U_y}}{\partial x} + \frac{\partial \overline{U_y^2}}{\partial y} + \frac{\partial \overline{P}}{\partial y} = 0, \tag{18}$$

as well as the continuity equation

$$\frac{\partial \overline{U}_x}{\partial x} + \frac{\partial \overline{U}_y}{\partial y} = 0. \tag{19}$$

Note that we neglected viscosity here. However, we cannot neglect the dissipation

$$\varepsilon = 2\nu \overline{\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}} \tag{20}$$

which appears in the second order moment equation (Klingenberg *et al.* 2020)

$$\begin{aligned}&\frac{\partial\left(\overline{U_x^3}+\overline{U_xU_y^2}+\overline{U_xU_z^2}+2\overline{PU_x}\right)}{\partial x}\\ &+\frac{\partial\left(\overline{U_x^2U_y}+\overline{U_y^3}+\overline{U_yU_z^2}+2\overline{PU_y}\right)}{\partial y}\\ &+\varepsilon = 0.\end{aligned} \tag{21}$$

The fact that the dissipation does not tend to zero in the inviscid limit $\nu \to 0$ is known as strong dissipative anomaly (Sreenivasan & Schumacher 2025).

Although we are not applying the Reynolds decomposition into mean and fluctuations here, it is worth briefly discussing two of its implications.

(i) After applying the Reynolds decomposition to (17) and (18), it is common to estimate the magnitude of all terms and neglect smaller terms (Pope 2001), which is referred to as the boundary layer approximation. We do not apply this approximation here as we want to keep the analysis as general as possible.

(ii) Applying the Reynolds decomposition to (20) yields

$$2\nu \overline{\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}} = 2\nu \frac{\partial \overline{U}_i}{\partial x_i}\frac{\partial \overline{U}_i}{\partial x_i} + 2\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}, \tag{22}$$

where the first term on the r.h.s. can be neglected if viscous terms are assumed to be small, and the second term is the classical definition of dissipation. Thus our definition (20) is equivalent to the classical definition if other viscous terms are neglected as is the case in this analysis.

Symmetry breaking

Before deriving the scaling laws, we have to consider symmetry breaking. To this end, the pressure can be eliminated by integrating (18) in y and taking the derivative in x of the result. This can then be inserted for the pressure gradient in (17). Integrating (17) again over y yields the integral invariant

$$I = \int_{-\infty}^{\infty}\left[\overline{U_x^2} - \left(\overline{U_y^2} - \overline{U_{y,\infty}^2}\right) - y\frac{\partial \overline{U_x U_y}}{\partial x}\right]\mathrm{d}y. \tag{23}$$

The transformation of the integral invariant must be equal to the identity transformation under the symmetries of the physical problem at hand. It thus breaks the symmetries, i.e. it implies a constraint on the possible symmetry transformations. The symmetries of (3) relevant here consist of a translation symmetry in space

$$\begin{aligned}\overline{T}_x:\quad t^* = t,\quad x_i^* = x_i + a_{x_i},\quad \overline{U_i}^* = \overline{U}_i,\\ H^*_{i_{\{n\}}} = H_{i_{\{n\}}},\quad \overline{P}^* = \overline{P},\quad \varepsilon^* = \varepsilon,\end{aligned} \tag{24}$$

scaling symmetries in time and space

$$\begin{aligned}\overline{T}_S:\quad t^* = \mathrm{e}^{a_{St}}t,\quad x_i^* = \mathrm{e}^{a_{Sx}}x_i,\quad \overline{U_i}^* = \mathrm{e}^{a_{Sx}-a_{St}}\overline{U}_i,\\ H^*_{i_{\{n\}}} = \mathrm{e}^{n(a_{Sx}-a_{St})}H_{i_{\{n\}}},\quad \overline{P}^* = \mathrm{e}^{2(a_{Sx}-a_{St})}\overline{P},\\ \varepsilon^* = \mathrm{e}^{2a_{Sx}-3a_{St}}\varepsilon,\end{aligned} \tag{25}$$

and a scaling of all statistical moments

$$\begin{aligned}\overline{T}_{Ss}:\quad t^* = t,\quad x_i^* = x_i,\quad \overline{U_i}^* = \mathrm{e}^{a_{Ss}}\overline{U}_i,\\ H^*_{i_{\{n\}}} = \mathrm{e}^{a_{Ss}}H_{i_{\{n\}}},\quad \overline{P}^* = \mathrm{e}^{a_{Ss}}\overline{P},\quad \varepsilon^* = \mathrm{e}^{a_{Ss}}\varepsilon,\end{aligned} \tag{26}$$

