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ENERGY DISSIPATION OF OLDROYD-B FLUIDS IN PLANE COUETTE FLOW

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ABSTRACT

We establish a mathematically rigorous bound on the infinite-time-averaged energy dissipation rate for Oldroyd-B fluids in plane Couette flow, which depends only on system parameters: the viscosity ratio, the Reynolds number, and the Weissenberg number. The bound is valid for all solutions within an identified parameter region, and is proven by extending the 'background flow method' for Newtonian fluids to a case where the energy is no longer quadratic, or even polynomial, by using a non-polynomial auxiliary functional. The bound recovers the steady dissipation rate and the best-known Newtonian result in their respective parameter regions. We also identify a parameter region where the total energy of Oldroyd-B fluids is bounded.

INTRODUCTION

Flows of viscoelastic fluids are encountered in a wide array of industries and display distinctive behaviours compared to their Newtonian counterparts. As such, understanding quantitative characteristics of these flows, such as the infinite-time-averaged energy dissipation rate, has attracted sustained interest. For Newtonian fluids, bounds on turbulent energy dissipation rates can be proven via the well-known 'background flow method' by Doering & Constantin (1992, 1994, 1995). However, due to the non-quadratic (and generally non-polynomial) polymer free energy, it is not obvious how to extend the aforementioned method to viscoelastic fluids, or if an analogous argument for obtaining rigorous bounds exists at all. To this end, we utilise the auxiliary functional method (Chernyshenko *et al.*, 2014; Chernyshenko, 2022), which considers general functionals of the underlying fields of dynamical systems described by partial differential equations, to prove rigorous bounds on quantities of interest averaged over long times.

A defining feature of viscoelastic flows is that, beyond the usual balance between kinetic energy production and viscous dissipation in Newtonian fluids, polymer chains can store energy through elastic stretching and dissipate it through relaxation. This additional mechanism has been found to sustain turbulent-like dynamics in parameter regimes where the Newtonian counterpart would be laminar. Depending on the regime, viscoelastic effects may lead to elevated energy dissipation rates (Rothstein & McKinley, 2001), increased pressure losses (Metzner & White, 1965), or substantial drag reduction even for small polymer concentrations (Metzner & Park, 1964; Jones & Maddock, 1966; Virk, 1975). Consequently, any energy analysis must account for both polymeric and viscous contributions (Shaqfeh, 1996; Rothstein & McKinley, 2001).

As a demonstration of their complexity, viscoelastic flows are numerically difficult to simulate even in parameter regions where they are expected to behave well. A central difficulty is the 'high Weissenberg number problem', where numerical schemes can lose convergence beyond a critical Weissenberg number (Keunings, 1986). Spurious instabilities have been reported even for plane Couette flow in regimes that are linearly stable in the inertia-less limit (Renardy, 1992, 1993; Keiller, 1992). While energy-consistent discretisations, the addition of polymeric stress diffusion, and positive-definite preserving polymer conformation-tensor formulations can improve robustness, they do not fully solve the problem (Lozinski & Owens, 2003; Thomases & Shelley, 2007; Balci *et al.*, 2011), and a loss of numerical stability can occur even at $Wi = \mathcal{O}(1)$ in more complex geometries (Lee *et al.*, 2021). These persistent numerical difficulties motivate mathematically rigorous results, such as bounds on the time-averaged energy dissipation rate, or boundedness of energy along trajectories.

In this paper, we derive an *a priori* upper bound on the infinite-time-averaged energy dissipation rate for Oldroyd-B fluids in plane Couette flow. A recent advance in this direction is the work of Binns & Wynn (2024), who proved global non-linear stability within a certain parameter region by constructing a non-polynomial Lyapunov functional. Building on their work, we now provide an upper bound on the dissipation rate in regimes where global stability has not been proven, notably for high Reynolds numbers.

