14th International Symposium on Turbulence and Shear Flow Phenomena (TSFP14)
Heidelberg, Germany, July 28–31, 2026

TURBULENT ENERGY CASCADES IN TRANSITIONAL MAGNETOHYDRODYNAMIC FLOWS

Nick Williams
Department of Mechanical and
Aerospace Engineering
The University of Manchester
Oxford Rd, Manchester M13 9PL
nicholas.williams-2@manchester.ac.uk

Alessandro de Rosis
Department of Mechanical and
Aerospace Engineering
The University of Manchester
Oxford Rd, Manchester M13 9PL

Alex Skillen
Department of Mechanical and
Aerospace Engineering
The University of Manchester
Oxford Rd, Manchester M13 9PL

ABSTRACT

Magnetohydrodynamic turbulence at high magnetic Reynolds number often departs from classical Kolmogorov scaling, driven by strong mean-field effects and pervasive magnetic reconnection across the cascade. We examine the transition to turbulence in a magnetohydrodynamic jet, with particular attention to the energy budgets and spectral characteristics of the flow during transition. Our analysis indicates that the production and cascade of turbulent energy during this transition is dominated by fluctuations in the magnetic field associated with reconnection.

INTRODUCTION

Magnetohydrodyamic (MHD) turbulence is a multiphysics phenomenon experienced by electrically conductive fluids in presence of magnetic fields, predominantly observed in astrophysical plasmas and fusion engineering applications. Depending on the physical characteristics of the flow, and the relative strength of the magnetic field, the fundamentals of the turbulence cascade can diverge significantly from non-conductive flows.

Foundational works in MHD from Kraichnan (1965) and Goldreich & Sridhar (1995) focused on understanding *how* MHD turbulence diverges from the established scaling laws of Kolmogorov (1941), with adjusted energy spectra scaling to match observational and satellite data, and a decomposition of energy spectra into field parallel $k_{\parallel}$ and field perpendicular $k_{\perp}$ components. Despite ongoing advancements, an accurate characterisation of the energy spectra in MHD turbulence remains an area of debate. Recent DNS studies, such as those of Dong *et al.* (2022), have provided further insights, with the energy spectra inertial range gradient shown to scale with the magnetic Reynolds number of the flow.

Progress has been made in understanding the energy cascades of fully developed MHD turbulence, but many fundamental aspects of MHD turbulence remain unexplored, such as turbulence transition. Whilst no real-world plasma in astrophysical or fusion applications is fully laminar, there are many physical processes that may inject energy into turbulence cascades from mean-field effects, such as the large-scale dynamo and magnetic reconnection. Magnetic reconnection events seen in solar flares and coronal mass ejections for example redistribute extreme amounts of magnetic energy, accelerating the local plasma, but it is not currently known if these events generate turbulent energy or simply redistribute energy from within existing turbulence cascades.

Using an ensemble of direct numerical simulations, we investigate the cascade of turbulent energy during magnetic reconnection, with focus on the transitional energy spectra of the turbulence and the mechanisms that redistribute energy into the turbulent cascades from the mean-fields.

GOVERNING EQUATIONS

The flow is governed by the incompressible Navier-Stokes equations with the Lorentz force:

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_{jj} u_i + b_j \partial_j b_i - \partial_i \frac{\|\boldsymbol{b}\|^2}{2}, \quad (1)$$

and the continuity equation:

$$\partial_i u_i = 0. \quad (2)$$

The evolution of the magnetic field is governed by the induction equation:

$$\partial_t b_i + u_j \partial_j b_i - b_j \partial_j u_i = \eta \partial_{jj} b_i, \quad (3)$$

and the divergence-free constraint:

$$\partial_i b_i = 0, \quad (4)$$

where u_i is the fluid velocity, b_i is the magnetic field strength, and ν and η are the fluid's kinematic viscosity and magnetic diffusivity. The Reynolds number is defined as $Re = \frac{UL}{\nu}$, and the magnetic Reynolds number as $Re_m = \frac{UL}{\eta}$. Indices $i, j = 1, 2, 3$ span Cartesian components, repeated indices imply summation, and $\partial_j \equiv \partial/\partial x_j$.