which is of purely statistical nature, i.e. this symmetry does not appear in the Navier-Stokes equations themselves but only in the statistical descriptions of turbulence such as (3). Applying the symmetry transformations (24)-(26) to the invariant (23) yields

$$I^* = \mathrm{e}^{3a_{Sx}-2a_{St}+a_{Ss}}I. \tag{27}$$

From the condition that the transformation of the symmetry breaking invariant be equal to the identity transformation,

$$3a_{Sx} - 2a_{St} + a_{Ss} = 0 \tag{28}$$

follows. The symmetry breaking (28) reduces the number of independent parameters by one. Eliminating a_{St} by use of (28) yields the characteristic condition

$$\begin{aligned}\frac{\mathrm{d}x}{a_{Sx}x+a_x} &= \frac{\mathrm{d}y}{a_{Sx}y}\\ &= -\frac{\mathrm{d}\overline{U_x^n U_y^m U_z^k}}{\left[\frac{1}{2}\left(n+m+k\right)\left(a_{Sx}+a_{Ss}\right)-a_{Ss}\right]\overline{U_x^n U_y^m U_z^k}}\\ &= -\frac{\mathrm{d}\varepsilon}{\frac{1}{2}\left(5a_{Sx}+a_{Ss}\right)\varepsilon}.\end{aligned} \tag{29}$$

Scaling laws

Integrating the characteristic condition (29) yields invariant coordinates and scaling laws for the statistical moments of the velocity. For plane jets, the invariant coordinate is

$$\eta = \frac{y}{x-x_0}, \tag{30}$$

which is the classical self-similar re-scaling of y. Here, $x_0 = -a_x/a_{Sx}$ is the virtual origin. As discussed in Nguyen *et al.* (2025), this directly implies that the jet half-width δ, implicitly defined by

$$\overline{U}_x(x,\delta(x)) = \frac{1}{2}\overline{U}_x(x,0) \tag{31}$$

grows linearly with x, i.e.

$$\delta(x) = S(x-x_0). \tag{32}$$

The constant S is usually referred to as the spreading rate of the jet. The scaling laws for the velocity moment obtained from integration of (29) read

$$\overline{U_x^n U_y^m U_z^k}(x,y) = \frac{\alpha_{x;n,m,k}\widetilde{\overline{U_x^n U_y^m U_z^k}}(\eta)}{(x-x_0)^{\frac{1}{2}(n+m+k)(1+a_{Ss}^*)-a_{Ss}^*}}, \tag{33}$$

where $\alpha_{x;n,m,k}$ is defined such that $\widetilde{\overline{U_x^{n+m+k}}}(0)=1$, and the parameter $a_{Ss}^* = a_{Ss}/a_{Sx}$ is new compared to classical scaling laws (Nguyen *et al.* 2025). The self-similar profiles $\widetilde{\overline{U_x^n U_y^m U_z^k}}$ collapse for all downstream locations and only depend on the similarity coordinate η. The corresponding scaling of the dissipation rate results as

$$\varepsilon(x,y) = \frac{\alpha_\varepsilon\widetilde{\varepsilon}(\eta)}{(x-x_0)^{\frac{1}{2}(5+a_{Ss}^*)}}, \tag{34}$$

where the coefficient α_ε is chosen such that $\widetilde{\varepsilon}(0)=1$.

Demanding that the mean streamwise velocity $\overline{U}_x$ decay in x yields the constraint

$$a_{Ss}^* < 1, \tag{35}$$

while the same argument for an infinite moment order $n+m+k\to\infty$ leads to

$$a_{Ss}^* > -1. \tag{36}$$

An even stronger constraint for large downstream distances x comes from considering the turbulent kinetic energy. The scaling of the turbulent kinetic energy can be derived as

$$\begin{aligned}k(x,y) = \frac{1}{2}\Big(&\overline{U_x^2}(x,y)+\overline{U_y^2}(x,y)+\overline{U_z^2}(x,y)\\ &-\overline{U}_x^2(x,y)-\overline{U}_y^2(x,y)\Big)\end{aligned} \tag{37}$$

yielding

$$k(x,y) = \frac{\alpha_{k,0}\widetilde{k}_0(\eta)}{x-x_0} - \frac{\alpha_{k,1}\widetilde{k}_1(\eta)}{(x-x_0)^{1-a_{Ss}^*}}, \tag{38}$$

