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EXPRESSING TURBULENT ENERGY AS GRADIENT INVARIANTS IN (MAGNETO)HYDRODYNAMICS AND BEYOND.

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ABSTRACT

In turbulent flows, the fluid element is deformed by the sharp velocity gradients generated by chaotic motion. In hydrodynamic (HD) turbulence, the connection between such deformations and the total energy was recently established through an exact identity linking the scale-integrated norm of the coarse-grained velocity gradient to the mean kinetic energy (D. Capocci, *Phys. Rev. Fluids*, 10(2), L022601 (2025)). Here, we first extend this framework to magnetohydrodynamics (MHD), thereby obtaining an exact decomposition of the total energy budget in terms of the invariants of the velocity and magnetic field gradient tensors, each admitting a precise physical interpretation. By using data from direct numerical simulations of homogeneous and isotropic MHD turbulence, we quantify this decomposition and compare it with the HD counterpart. We find that vortical motion contributes about 35% of the total energy, contractile fluid motion about 19%; moreover, two approximate equipartitions are observed: one between extensional motion and current-carrying structures, each accounting for about 15%, and another between magnetic stretching and compression, each contributing approximately 7%. Secondly, by embedding the gradient-based expansion of kinetic energy into the Navier–Stokes equation, we derive an exact scale-by-scale balance containing a contribution related to local flow topology, which simplifies further in two-dimensional turbulence.

BRIDGING KINETIC ENERGY AND GRADIENT NORM

Among the many physical systems in which turbulence arises, two features remain central: the nonlinear character of the governing equations and the multiscale nature of the resulting dynamics. A turbulent flow is characterised by the generation of small-scale vortical motion from energy injected at large scales, a process commonly referred to as the energy cascade. This interscale transfer produces velocity gradients at all scales, which in turn determine the deformations that the element of fluid undergoes. This process is at the core of turbulent transport and dissipation, and it is relevant across a wide range of applications, from geophysical and environmental flows (Buaria *et al.*, 2019; Thorpe, 2005; Pedlosky, 2013) to astrophysics (Cho *et al.*, 2003; Goldstein *et al.*, 1995; Goldreich & Sridhar, 1997). Fig. 1 shows an example of such flows. Yet, despite its fundamental importance, the precise relation between kinetic energy and the local geometry of the flow remains only partially understood. To address this knowledge gap, in Capocci (2025) an exact expression connecting the total kinetic energy and the velocity gradient norm across the scales was derived. This identity expresses the total energy budget in terms of the intensity of vortical motion and strain motion deformations. More generally, the flexibility due to the kinematic root of this formalism allows for its application to other types of (turbulent) fields. As discussed in the Supplementary Material of (Capocci, 2025), this expression can be used to characterise not only the kinematics, but also the dynamics, of the fields.

Figure 1. Visualisation of the magnitude of the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ from numerical simulations of magnetohydrodynamic turbulence (Capocci *et al.*, 2025*a*).

The aim of this work is twofold: (i) we first apply the aforementioned formalism to magnetohydrodynamic (MHD) turbulence. This approach can quantify the role of contractile and extensional strain deformations alongside the vortical motion in the energy balance while also including the contributions of the magnetic field stretching and compression, and current-carrying structures. (ii) we then extend the purely kinematic nature of the abovementioned energy decomposition by embedding it into the Navier–Stokes equation. This yields an exact dynamical identity for the kinetic-energy evolution, involving a term related to flow topology. In two-dimensional turbulence, the identity simplifies further and reduces to a particularly simple exact balance.

In the previous investigation (Capocci, 2025), an exact identity linking the total kinetic energy, $\langle E_K \rangle = \langle u_i u_i \rangle/2$, to the coarse-grained velocity gradient across the scales was derived, where $\langle \cdot \rangle$ denotes the volume average and Einstein summation is assumed. The identity reads

$$\langle E_K \rangle = \frac{1}{2}\int_0^\infty \mathrm{d}\theta \, \langle \partial_j \overline{u}_i^{\sqrt{\theta}} \, \partial_j \overline{u}_i^{\sqrt{\theta}} \rangle \tag{1}$$