We proceed by considering the aforementioned auxiliary functional method (Chernyshenko *et al.*, 2014; Chernyshenko, 2022). For an ordinary differential equation described by $\dot{x} = f(x)$ and a quantity of interest $F(x)$, if there exists a bounded differentiable function $V(x)$ such that an inequality of the form $F(x)+\nabla V(x)\cdot f(x)\leq\kappa$ holds for all states x, then evaluating this along a solution trajectory and taking the infinite-time average yields $\overline{F}\leq\kappa$. The central challenge in this method is finding such a $V(x)$.

For partial differential equations, V is taken to be a functional of the underlying fields. Its Lie derivative, $\mathscr{L}_V[\mathbf{u},\mathbf{c}]$, is the rate of change of V obtained by evaluating its functional derivative using the right-hand sides of the governing equations (5). To bound the energy dissipation rate $\mathscr{E}$, we seek an auxiliary functional V and a constant κ such that the inequality

$$\mathscr{E}[\mathbf{u},\mathbf{c}]+\mathscr{L}_V[\mathbf{u},\mathbf{c}]\leq\kappa \tag{1}$$

holds for all admissible states $(\mathbf{u},\mathbf{c})$.

Crucially, for any solution $(\mathbf{u}(t),\mathbf{c}(t))$ satisfying the governing equations and boundary conditions, the Lie derivative is equal to the time derivative of V

$$\mathscr{L}_V[\mathbf{u}(t),\mathbf{c}(t)]=\frac{\mathrm{d}}{\mathrm{d}t}V[\mathbf{u}(t),\mathbf{c}(t)]. \tag{2}$$

Hence, if V remains uniformly bounded along the trajectory for all $t\geq 0$, then the infinite-time average of $\mathrm{d}V/\mathrm{d}t$ is equal to zero. Taking the infinite-time average of (1) therefore yields

$$\overline{\mathscr{E}}\leq\kappa. \tag{3}$$

For Navier-Stokes shear flows, the background flow method (Doering & Constantin, 1992, 1994, 1995) can be interpreted as a special case of the auxiliary functional method, in which the auxiliary functional is chosen to be a particular perturbation kinetic energy relative to a prescribed background flow $\mathbf{b}$. Writing $\mathbf{u}=\mathbf{b}+\mathbf{v}$, where $\mathbf{b}$ is time-independent and satisfies the same boundary conditions as $\mathbf{u}$, one defines

$$V_{\text{Newtonian}}[\mathbf{u};\mathbf{b}]:=\frac{1}{2}\|\mathbf{v}\|_2^2=\frac{1}{2}\|\mathbf{u}\|_2^2-\langle\mathbf{b},\mathbf{u}\rangle+\frac{1}{2}\|\mathbf{b}\|_2^2, \tag{4}$$

where the standard L^2 norm and inner product are defined by $\|\mathbf{A}\|_2^2=\langle\mathbf{A},\mathbf{A}\rangle$ and $\langle\mathbf{A},\mathbf{B}\rangle=\int_\Omega A_{ij}B_{ij}\,\mathrm{d}\mathbf{x}$. Thus, relative to the physical kinetic energy $\|\mathbf{u}\|_2^2/2$, the auxiliary functional differs by a linear cross term involving $\mathbf{b}$ and $\mathbf{u}$, plus a constant depending only on $\mathbf{b}$. The background flow $\mathbf{b}$ therefore acts as a tuning parameter, and by choosing its shape appropriately, typically within thin boundary layers, tractable bounds on dissipation can be derived. Extending this approach to viscoelastic systems requires addressing the non-linear structure of polymer dynamics, for which polynomial auxiliary functionals may no longer be sufficient.

PROBLEM FORMULATION

We consider an incompressible Oldroyd-B fluid with velocity $\mathbf{u}$ and polymer conformation tensor $\mathbf{c}$ in plane Couette flow. Under standard non-dimensionalisation with channel height ℓ, top plate velocity U, and the associated timescale ℓ/U, the governing equations of the system are given by

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\frac{\beta}{Re}\nabla^2\mathbf{u}+\frac{1-\beta}{ReWi}\nabla\cdot\mathbf{c} \tag{5a}$$

$$\nabla\cdot\mathbf{u}=0 \tag{5b}$$

$$\frac{\partial\mathbf{c}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{c}=\mathbf{c}\cdot\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\cdot\mathbf{c}+\frac{1}{Wi}(\mathbf{I}-\mathbf{c}). \tag{5c}$$