To isolate the effects of turbulent fluctuations from the mean flow, a Reynolds decomposition can be applied to both governing equations:

$$\underbrace{\partial_t \overline{u_i}}_{\text{time derivative}} + \underbrace{\overline{u_j}\frac{d\overline{u_i}}{dx_j}}_{\text{kinetic advection}} - \underbrace{\overline{b_j}\frac{d\overline{b_i}}{dx_j}}_{\text{magnetic advection}} =$$
$$-\underbrace{\frac{d\overline{p}}{dx_i}}_{\text{pressure gradient}} + \underbrace{\nu\frac{d^2\overline{u_i}}{dx_j dx_j}}_{\text{diffusion}} + \underbrace{\frac{d\,\overline{b_i' b_j'}}{dx_j}}_{\text{magnetic stresses}} - \underbrace{\frac{d\,\overline{u_i' u_j'}}{dx_j}}_{\text{kinetic stresses}} - \underbrace{\frac{1}{2}\frac{d(\overline{b_i b_i})}{dx_i}}_{\text{magnetic pressure}}, \quad (5)$$

$$\underbrace{\partial_t \overline{b_i}}_{\text{time derivative}} + \underbrace{\overline{u_j}\partial_j\overline{b_i} - \overline{b_j}\partial_j\overline{u_i}}_{\text{magnetic induction}} = \underbrace{\eta\partial_{jj}\overline{b_i}}_{\text{magnetic diffusion}} + \underbrace{\partial_j\,\overline{u_i' b_j'} - \partial_j\,\overline{b_i' u_j'}}_{\text{turbulent electromotive stresses}}, \quad (6)$$

where overbars denote averaged quantities and primes denote fluctuations. The effects of turbulent fluctuations are now isolated in the magnetic stresses, Reynolds stresses and turbulent electromotive force terms. To investigate the evolution of turbulent energy, we also present the transport equations for turbulent kinetic energy 'k' and turbulent magnetic energy 'm':

$$\underbrace{\partial_t k}_{\text{time derivative}} + \underbrace{\overline{u_j}\partial_j k}_{\text{mean-field advection}} - \underbrace{\overline{b_j}(\overline{u_i'\partial_j b_i'})}_{\text{mean-field transfer}} = -\underbrace{\overline{u_i'\partial_i p'}}_{\text{kinetic pressure transport}} + \underbrace{\nu(\overline{u_i'\partial_{jj}u_i'})}_{\text{kinetic dissipation}}$$
$$\underbrace{-\overline{u_i'u_j'}\partial_j\overline{u_i} + \overline{u_i'b_j'}\partial_j\overline{b_i}}_{\text{kinetic production}} - \underbrace{\overline{u_i'u_j'\partial_j u_i'}}_{\text{turbulent transport}} + \underbrace{\overline{u_i'b_j'\partial_j b_i'}}_{\text{turbulent transfer}} - \underbrace{\frac{1}{2}\overline{u_i'\partial_i b_j' b_j'}}_{\text{magnetic pressure transport}}, \quad (7)$$

$$\underbrace{\partial_t m}_{\text{time derivative}} + \underbrace{\overline{u_j}(\overline{b_i'\partial_j b_i'}) - \overline{b_j}(\overline{b_i'\partial_j u_i'})}_{\text{mean-field transfer}} - \underbrace{\overline{b_i'b_j'}\partial_j\overline{u_i} + \overline{b_i'u_j'}\partial_j\overline{b_i}}_{\text{magnetic production}}$$
$$-\underbrace{\overline{b_i'b_j'\partial_j u_i'} + \overline{u_j'b_i'\partial_j b_i'}}_{\text{turbulent transfer}} = \underbrace{\eta(\overline{b_i'\partial_{jj}b_i'})}_{\text{magnetic dissipation}}. \quad (8)$$

To explore the transition to MHD turbulence we initialise a jet flow with a single unstable wavemode in the streamwise velocity field and a low-level overlay of random noise concentrated in the core of the jet. These conditions follow Mak *et al.* (2017).