where $\overline{u}_i^\ell = G^\ell * u_i$ denotes the filtered, or equivalently coarse-grained, velocity field obtained by convolving u_i with a low-pass kernel G^ℓ related to the filter scale ℓ. The filter kernel is required to satisfy only mild conditions, such as evenness and unit integral (Pope, 2000). Eq. (1) is purely kinematic: it requires only differentiability of the velocity field together with either periodic boundary conditions or an infinite domain.[1] The extension to wall-bounded flows would require additional analysis regarding the wall-coarse graining interaction (Sagaut, 2005). In addition, we use the terminology of kinetic energy only to identify the relevant quadratic form and the velocity field plays only the role of a test field, and no dynamical assumption enters the derivation. As discussed in the Supplementary Material of (Capocci, 2025), this identity does not depend on the specific choice of filter kernel, although for brevity the original derivation was presented using the Gaussian filter. However, the same filter-independence does not generally extend to the mean resolved-scale kinetic energy, $\langle E_K^\ell \rangle = \langle \overline{u}_i^\ell \overline{u}_i^\ell \rangle/2$, namely the kinetic energy associated with the coarse-grained velocity field. For this reason, in what follows we specialise the analysis to the Gaussian filter kernel.

The above identity can be extended to the resolved-scale kinetic energy, $\langle E_K^\ell \rangle$, namely the kinetic energy associated with the low-pass filtered velocity field. Moreover, the velocity-gradient tensor can be decomposed into its symmetric and antisymmetric parts, namely the strain-rate tensor S_{ij} and the vorticity vector ω_i, through the relation $\partial_j u_i = S_{ij} - \varepsilon_{ijk}\omega_k/2$, where ε_{ijk} is the Levi-Civita tensor. This leads to

$$\langle E_K^\ell \rangle = \frac{1}{2}\int_{\ell^2}^\infty \mathrm{d}\theta \, \langle \partial_j \overline{u}_i^{\sqrt{\theta}} \, \partial_j \overline{u}_i^{\sqrt{\theta}} \rangle \tag{2}$$

$$= \frac{1}{2}\int_{\ell^2}^\infty \mathrm{d}\theta \left(\langle \overline{S}_{ij}^{\sqrt{\theta}} \, \overline{S}_{ij}^{\sqrt{\theta}} \rangle + \frac{1}{2}\langle ||\overline{\boldsymbol{\omega}}^{\sqrt{\theta}}||^2 \rangle \right) \tag{3}$$

where the vorticity is written in vector form for compactness. Note that eq. (1) is obtained by setting $\ell = 0$ in the above equation since $E_K^{\ell=0} = E_K$.

To obtain a more direct physical interpretation of Eq. (3), the strain-rate contribution, i.e. the first term on the right-hand side, can be evaluated in the so-called *principal-axis frame*. In this frame, the strain-rate tensor becomes diagonal. Moreover, since the strain-rate tensor is a symmetric tensor, it admits real eigenvalues λ_i with $i \in \{1,...,d\}$ where d is the dimension of the velocity vector. In this form, the resolved kinetic energy is decomposed into contributions associated with the intensity of strain deformations and vortical motion.

Notably, eq. (1) is a particular case of a more general identity, which can be proved by extending the same procedure of (Capocci, 2025). For two arbitrary vector fields f_i and g_i, the identity reads

$$\langle f_i \, g_i \rangle = \int_0^\infty \mathrm{d}\theta \, \langle \partial_j \overline{f}_i^{\sqrt{\theta}} \, \partial_j \overline{g}_i^{\sqrt{\theta}} \rangle \tag{4}$$

Thus, the scalar product of two differentiable fields can be expressed in terms of the contraction between their coarse-grained gradients across the scales. Moreover, by applying the strain–vorticity decomposition introduced above, scalar products of this type can be recast in terms of the mutual alignments of the irrotational field lines and the curls of the two fields. A more detailed investigation of Eq. (4), in the context of real analysis, will be presented in future work (Capocci, 2026).