The system is governed by three non-dimensional parameters: the Reynolds number $Re=\rho U\ell/\eta$, viscosity ratio $\beta=\eta_s/\eta$, and the Weissenberg number $Wi=U\lambda/\ell$, where the total viscosity η is the sum of polymer viscosity η_p and solvent viscosity η_s, so that $0<\beta\leq 1$, and λ is the characteristic relaxation time of the polymer dumbbells. No-slip boundary conditions are assumed at the top and bottom plates, with periodicity in streamwise and spanwise directions. Since the conformation tensor $\mathbf{c}$ is constructed as the covariance matrix of the end-to-end vector of the polymer dumbbells, it is symmetric positive definite, *i.e.* $\mathbf{c}=\mathbf{c}^\top\succ 0$.

We follow the well-established formulation of total energy of Oldroyd-B fluids (Hu & Lelièvre, 2007)

$$E:=\frac{1}{2}\|\mathbf{u}\|_2^2+\frac{1-\beta}{2ReWi}\int_\Omega \operatorname{tr}(\mathbf{c}-\mathbf{I})-\log\det\mathbf{c}\,\mathrm{d}\mathbf{x} \tag{6}$$

which is the weighted sum of the fluid's kinetic energy and the potential energy stored in the polymer coils by elastic stretching. By computing the time derivative of (6) we arrive at the energy equation

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t}=&-\frac{\beta}{Re}\|\nabla\mathbf{u}\|_2^2-\frac{1-\beta}{2ReWi^2}\int_\Omega\operatorname{tr}\left(\mathbf{c}+\mathbf{c}^{-1}-2\mathbf{I}\right)\,\mathrm{d}\mathbf{x}\\&+\frac{\beta}{Re}\int_{y=1}\frac{\partial u_x}{\partial y}\,\mathrm{d}A+\frac{1-\beta}{ReWi}\int_{y=1}c_{xy}\,\mathrm{d}A.\end{aligned} \tag{7}$$

The last two boundary integrals in (7) represent the mechanical power input through the moving upper wall. The first two terms are negative semidefinite and we therefore interpret them as the energy dissipation rate

$$\mathscr{E}:=\frac{\beta}{Re}\|\nabla\mathbf{u}\|_2^2+\frac{1-\beta}{2ReWi^2}\int_\Omega\operatorname{tr}\left(\mathbf{c}+\mathbf{c}^{-1}-2\mathbf{I}\right)\,\mathrm{d}\mathbf{x}. \tag{8}$$

To establish an upper bound on the infinite-time-averaged energy dissipation rate $\overline{\mathscr{E}}$, we now construct an auxiliary functional $V[\mathbf{u},\mathbf{c}]$ which is bounded along trajectories. Following the auxiliary functional method outlined previously, demonstrating an inequality of the form $\mathscr{E}+\Lambda\frac{\mathrm{d}V}{\mathrm{d}t}\leq\kappa$, where $\Lambda>0$ is a constant balancing parameter, ensures that $\overline{\mathscr{E}}$ is bounded by κ for all admissible solutions.

AUXILIARY FUNCTIONAL

We note that the background flow method is, in fact, an extension to the classical energy stability analysis where $\mathbf{b}$ in (4) is chosen as the steady solution. It is therefore natural to first consider the global stability result by Binns & Wynn (2024) where they constructed a non-polynomial Lyapunov functional in the form

$$L[\mathbf{v},\mathbf{d}]=\frac{\alpha}{1-\beta}ReWi\,K[\mathbf{v}]+H[\mathbf{d}] \tag{9}$$

where in addition to $\mathbf{u}=\mathbf{b}+\mathbf{v}$, the conformation tensor $\mathbf{c}$ is decomposed as $\mathbf{c}=\mathbf{c}_b+\mathbf{d}$ where $\mathbf{c}_b$ is a polymer background profile of our choice, which in the stability analysis is taken to be the steady solution. The fluid 'perturbation kinetic energy' and the polymer 'relative entropy' are defined as