$$\boldsymbol{u}(\boldsymbol{x}, t=0) = \begin{pmatrix} \mathrm{sech}^2(x_2) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 10^{-3}\sin(\sigma x_1) + \alpha \\ 10^{-3}\sin(\sigma x_1) + \alpha \\ \alpha \end{pmatrix} \mathrm{e}^{-x_2^2}, \quad (9)$$
$$\boldsymbol{b}(\boldsymbol{x}, t=0) = \begin{pmatrix} b_0 \\ 0 \\ 0 \end{pmatrix}, \quad (10)$$

Where $\sigma = 0.9$ is an unstable wavemode and $\alpha(\boldsymbol{x}) \sim \mathcal{U}(-10^{-5}, 10^{-5})$ is a uniformly distributed random field representing small-scale background fluctuations.

The governing equations are solved on a periodic $512 \times 1024 \times 512$ Cartesian grid of volume $\Omega = L_{x_1} \times L_{x_2} \times L_{x_3}$, where $L_{x_1} = L_{x_3} = 4\pi/\sigma$ and $L_{x_2} = 20$, with $x_2 \in [-10, 10]$. The solver is an in-house DNS code developed for this project, utilising a 6th order central finite difference scheme coupled with a spectral Poisson solver. Time is advanced using a 4th order explicit Runge-Kutta scheme. The divergence free condition of the magnetic field is enforced using the projection method of Brackbill & Barnes (1980). The solver is a CPU-based C++ code with efficient parallelization of the spectral Poisson solver achieved using HeFFTe and FFTW. Scalable data I/O is achieved using parallel HDF5.

Lengths, velocities, and time are nondimensionalised with respect to $L_0 = L_{x_1}/(4\pi/\sigma)$, $u_0 = \max[\mathrm{sech}^2(x_2)]$, and $t_0 = L_0/u_0$. The Reynolds and magnetic Prandtl numbers are $\mathrm{Re} = u_0 L_0/\nu = 3500$ and $\mathrm{Pm} = \nu/\eta = 1$, respectively, and the initial pre-dynamo Lunquist number is $S = L_0 b_0/\eta = 52.5$.

RESULTS & DISCUSSION

We initiate the simulation and allow the flow to evolve until the hydrodynamic instability emerges from the jet, driving the production of large vortex structures. This vorticity stretches the initial magnetic field, facilitating energy transfer from the kinetic to magnetic domain via the mean-field electromotive force.

When the vortex edges begin to merge magnetic reconnection events begins in regions of magnetic shear, gradually intensifying as increasing quantities of magnetic flux are annihilated and energy is transferred from the magnetic field back to the kinetic domain.

The evolution of the jet is presented in Figure 1, showing the visual evolution of the reconnecting & turbulent structures, along with the associated spectral behaviour of the flow at each time instant. The energy spectra for the fully turbulent flow is expectedly different than traditional Kolmogorov scaling due to the eddy alignment with the magnetic field and the 'recycling' of energy up the cascade from reconnecting magnetic eddies, however the development of the spectra during the transition gives significant insight into how the energy cascades during this reconnection-driven turbulence transition. As far as we are aware, this is the first time the evolution of MHD energy spectra during reconnection-driven turbulence transition has been recorded in an evolving flow (see Karimabadi *et al.* (2013) & Kowal *et al.* (2017) for similar, statistically stationary studies).

In addition to the energy spectra, the spectra of the forces that facilitate the inter-domain energy transfer are also presented in Figure 1. The instantaneous Lorentz force; $\nabla \times b_i \times b_i$, turbulent Lorentz force; $\nabla \times b' \times b'$, instantaneous electromotive force; $u_i \times b_i$, and turbulent electromotive force; $u' \times b'$ are all presented for each time instant. The evolution of the Lorentz force, which facilitates magnetic to kinetic energy transfer, is insightful for understanding how the energy transfer changes during the transition from mean-field dominated to turbulence dominated, and for understanding the physical scales on which the force mainly acts. In the early points of transition we see large 'bumps' the $k = 10 - 30$ wavenumber band of the spectra, likely the wavenumber range associated with the growing magnetic reconnection, and very little energy transfer at higher wavenumbers. Later in the transition, once turbulent structures have emerged and the cascading of energy has begun, the spectra of all the forces takes a much smoother profile and extends across the entire band of wavenumbers. The huge jump in the turbulent Lorentz force from $t = 82$ to $t = 88$ by nearly two orders of magnitude is indicative that significant amounts of magnetic turbulence has been produced in this period, with the turbulent electromotive force increasing by less than one order of magnitude for the same period.