ENERGY BUDGET DECOMPOSITION IN HD AND MHD

In this section we apply the methodology of the previous section to three-dimensional incompressible MHD, where the total energy budget is expanded in terms of coarse-grained gradient tensors of the velocity and magnetic field. The governing equations are

$$\partial_t u_i = -\partial_j \left(u_i u_j\right) + \partial_j \left(b_i b_j\right) - \partial_i \left(p + \frac{b^2}{2}\right) + \nu \Delta u_i + f_i \tag{5}$$

$$\partial_t b_i = -\partial_j \left(b_i u_j\right) + \partial_j \left(u_i b_j\right) + \eta \Delta b_i \tag{6}$$

$$\partial_i u_i = 0 \tag{7}$$

$$\partial_i b_i = 0 \tag{8}$$

where u_i is the velocity, b_i is the magnetic field, p is the kinetic pressure divided by the uniform mass density ρ, Δ is the Laplacian operator, f_i is a (large-scale) mechanical forcing, ν and η are respectively the kinematic viscosity and magnetic diffusivity. In the present work we consider the case $\eta = \nu$. The magnetic field is expressed in Alfvén units $b_i/\sqrt{4\pi\rho} \to b_i$. Since the total energy is given by the sum of the kinetic and magnetic contributions, namely $E = \left(u_i u_i + b_i b_i\right)/2$, we apply the gradient-based decomposition to both contributions separately. The gradient-based expression for the magnetic energy reads simply

$$\langle E_M^\ell \rangle = \frac{1}{2}\int_{\ell^2}^\infty \mathrm{d}\theta \, \langle \partial_j \overline{b}_i^{\sqrt{\theta}} \, \partial_j \overline{b}_i^{\sqrt{\theta}} \rangle. \tag{9}$$

Before proceeding to further analysis, we first verify that, applied to data from DNS, the gradient-based formulation of eqs. (2) and (9) reproduces *exactly* the standard scalar-product definition of the kinetic and magnetic energies specifically $\langle E_K^\ell \rangle = \langle \overline{u}_i^\ell \overline{u}_i^\ell \rangle/2$ and $\langle E_M^\ell \rangle = \langle \overline{b}_i^\ell \overline{b}_i^\ell \rangle/2$. For this reason we consider data from direct numerical simulations (DNS). Our database, specifically run A1 of Capocci *et al.* (2025*a*), describes stationary homogeneous and isotropic MHD turbulence where eqs. (5)–(8) are evolved in a triply-periodic box with a stochastic forcing active in the wavenumber band $k \in [2.5, 5.0]$ with $k = \pi/\ell$; the midpoint of this interval defines the forcing scale L_f. Fig. 2 shows that, in both panels, the profiles obtained from the gradient-based expressions are indistinguishable from those computed using the standard definitions.

Having established the validity of the gradient-based expressions, we now decompose the magnetic-field gradient tensor in analogy with the velocity-gradient tensor, by separating its symmetric and antisymmetric parts, viz $\partial_j b_i = \Sigma_{ij} -$

[1] Specifically, the identity in eq. (1) relies on the very weak condition $\lim_{\ell\to\infty} \langle \overline{u}_i^\ell \overline{u}_i^\ell \rangle = 0$.

$\varepsilon_{ijk} j_k/2$, where Σ_{ij} is the magnetic strain-rate tensor and $\boldsymbol{j}$ is the electric current density vector. As anticipated in the previous section, we evaluate the contractions involving the symmetric tensors in their respective principal-axis frames in which they become diagonal. Although the symmetric parts of the velocity and magnetic gradients do not, in general, share the same principal frame, the contractions appearing here are scalar invariants and may therefore be combined in the same budget equation. Their associated eigenvalues must, however, be interpreted separately, since they generally correspond to different eigenvectors. By virtue of the solenoidality of the fields, see eqs. (7)–(8), the eigenvalues of both the strain-rate and magnetic strain-rate tensors individually sum to zero at every filtering scale. For the strain-rate tensor, this implies that the smallest eigenvalue $\overline{\lambda}_1^{\ell} \leq 0$ is associated with contraction along its eigenvector, whereas $\overline{\lambda}_3^{\ell} \geq 0$ corresponds to an extensional direction. The intermediate eigenvalue $\overline{\lambda}_2^{\ell}$ may be either positive or negative and is therefore referred to as *indefinite-type*. An analogous classification applies to the magnetic strain-rate tensor. However, whereas the velocity strain-rate tensor directly describes local fluid deformation, the interpretation of the magnetic strain-rate tensor eigenvalues $\overline{\mu}_i^{\ell}$ is similar. In this case, extensional and contractile directions are associated with the local stretching and compression of the magnetic field.