where $\alpha_{k,i}$ is chosen such that $\widetilde{k}_i(0)=1$. It is clear that when $a_{Ss}^*>0$, there is some value of x for which the second term becomes larger than the first, turning the turbulent kinetic energy negative. To avoid such unphysical behavior, a scaling with $a_{Ss}^*>0$ must be restricted to a certain range of x where the upper bound cannot be too large. A scaling that applies to the whole self-similar range thus needs to satisfy

$$a_{Ss}^* < 0. \tag{39}$$

NON-EQUILIBRIUM DISSIPATION SCALING

To investigate the physical meaning of a_{Ss}^*, we will analyse how it is connected to non-equilibrium dissipation scaling in the following. If $a_{Ss}^*\neq 0$, the scaling laws (33) and (34) deviate from the classical scalings

$$\overline{U}_x \sim x^{-\frac{1}{2}} \tag{40}$$

and

$$\varepsilon \sim x^{-\frac{5}{2}}. \tag{41}$$

In recent studies by Layek & Sunita (2018) and Cafiero & Vassilicos (2019), it was theorized that such a non-classical scaling in the plane jet is due to the dissipation scaling

$$D_0(x) \sim \frac{K_0^{3/2}(x)}{\delta(x)}\left(\frac{\mathrm{Re}_G}{\mathrm{Re}_\delta(x)}\right)^m, \tag{42}$$

where D_0 and K_0 are the streamwise scaling factors of dissipation and turbulent kinetic energy respectively, and m is some real parameter between zero and one. If $m=0$, the equilibrium scaling (1) is obtained in (42), while for $m=1$, (42) describes the non-equilibrium dissipation scaling (2). Naturally, the question arises, if and how the parameters m and a_{Ss}^* are related.

As discussed before, the scaling of the turbulent kinetic energy should be dominated by the first term on the r.h.s. in (38) in any physically plausible setting. We will proceed under the assumption that this is true, yielding

$$k(x,y) \approx \frac{\alpha_{k,0}\widetilde{k}_0(\eta)}{x-x_0}. \tag{43}$$

The scaling of the local Reynolds number thus follows as

$$\mathrm{Re}_\delta(x) = \frac{\sqrt{k(x,0)}\delta(x)}{\nu} = \frac{\sqrt{\alpha_{k,0}}S}{\nu}(x-x_0)^{\frac{1}{2}}. \quad (44)$$

With the usual definition of the global Reynolds number

$$\mathrm{Re}_G = \frac{U_b h}{\nu} = \frac{1}{\nu}, \quad (45)$$

where we set the inlet bulk velocity U_b and the slit height h to unity without loss of generality, the centerline scaling of the turbulent kinetic energy then follows by definition as

$$k(x,0) = \frac{1}{\delta^2(x)}\left(\frac{\mathrm{Re}_G}{\mathrm{Re}_\delta(x)}\right)^{-2}. \quad (46)$$

In a similar way, the dissipation scaling law (34) can be rewritten as

$$\varepsilon(x,0) = \frac{\widetilde{C}_\varepsilon}{\delta^4(x)}\left(\frac{\mathrm{Re}_G}{\mathrm{Re}_\delta(x)}\right)^{a_{Ss}^*-3}, \quad (47)$$

where we again only consider the centerline, i.e. $y,\eta = 0$. The coefficients have been lumped into $\widetilde{C}_\varepsilon$ as

$$\widetilde{C}_\varepsilon = \alpha_\varepsilon \sqrt{\alpha_{k,0}}^{a_{Ss}^*-3} S^{a_{Ss}^*+1}. \quad (48)$$

Combining both (46) and (47) yields

$$\varepsilon(x,0) = \widetilde{C}_\varepsilon \frac{k^{\frac{3}{2}}(x,0)}{\delta(x)}\left(\frac{\mathrm{Re}_G}{\mathrm{Re}_\delta(x)}\right)^{a_{Ss}^*}, \quad (49)$$

which is the same as (42). In consequence, a_{Ss}^* and m encode the same relation between dissipation, turbulent kinetic energy and the jet half-width as long as the assumption (43) is accurate.