To further understand how the energy is redistributed during the transition, we present the full turbulent energy budget for both domains in Figure 2. There are many mechanisms that facilitate energy transfer during the transition; turbulence production via the mean-field, turbulent dissipation via viscous and resistive forces, conservative turbulent transport, and conservative inter-domain transfer. All the budget terms displayed have some impact on the turbulence transition, but the electromotive interaction with the magnetic mean-shear, $\overline{b'_i u'_j}\partial_j \overline{b_i}$, clearly dominates the production of turbulent energy, meaning most turbulent energy is initially created as magnetic fluctuations before cascading into the kinetic domain.

SUMMARY

We have analysed the characteristics of the reconnection-driven turbulence transition in a magnetohydrodynamic jet flow, with focus on the evolution of the spectra and the turbulent energy budgets. We find a dominant mechanism that produces magnetic fluctuations during the transition, giving a steeper spectra of magnetic fluctuations that gradually softens into the expected MHD curve as energy is cascaded between domains.

We also find that the turbulent Lorentz force is significantly more impactful that the turbulent electromotive force in the inter-domain cascade of turbulent energy.

In future work we intend to investigate the universality of our results across a broad parameter range of Lundquist and magnetic Reynolds numbers.

ACKNOWLEDGEMENTS

The authors would like to thank the EPSRC for the computational time made available on the UK supercomputing facility ARCHER/ARCHER2 via the UK Turbulence Consortium (EP/X035484/1).

The authors would also like to thank the EPSRC for the funding made available via the Collaborative Computational Project in Nuclear Thermal Hydraulics (CCP-NTH) which has facilitated this submission and the associated costs (EP/T026685/1).

REFERENCES

Brackbill, J. U. & Barnes, D.C. 1980 The Effect of Nonzero V . B on the Numerical Solution of the Magnetohydrodynamic Equations* .

Dong, Chuanfei, Wang, Liang, Huang, Yi-Min, Comisso, Luca, Sandstrom, Timothy A. & Bhattacharjee, Amitava 2022 Reconnection-driven energy cascade in magnetohydrodynamic turbulence. *Science Advances* **8** (49), eabn7627.

Goldreich, P. & Sridhar, S. 1995 Toward a theory of interstellar turbulence. 2: Strong alfvenic turbulence. *The Astrophysical Journal* **438**, 763.

Karimabadi, H., Roytershteyn, V., Wan, M., Matthaeus, W. H., Daughton, W., Wu, P., Shay, M., Loring, B., Borovsky, J., Leonardis, E., Chapman, S. C. & Nakamura, T. K. M. 2013 Coherent structures, intermittent turbulence, and dissipation in high-temperature plasmas. *Physics of Plasmas* **20** (1), 012303.

Kolmogorov, A. 1941 The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds' Numbers. *Akademiia Nauk SSSR Doklady* **30**, 301–305, aDS Bibcode: 1941DoSSR..30..301K.

Kowal, Grzegorz, Falceta-Gonçalves, Diego A., Lazarian, Alex & Vishniac, Ethan T. 2017 Statistics of Reconnection-driven Turbulence. *The Astrophysical Journal* **838** (2), 91.

Kraichnan, Robert H. 1965 Inertial-Range Spectrum of Hydromagnetic Turbulence. *The Physics of Fluids* **8** (7), 1385–1387, publisher: AIP Publishing.

Mak, J., Griffiths, S. D. & Hughes, D. W. 2017 Vortex disruption by magnetohydrodynamic feedback. *Physical Review Fluids* **2** (11), 113701.