$$\begin{aligned} K[\mathbf{v}] &= \frac{1}{2}\|\mathbf{v}\|_2^2 \\ H[\mathbf{d}] &= \frac{1}{2}\int_\Omega \operatorname{tr}(\mathbf{I}+\alpha\mathbf{d}) - 3 - \log\det(\mathbf{I}+\alpha\mathbf{d})\ \mathrm{d}\mathbf{x}, \end{aligned} \tag{10}$$

where α is a constant that ensures $\mathbf{I}+\alpha\mathbf{d} \succ 0$ for any possible solution to (5). Binns & Wynn (2024) showed that Oldroyd-B fluids in plane Couette flow are globally stable, *i.e.* $(\mathbf{u},\mathbf{c}) \to (\mathbf{u}_{\text{steady}}, \mathbf{c}_{\text{steady}})$, within the parameter region

$$c_1\left(\frac{1-\beta}{1-Wi}\right)Wi + c_2\,Re < \beta, \qquad 0<\beta<1, \tag{11}$$

where $c_1, c_2 > 0$ are certain constants. Within this parameter range, the dissipation of the steady flow can be computed by substituting the steady solutions to the flow field and the polymer conformation tensor into (8) to give $\mathscr{E}_{\text{steady}} = 1/Re$, which is equal to the corresponding Newtonian ($\beta = 1$) value.

We choose our auxiliary functional to be in a form similar to $L[\mathbf{v},\mathbf{d}]$, taking $\alpha = 1$ and $\mathbf{c}_b = \mathbf{I}$, which corresponds to the relaxed polymer configuration:

$$V[\mathbf{v},\mathbf{c}] := \frac{1}{2}\|\mathbf{v}\|_2^2 + \frac{1-\beta}{2ReWi}\int_\Omega \operatorname{tr}(\mathbf{c}-\mathbf{I}) - \log\det\mathbf{c}\,\mathrm{d}\mathbf{x}. \tag{12}$$

We now introduce the notation for the weighted inner product and norm

$$\langle \mathbf{A},\mathbf{B}\rangle_{\mathbf{M}} := \int_\Omega \operatorname{tr}\left[\mathbf{A}^\top(\mathbf{I}+\mathbf{M})^{-1}\mathbf{B}\right]\ \mathrm{d}\mathbf{x}, \quad \|\mathbf{A}\|_{\mathbf{M}}^2 = \langle \mathbf{A},\mathbf{A}\rangle_{\mathbf{M}} \tag{13}$$

defined for any vectors or matrices $\mathbf{A}$, $\mathbf{B}$, and any matrix satisfying $\mathbf{M} \succ -\mathbf{I}$. We retain the usual notation for L^2 inner product and norm $\|\cdot\|_2$, which may be written under the new weighted inner product notation as $\|\cdot\|_{\mathbf{0}}$. The energy dissipation rate can now be written as

$$\mathscr{E} = \frac{\beta}{Re}\|\nabla\mathbf{u}\|_2^2 + \frac{1-\beta}{2ReWi^2}\|\mathbf{c}-\mathbf{I}\|_{\mathbf{c}-\mathbf{I}}^2, \tag{14}$$

which highlights $\mathbf{I}$ as the relaxed polymer state. We construct the fluid background flow to be a pure-shear profile, which takes the form $\mathbf{b} = b(y)\,\mathbf{e}_x$ and

$$b(y) = \begin{cases} \frac{1}{2\delta}y, & 0 \le y < \delta, \\ \frac{1}{2}, & \delta \le y \le 1-\delta, \\ \frac{1}{2} + \frac{1}{2\delta}(y-(1-\delta)), & 1-\delta < y \le 1. \end{cases} \tag{15}$$

Differentiating $V[\mathbf{v},\mathbf{c}]$ along trajectories, and using incompressibility and boundary conditions, gives