However, due to them applying the boundary layer approximation, both Layek & Sunita (2018) and Cafiero & Vassilicos (2019) obtain scalings different to (33) for the mean centerline velocity, specifically

$$\overline{U}_x(x,0) = \frac{\alpha_x}{(x-x_0)^{\frac{1}{2}\frac{1+m}{1+2m}}} \quad (50)$$

instead of

$$\overline{U}_x(x,0) = \frac{\alpha_x}{(x-x_0)^{\frac{1}{2}(1-a_{Ss}^*)}}. \quad (51)$$

Furthermore, while the present approach always yields a linear spreading of the jet, i.e. (32), the theory of Layek & Sunita (2018) and Cafiero & Vassilicos (2019) yields an m-dependent spreading

$$\delta(x) = S(x-x_0)^{\frac{1+m}{1+2m}}, \quad (52)$$

which only becomes linear for $m = 0$. The scaling law of the dissipation is also different, with the boundary layer approximation resulting in

$$\varepsilon(x,0) = \frac{\alpha_\varepsilon}{(x-x_0)^{\frac{5+11m}{2+4m}}}, \quad (53)$$

contrasting the scaling (34) derived here. It is therefore clear that although m and a_{Ss}^* encode the same non-equilibrium dissipation scaling (42)/(49), they are not the same parameter, and, hence, mimic different physics. There is an argument from Cafiero & Vassilicos (2019), that m cannot be negative and since it only relies on (42), it equally applies to a_{Ss}^*. This argument lies in the fact, that the ratio between the integral scale, represented by the half-width δ, and the Taylor microscale λ can be expressed as

$$\frac{\delta^2}{\lambda^2} = \frac{\widetilde{C}_\varepsilon}{10}\mathrm{Re}_G^m \mathrm{Re}_\delta^{1-m}. \quad (54)$$

This ratio cannot decrease with both global Reynolds number, thus $m \geq 0$, and local Reynolds number, thus $m \leq 1$. However, this argument hinges on $\widetilde{C}_\varepsilon$ being independent of Re_G, which from our viewpoint does not necessarily have to be true. We thus consider negative values of m and a_{Ss}^* as possible, which was similarly argued in Layek & Sunita (2018).

Ideal scaling

While in (42) and (49), m and a_{Ss}^* were introduced as variable parameters, the ideal non-equilibrium scaling (2) follows for $m, a_{Ss}^* = 1$. There is an argument rooted in symmetry theory to support this too. Due to the aforementioned dissipative anomaly, even for a negligibly small, but non-zero viscosity ν, the dissipation rate ε does not become negligibly small and thus influences the flow significantly. In consequence, it can be argued that ν acts as a second symmetry breaking invariant in addition to the integral invariant (23). The result is a coupling of the scaling symmetries of time and space (Klingenberg *et al.* 2020) as

$$2a_{Sx} - a_{St} = 0. \quad (55)$$

In conjunction with the integral symmetry breaking (28), this leads to

$$a_{Sx} = a_{Ss} \quad \Rightarrow \quad a_{Ss}^* = 1. \quad (56)$$

Additionally, Layek & Sunita (2018) have shown that a non-zero viscosity directly leads to the non-equilibrium dissipation scaling with $m = 1$ for the boundary layer approximation too. However, they attributed this to low Reynolds number flows, not to additional symmetry breaking rooted in the dissipative anomaly.

Experimental validation

As discussed in Cafiero & Vassilicos (2019), there is some experimental evidence for non-equilibrium type scaling of the form (50), (52), and (53) with $m = 1$ in the data of Antonia *et al.* (1980) and their own measured data. For the scaling laws (33) and (34) presented here, such an agreement with experiments could not be observed, neither for a variable non-zero

a_{Ss}^*, nor for $a_{Ss}^* = 1$. Additionally, it has to be kept in mind that $a_{Ss}^* = 1$ leads to a constant, non-decaying centerline velocity, see (51), which is not physical, at least over long ranges of x. Therefore, applying the boundary layer approximation seems to be the more physical approach in this case.

CONCLUSION AND OUTLOOK

Lie-symmetry theory provides a promising pathway to formalize the theory of self-similarity in free shear flows. The scaling laws can be determined from first principles up to a parameter a_{Ss}^* that must be found empirically. We presented an analysis which omits the Reynolds decomposition and the boundary layer approximation. The parameter a_{Ss}^* of the scaling laws encodes non-equilibrium dissipation scaling in the same way as the parameter m from previous approaches to the same problem by Layek & Sunita (2018) and Cafiero & Vassilicos (2019). In contrast to the present approach, these approaches did apply the Reynolds decomposition and the boundary layer approximation. We found that viscosity induced symmetry breaking yields the ideal non-equilibrium scaling $m, a_{Ss}^* = 1$ in both approaches. However, only the results of the previous approach by Layek & Sunita (2018) and Cafiero & Vassilicos (2019) could be validated experimentally.

Further research on the topic might focus in investigating the link between the strong dissipative anomaly, the ensuing symmetry breaking through viscosity and non-equilibrium dissipation scaling. Finding such a link might provide further insights into the physics of dissipation, which is a fundamental mechanism in turbulence.