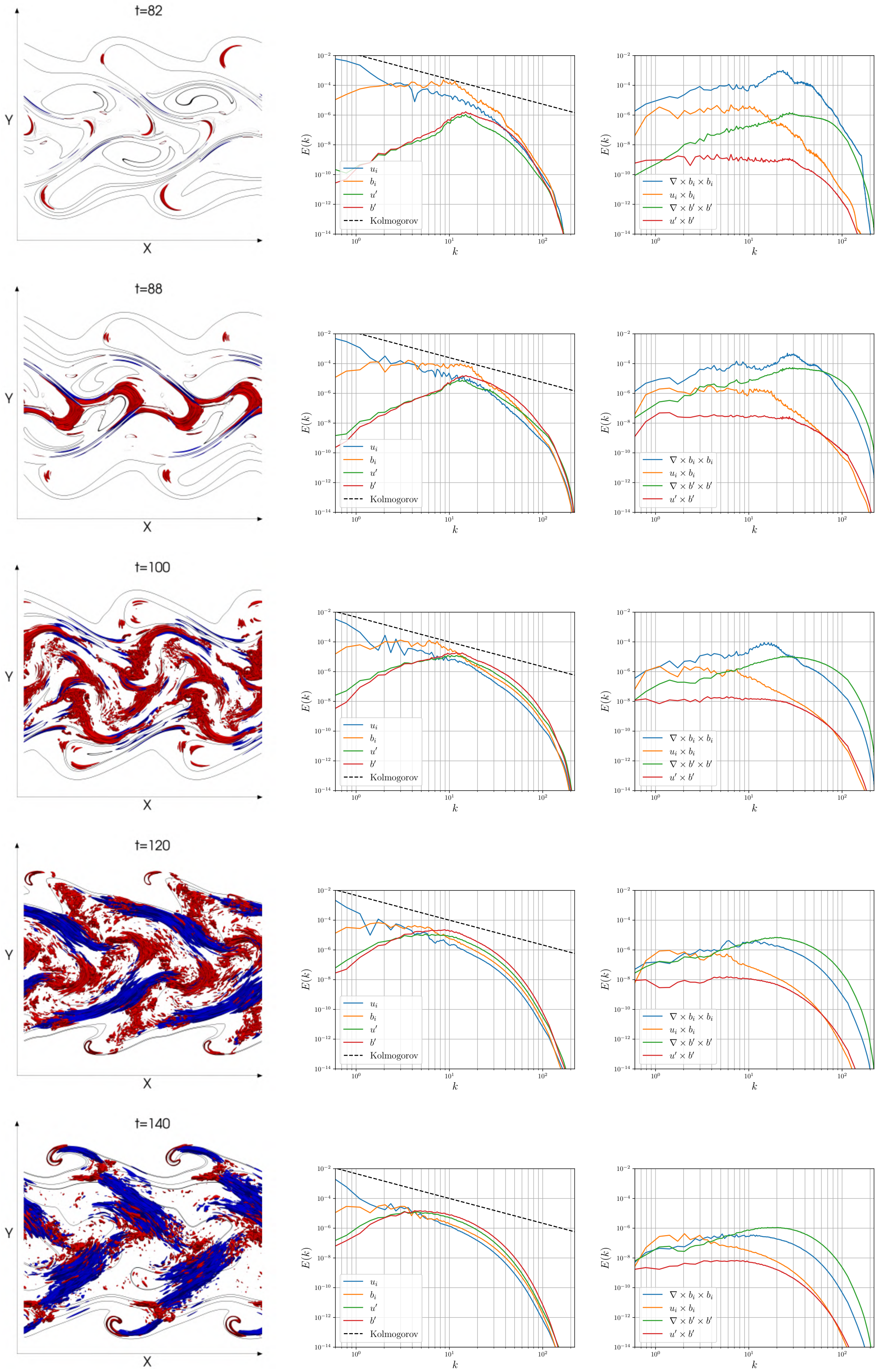

Figure 1: Evolution of the jet throughout the turbulence transition. Visualisation (left column) shows mean-magnetic field lines (black), and the 99th percentile of the current density (blue) and second invariant of the velocity gradient (red). Energy spectra of the instantaneous fields and turbulent fluctuations are shown in the middle column. Spectra of both the instantaneous and turbulent Lorentz force and electromotive force is shown in the right column. Each row corresponds to the same time interval.