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} =& -\frac{\beta}{2Re}\left(\|\nabla\mathbf{v}\|_2^2 + \|\nabla\mathbf{u}\|_2^2\right) - \frac{1-\beta}{2ReWi^2}\|\mathbf{c}-\mathbf{I}\|_{\mathbf{c}-\mathbf{I}}^2 \\ & - \int_\Omega b' v_x v_y\,\mathrm{d}\mathbf{x} + \frac{1-\beta}{ReWi}\langle(\mathbf{c}-\mathbf{I})\mathbf{c}, \nabla\mathbf{b}\rangle_{\mathbf{c}-\mathbf{I}} + \frac{\beta}{2Re}\|\nabla\mathbf{b}\|_2^2. \end{aligned} \tag{16}$$

AN ANALYTICAL BOUND ON DISSIPATION

We now demonstrate that the inequality

$$\mathscr{E} + \Lambda\frac{\mathrm{d}V}{\mathrm{d}t} \le \kappa \tag{17}$$

holds within an identified parameter region for all admissible solutions, where

$$\begin{aligned} \mathscr{E} + \Lambda\frac{\mathrm{d}V}{\mathrm{d}t} =& \Lambda\left(-\frac{\beta}{2Re}\|\nabla\mathbf{v}\|_2^2 - \int_\Omega b' v_x v_y\,\mathrm{d}\mathbf{x} + \frac{\beta}{2Re}\|\nabla\mathbf{b}\|_2^2\right) \\ & + \frac{(2-\Lambda)\beta}{2Re}\|\nabla\mathbf{u}\|_2^2 + \Lambda\frac{1-\beta}{ReWi}\int_\Omega b' c_{xy}\,\mathrm{d}\mathbf{x} \\ & - \frac{(\Lambda-1)(1-\beta)}{2ReWi^2}\|\mathbf{c}-\mathbf{I}\|_{\mathbf{c}-\mathbf{I}}^2. \end{aligned} \tag{18}$$

Note that $\mathbf{b}$ has been constructed such that b' is only non-zero within the boundary layers of thickness δ (see (15)). We bound the cross term in the first line of (18) and absorb it into the negative $\|\nabla\mathbf{v}\|_2^2$ term, leaving only the constant background term with the following estimate (Doering & Constantin, 1992)

$$\begin{aligned} \left|\int_\Omega b' v_x v_y\,\mathrm{d}\mathbf{x}\right| &\le b'\left|\int_0^\delta v_x v_y\,\mathrm{d}\mathbf{x} + \int_{1-\delta}^1 v_x v_y\,\mathrm{d}\mathbf{x}\right| \\ &\le \frac{\delta}{4}\left(\frac{1}{2\sqrt{2}}\left\|\frac{\partial v_x}{\partial x}\right\|_2^2 + \frac{\sqrt{2}}{2}\left\|\frac{\partial v_y}{\partial y}\right\|_2^2\right) \\ &\le \frac{\delta}{8\sqrt{2}}\|\nabla\mathbf{v}\|_2^2. \end{aligned} \tag{19}$$

Hence, we wish to choose the 'boundary layer' thickness to be sufficiently small such that

$$-\frac{\beta}{2Re}\|\nabla\mathbf{v}\|_2^2 - \int_\Omega b' v_x v_y\,\mathrm{d}\mathbf{x} \le \left(-\frac{\beta}{2Re} + \frac{\delta}{8\sqrt{2}}\right)\|\nabla\mathbf{v}\|_2^2 \le 0. \tag{20}$$

One sufficient choice is

$$0 < \delta \le \min\left\{\frac{1}{2}, \frac{4\sqrt{2}\beta}{Re}\right\}. \tag{21}$$

We also wish to choose the balancing parameter Λ to satisfy $\Lambda \ge 2$ so that the $(2-\Lambda)\beta\|\nabla\mathbf{u}\|_2^2/(2Re)$ term in (18) is upper-bounded by 0. We then proceed to bound the polymeric terms, as shown in the final line of (18). Via the standard Rayleigh quotient bound for positive-definite matrices we have

$$|c_{xy}| \le \frac{a}{2}\|\mathbf{c}-\mathbf{I}\|_{\mathbf{c}-\mathbf{I}}^2 + \frac{2-\sqrt{2}}{a}, \qquad \forall a \ge 1. \tag{22}$$