ACKNOWLEDGEMENTS

M.O. and N.B. acknowledge being funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under project number 577383178 (OB 96/59-1).

REFERENCES

Antonia, Robert A, Satyaprakash, BR & Hussain, AKMF 1980 Measurements of dissipation rate and some other characteristics of turbulent plane and circular jets. *The Physics of Fluids* **23** (4), 695–700.

Avsarkisov, V, Oberlack, Martin & Hoyas, Sergio 2014 New scaling laws for turbulent poiseuille flow with wall transpiration. *Journal of Fluid Mechanics* **746**, 99–122.

Cafiero, Gioacchino & Vassilicos, JC 2019 Non-equilibrium turbulence scalings and self-similarity in turbulent planar jets. *Proceedings of the Royal Society A* **475** (2225), 20190038.

Dairay, Thibault, Obligado, Martin & Vassilicos, John Christos 2015 Non-equilibrium scaling laws in axisymmetric turbulent wakes. *Journal of Fluid Mechanics* **781**, 166–195.

George, William K 1995 Some new ideas for similarity of turbulent shear flows. *Turbulence, Heat and Mass Transfer* **1**, 13–24.

Klingenberg, Dario & Oberlack, Martin 2022 Statistically invariant eddy viscosity models. *Physics of Fluids* **34** (5), 055114.

Klingenberg, Dario, Oberlack, Martin & Plümacher, Dominik 2020 Symmetries and turbulence modeling. *Physics of Fluids* **32** (2), 025108.

Layek, GC & Sunita 2018 Non-kolmogorov dissipation in a turbulent planar jet. *Physical Review Fluids* **3** (12), 124605.

Nguyen, Cat Tuong, Benedikt, Nils & Oberlack, Martin 2025 Hidden intermittency in turbulent jet flows. *Flow, Turbulence and Combustion* pp. 1–22.

Nguyen, Cat Tuong & Oberlack, Martin 2025 Passive scalar statistics in a turbulent round jet: symmetry theory and direct numerical simulation. *Journal of Fluid Mechanics* **1013**, A33.

Oberlack, Martin 2001 A unified approach for symmetries in plane parallel turbulent shear flows. *Journal of Fluid Mechanics* **427**, 299–328.

Oberlack, Martin, Hoyas, Sergio, Kraheberger, Stefanie V, Alcántara-Ávila, Francisco & Laux, Jonathan 2022 Turbulence statistics of arbitrary moments of wall-bounded shear flows: A symmetry approach. *Physical Review Letters* **128** (2), 024502.

Oberlack, Martin & Rosteck, Andreas 2010 New statistical symmetries of the multi-point equations and its importance for turbulent scaling laws. *Discrete Continuous Dyn. Syst* **3**, 451–471.

Pope, Stephen B 2001 Turbulent flows. *Measurement Science and Technology* **12** (11), 2020–2021.

Sadeghi, H, Oberlack, M & Gauding, M 2018 On new scaling laws in a temporally evolving turbulent plane jet using lie symmetry analysis and direct numerical simulation. *Journal of Fluid Mechanics* **854**, 233–260.

Sadeghi, H, Oberlack, M & Gauding, M 2021 New symmetry-induced scaling laws of passive scalar transport in turbulent plane jets. *Journal of Fluid Mechanics* **919**, A5.

Seoud, RE & Vassilicos, JC 2007 Dissipation and decay of fractal-generated turbulence. *Physics of fluids* **19** (10).

Sreenivasan, Katepalli R & Schumacher, Jörg 2025 What is the turbulence problem, and when may we regard it as solved? *Annual Review of Condensed Matter Physics* **16** (1), 121–143.

Vassilicos, J Christos 2015 Dissipation in turbulent flows. *Annual review of fluid mechanics* **47** (1), 95–114.

Vassilicos, John Christos & Laval, Jean-Philippe 2025 Scale-by-scale nonequilibrium in turbulent flows. *Coarse Graining Turbulence* (1), 333–354.

Wacławczyk, Marta, Staffolani, Nicola, Oberlack, Martin, Rosteck, Andreas, Wilczek, Michael & Friedrich, Rudolf 2014 Statistical symmetries of the lundgren-monin-novikov hierarchy. *Physical Review E* **90** (1), 013022.

Yasuda, Tatsuya & Vassilicos, John Christos 2018 Spatio-temporal intermittency of the turbulent energy cascade. *Journal of Fluid Mechanics* **853**, 235–252.