Having clarified this, we return to eq. (3) and express the strain-rate contraction in terms of the eigenvalues

$$\begin{aligned}\langle E_K^{\ell}\rangle &= \frac{1}{2}\int_{\ell^2}^{\infty}\mathrm{d}\theta\,\langle(\overline{\lambda}_1^{\sqrt{\theta}})^2+(\overline{\lambda}_2^{\sqrt{\theta}})^2+(\overline{\lambda}_3^{\sqrt{\theta}})^2\rangle+\frac{1}{2}\langle||\overline{\boldsymbol{\varpi}}^{\sqrt{\theta}}||^2\rangle \\ &= \frac{1}{2}\left(P_1^{\ell}+P_2^{\ell}+P_3^{\ell}\right)+\frac{R^{\ell}}{2} \qquad (10)\end{aligned}$$

where each P_i^{ℓ} represents the contribution of the $i-$th strain-rate eigenvalue to the resolved-scale kinetic energy, while R^{ℓ} represents the contribution of vortical motion. Although in eq. (3) the strain-rate contribution and the vorticity-squared term are related through the Betchov relation (Betchov, 1956), i.e. $P_1^{\ell}+P_2^{\ell}+P_3^{\ell}=R^{\ell}$, so that one of them could be expressed in terms of the other with a different prefactor, we keep the four contributions separate in order to emphasise the distinct physical roles of strain and vortical motion. By contrast, in Capocci (2025) the energy budget was expressed solely in terms of the strain-rate tensor eigenvalues.

The magnetic-energy counterpart is obtained in a straightforward way

$$\begin{aligned}\langle E_M^{\ell}\rangle &= \frac{1}{2}\int_{\ell^2}^{\infty}\mathrm{d}\theta\,\langle(\overline{\mu}_1^{\sqrt{\theta}})^2+(\overline{\mu}_2^{\sqrt{\theta}})^2+(\overline{\mu}_3^{\sqrt{\theta}})^2\rangle+\frac{1}{2}\langle||\overline{\boldsymbol{j}}^{\sqrt{\theta}}||^2\rangle \\ &= \frac{1}{2}\left(M_1^{\ell}+M_2^{\ell}+M_3^{\ell}\right)+\frac{J^{\ell}}{2} \qquad (11)\end{aligned}$$

where each M_i^{ℓ} represents the contribution of the $i-$th magnetic strain-rate tensor eigenvalue to the resolved-scale magnetic energy, whereas J^{ℓ} denotes the contribution associated with the electric current density. Similarly to the kinetic terms, these contributions obey a Betchov constraint i.e. $M_1^{\ell}+M_2^{\ell}+M_3^{\ell}=J^{\ell}$. The decomposition in eqs. (10)–(11) can therefore be interpreted as an exact *gradient invariants decomposition* of the total energy budget, whose contributions are associated with the multi-scale gradient organisation of the velocity and magnetic fields, namely extensional and contractile motion, vortical motion, magnetic stretching and compression, and *current-carrying* structures across the scales. In this regard, the coarse-graining of the magnetic strain-rate tensor eigenvalues was already considered in Consolini *et al.* (2015); Quattrociocchi *et al.* (2019) in the context of in situ measurements of astrophysical plasma fields.

Table 1. Percentages of the field structure contributions to the total energy in both HD and MHD. These percentages correspond to the small-scale saturation values of fig. 3 as indicated by the symbols. The percentages are rounded.

Field structure	HD	MHD	Symbol
Fluid compression	27%	19%	$P_1^{\ell}/2$
Fluid extension	20%	15%	$P_3^{\ell}/2$
Vortical motion	50%	35%	$R^{\ell}/2$
Indefinite-type (kinetic)	2%	1%	$P_2^{\ell}/2$
Mag. field contraction	-	7%	$M_1^{\ell}/2$
Mag. field stretching	-	7%	$M_3^{\ell}/2$
Current-carrying structures	-	15%	$J^{\ell}/2$
Indefinite-type (magnetic)	-	$<1\%$	$M_2^{\ell}/2$