Again, since b' is only supported in the boundary layers, let $\Omega_\delta := \{\mathbf{x}\in\Omega : y \in [0,\delta]\cup[1-\delta,1]\}$, we have

$$\begin{aligned} &\Lambda\frac{1-\beta}{ReWi}\int_\Omega b' c_{xy}\,\mathrm{d}\mathbf{x} \\ \le& \frac{\Lambda(1-\beta)}{2\delta ReWi}\int_{\Omega_\delta}|c_{xy}|\ \mathrm{d}\mathbf{x} \\ \le& \frac{\Lambda(1-\beta)}{2\delta ReWi}\left[\frac{a}{2}\|\mathbf{c}-\mathbf{I}\|_{\mathbf{c}-\mathbf{I}}^2 + \frac{2\delta\left(2-\sqrt{2}\right)}{a}\right] \end{aligned} \tag{23}$$

which holds for any $a \geq 1$. We choose a to satisfy

$$1 \leq a \leq \frac{2\delta\left(1-\Lambda^{-1}\right)}{Wi} \tag{24}$$

such that the first term on the right-hand side of (23) can be upper-bounded by the final negative term in (18). Collecting the above restrictions on δ, Λ, and a, we make the following choices

$$\delta = \min\left\{\frac{1}{2}, \frac{4\sqrt{2}\beta}{Re}\right\}, \quad \Lambda = \frac{2}{1-Wi}, \quad a = \frac{2\delta}{Wi}\left(1-\frac{1}{\Lambda}\right) \tag{25}$$

such that (18) has a finite upper bound, subject to $a \geq 1, \delta \leq 1/2$. We collect these observations and restrictions to produce the main analytical result of the paper:

Theorem 1. *Let* $0 < Wi < 1$ *and* $0 < \beta < 1$. *If*

$$0 < Re < 4\sqrt{2}\beta\left(1+Wi^{-1}\right) \tag{26}$$

then the upper bound on the infinite-time-averaged energy dissipation rate

$$\overline{\mathscr{E}} \leq \frac{1}{1-Wi}\left(\frac{1}{8\sqrt{2}} + \frac{(\sqrt{2}-1)(1-\beta)}{2\beta(1+Wi)}\right)\max\left\{1, \frac{8\sqrt{2}\beta}{Re}\right\} \tag{27}$$

holds for all admissible solutions to the Oldroyd-B equations (5).

DISCUSSION

We have proved an *a priori* bound on the energy dissipation rate of Oldroyd-B fluids in plane Couette flow. In particular, (26) implies that the bound remains valid for arbitrarily large Reynolds numbers provided the Weissenberg number is sufficiently small and β is bounded away from zero. In the weakly elastic, near-Newtonian limit ($Wi \to 0$, $\beta \to 1$), the bound reduces to the classical Newtonian Couette bound $\overline{\mathscr{E}} \leq \left(8\sqrt{2}\right)^{-1}$ by Doering & Constantin (1992), with small additional $\mathscr{O}(Wi)$ and $\mathscr{O}(1-\beta)$ polymeric contributions that increase the provable dissipation bound. The proof of Theorem 1 relies on showing that the auxiliary functional V is bounded along trajectories in the parameter region (26). Since V differs from the total energy (6) only by bounded background terms, it follows that the total energy of the system is also bounded along trajectories in (26).

At low Reynolds numbers, the bound approaches the steady dissipation rate $1/Re$ only in the limiting Newtonian case, revealing the potential sub-optimality of the high-Re-focused construction. The source of this deficiency is straightforward: our proof controls the perturbations around a background velocity profile $\mathbf{b}$ and a fixed polymer background $\mathbf{c}_b = \mathbf{I}$. While the velocity background matches the steady Couette profile at small Re, the choice $\mathbf{c}_b = \mathbf{I}$ does not reflect the steady polymer configuration except when $Wi = 0$. This motivates using steady backgrounds $(\mathbf{b}_{\text{steady}}, \mathbf{c}_{\text{steady}})$ in the auxiliary functional, in the spirit of the global stability proof by Binns & Wynn (2024). In the parameter region (11) where the flow is provably stable, this recovers the analytical steady dissipation $1/Re$ up to a vanishing $\mathscr{O}(\Lambda^{-1})$ term