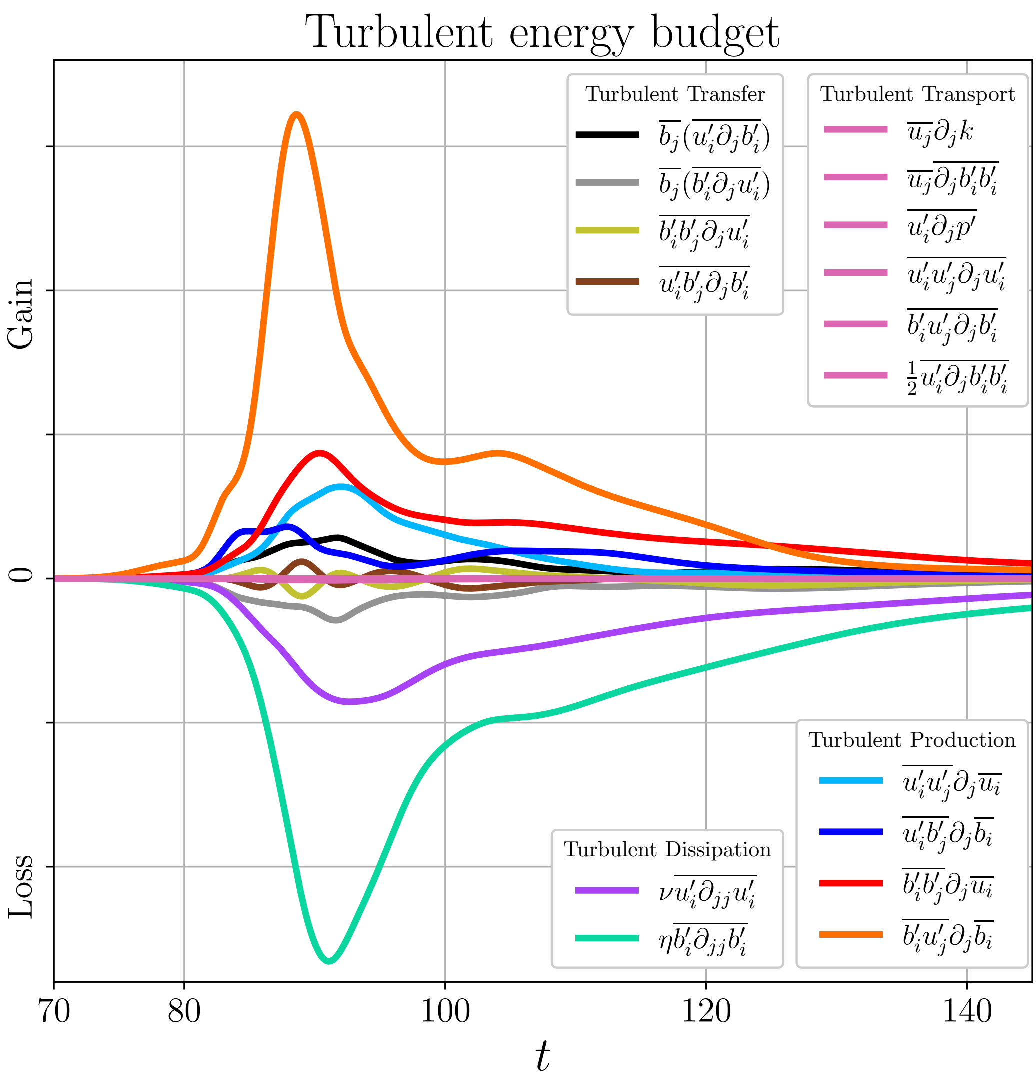

Figure 2: Collated turbulent energy budget throughout the turbulence transition. Turbulent energy is produced mainly as magnetic fluctuations through the $\overline{b_i'u_j'}\partial_j\overline{b_i}$ mode before cascading between domains via the turbulent transfer terms. Turbulent transport terms, whilst included, are comparatively negligible.