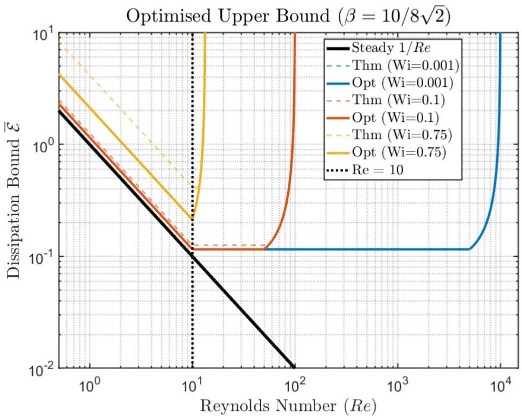

Figure 1. The scaling with Re of the upper bounds on dissipation $\overline{\mathscr{E}}$ obtained from the analytical bound (27) (dashed lines) and the ‘optimal’ bounds found by optimising over δ, a, Λ (solid lines) at selected Weissenberg numbers, for $\beta = 10/(8\sqrt{2})$. The steady dissipation rate $1/Re$ is shown with a solid black line. The critical Reynolds number $Re = 8\sqrt{2}\beta = 10$ separating the two cases of (27) is indicated with a vertical dotted line.

as the balancing parameter tends to infinity $\Lambda \to \infty$, showing that the same auxiliary functional framework can capture both steady and unsteady (high Re) regimes via different background choices.

At intermediate parameter values, one may expect further improvements from picking intermediate polymer backgrounds $\mathbf{c}_b$ interpolating between $\mathbf{I}$, which resembles the polymer configuration in the bulk of a ‘well-mixed’ turbulent flow, and $\mathbf{c}_{\text{steady}}$. In practice, exploring this interpolation is technically involved due to the interaction of the coupled optimisation choices ($\mathbf{b}$, $\mathbf{c}_b$, δ, a, Λ). As a first step, one can tighten the explicit analytical bound (27) by numerically optimising over the remaining scalar parameters (a, δ, Λ) while keeping $\mathbf{c}_b = \mathbf{I}$ and the piecewise linear choice of $\mathbf{b}$, as shown in Figure 1. This optimisation confirms that our closed-form parameter choices, especially for Λ, are indeed conservative, particularly as Wi increases. It also illustrates that the choice $\mathbf{c}_b = \mathbf{I}$ is not optimal for small Reynolds numbers.

A central limitation of our analytical bound is the restriction $Wi < 1$. The analysis makes the origin of this constraint clear: the admissible region is controlled by the worst-case polymer configurations that maximise the near-wall production term involving c_{xy} relative to the linear relaxation dissipation of the Oldroyd-B constitutive equation. In the current construction, this term is handled by a global estimate on $\left|c_{xy}\right|$ in terms of a quadratic free-energy-like quantity via (23), which is necessarily conservative because it allows conformation configurations that are algebraically admissible but may not be dynamically attainable, particularly in the presence of walls. In contrast, the wall dynamics yield a substantially stronger constraint. Under mild regularity assumptions, one can show that the conformation tensor at the walls satisfies the time-average inequality

$$\overline{c_{xy}^2\big|_{\text{wall}}} \leq \frac{1}{2}\overline{\operatorname{tr}\left[(\mathbf{c}-\mathbf{I})\,\mathbf{c}^{-1}(\mathbf{c}-\mathbf{I})\right]_{\text{wall}}} \tag{28}$$

which controls c_{xy}^2 directly by the polymeric free-energy density at the wall. This is significantly tighter than a bound on $|c_{xy}|$ and prevents the near-wall production term from being saturated by arbitrarily large shear configurations, indicating that the wall constraint suppresses the most adverse polymer states. The physical mechanism is intuitive: the wall impedes wall-normal polymer stretching so that, after an initial transient, $c_{yy}\big|_{\text{wall}} = 1$. With c_{yy} fixed and $\mathbf{c} \succ 0$, positive-definiteness couples c_{xy} to the remaining stretch components, so large c_{xy} necessarily carries 'dissipative cost'. The steady Couette conformation $\mathbf{c}_{\text{steady}}$ is consistent with this inequality, giving hope that it genuinely captures the near-wall physics. From a similar argument, even if a bound in the form $c_{yy} \leq 1+\rho$ can be proven near the wall, then one would prove energy-boundedness and bounds on dissipation for the Oldroyd-B model at arbitrarily large Weissenberg numbers.

Under the current model, it is not clear whether (28) holds only at the walls or in a vicinity of them. An alternative approach is to consider other constitutive models, such as those augmented with a spatial polymeric stress diffusion term (Constantin & Kliegl, 2012), which penalises large polymer stress gradients and thus gives hope that the tight wall inequality can be 'carried' into some regions near the walls. However, since the magnitude of polymeric stress diffusion is often extremely small (El-Kareh & Leal, 1989), it is not obvious whether the inclusion of such terms leads to a substantial effect on the tightness of the bound or the size of the admissible region.

A more promising approach is to use models whose damping is superlinear, such as the FENE (finite extensible nonlinear elastic) models. Perhaps the most well-known of these is the FENE-P model, which provides a closure to the FENE dynamics via the Peterlin function. In the FENE-P model, the damping on polymer chains diverges superlinearly as the polymers are stretched to their limits. As such, unlike the Oldroyd-B model where the polymeric production and dissipation are both $\mathscr{O}(\mathbf{c})$ at large extensions, which leads to the restriction $a \sim (ReWi)^{-1} \geq 1$, FENE-P allows for any a, and thus, $ReWi$ can be arbitrarily large, resulting in a bound that is valid everywhere in parameter space. We reserve the full proof and explicit bound for a separate paper.

Finally, from a purely methodological perspective, we have shown that the background flow method for Newtonian fluids can be rigorously extended to systems where the energy is not quadratic, or even polynomial. In the Newtonian setting, the background flow method can be interpreted neatly as a search for a suitable auxiliary functional V constructed as a linear perturbation of the kinetic energy about a prescribed background. In the non-Newtonian case presented, where the total energy is no longer polynomial, it becomes less obvious how to construct such a functional. This non-triviality is resolved, as demonstrated above, by picking a suitable background polymer field. We note that while attempts have been made to introduce a quadratic form of polymer energy (Balci *et al.*, 2011), the manipulations involved render the right-hand side of the constitutive equation non-polynomial, and it is not clear whether this simplifies the analysis in our case.

CONCLUSION

In this paper, with a non-polynomial auxiliary functional, we provide the first mathematically rigorous upper bound on the infinite-time-averaged energy dissipation rate of Oldroyd-B fluids in plane Couette flow. The bound is valid for arbitrarily high Reynolds numbers given sufficiently small Weissenberg numbers and depends explicitly only on system parameters. It also recovers the best-known Newtonian result and the analytical steady dissipation within their respective parameter regions.

As part of the proof, we show that the auxiliary functional V is bounded, which is required for the auxiliary functional method. Since V is effectively a perturbation energy and the background fields $\mathbf{b}$, $\mathbf{c}_b$ are both bounded, it immediately follows that the total energy (6) is also bounded along trajectories. This is the first known proof that, if the total energy is initially bounded, then it remains uniformly bounded for a region of parameter space larger than the region in which steady behaviour is provable. This helps distinguish high-Weissenberg-number-problem-type numerical anomalies from genuine flow behaviour.

One central contribution is extending the background flow method to cases where energy is no longer quadratic, or even polynomial. This is the first application of a non-polynomial auxiliary functional in obtaining rigorous bounds in Navier-Stokes type problems, and provides a basis for bounding non-polynomial quantities of interest in dynamical systems, particularly when a non-polynomial Lyapunov functional is already known.

These three results—extending the background flow method to non-polynomial cases, a rigorous upper bound on the energy dissipation rate, and boundedness of energy in Oldroyd-B shear flow—are the main contributions of this work.

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