At this stage, we quantify eqs. (10)–(11) using the aforementioned DNS data and compare the resulting decomposition with the purely HD case. For the latter, we use the same dataset of forced homogeneous and isotropic turbulence, i.e. run HL2 from Biferale *et al.* (2024), employed in the earlier investigation (Capocci, 2025). Fig. 3 shows the quantification of each contribution from eqs. (10) and (11), normalised by the corresponding resolved-scale total energy, and displayed as functions of L_f/ℓ, namely the forcing scale divided by the filtering scale. Each contribution is normalised by the corresponding resolved-scale total energy, so as to quantify its relative weight in the energy budget at the same filtering scale. Of particular relevance are the small-scale saturation values of the various contributions, as they represent the contribution of each field structure to the full, i.e. unfiltered, mean energy. In this regard, to summarise the analysis that follows, table 1 reports the percentages relative to the saturation values.

Starting from the hydrodynamic (HD) case, we simply recover the same values for P_i^{ℓ} reported in Capocci (2025). Specifically, the purely contractile contribution accounts for approximately 27% of the corresponding resolved-scale energy, whereas the purely extensional one contributes about 20%. By contrast, the indefinite-type deformation, associated with either compression or extension, contributes less than 2% to the energy budget. The exact 50% contribution of vortical motion follows from incompressibility and homogeneity, as formalised by the aforementioned Betchov relation. Overall, the contributions are only weakly scale dependent, with this being particularly evident for P_2^{ℓ}.

Turning to the MHD case, one generally observes a stronger scale dependence of the individual contributions than in HD. In addition, the vortical contribution is weaker than in hydrodynamics, reaching only about 35% of the energy budget and thus indicating a reduced vorticity intensity in MHD. Moreover, its peak at $L_f/\ell \approx 1$ reflects the direct influence of energy injection at the forcing scale. Similarly to the hydrodynamic case, contractile motion remains dominant over extensional motion, carrying roughly 19% of the total energy budget. It is also noteworthy that, at small scales, $J^\ell \approx P_3^\ell$ indicating that the energy associated with the electric current density nearly coincides with that carried by extensional motion, both contributing approximately 15%. Whether this reflects an exact underlying identity will be investigated elsewhere. Another remarkable feature is that $M_1^\ell \approx M_3^\ell$ at all scales, showing an approximate equipartition between the energy carried by compression and stretching of the magnetic field. A signature consistent with this behaviour was reported by Dallas & Alexakis (2013), where an approximate equipartition between the sign-definite magnetic strain-rate eigenvalues at small scales, specifically μ_1 and μ_3, was observed in the context of MHD turbulence decay. Finally, like the HD case, the indefinite-type contributions P_2^ℓ and M_2^ℓ, associated with the kinetic and magnetic energies respectively, are strongly depleted, especially the latter, and together account for less than 2% of the total energy budget. We observe that the indefinite-type deformation contributions, hence the eigenvalues $\overline{\lambda}_2^\ell$ and $\overline{\mu}_2^\ell$, do not play a significant role in the energy budget, however, their sign determines the overall topology of the fields in (M)HD turbulence (Dallas & Alexakis, 2013; Gao *et al.*, 2026).

APPLICATION TO NAVIER-STOKES DYNAMICS

Up to this point, the discussion has remained purely kinematic. We now incorporate the gradient-based framework into the Navier–Stokes dynamics, thereby extending the identity of eq. (1) beyond a (kinematic) decomposition of kinetic energy. In particular, this step develops further the discussion outlined in the Supplemental Material of Capocci (2025). The incompressible Navier–Stokes equation reads

$$\partial_t u_i = -\partial_j \left(u_i u_j\right) - \partial_i p + \nu \Delta u_i \tag{12}$$
$$\partial_i u_i = 0 \tag{13}$$

where we retain the same notation introduced above for the MHD equations, with the obvious simplification to the purely hydrodynamic case. To focus on the methodology, we do not consider any forcing acting on the system. Since the present approach is formulated across the filtering scales, we apply the coarse-graining operation to the full Navier–Stokes (NS) equation (Germano, 1992)

$$\partial_t \overline{u}_i^\ell = -\partial_j \left(\overline{u}_i^\ell \overline{u}_j^\ell\right) - \partial_i \overline{p}^\ell + \nu \Delta \overline{u}_i^\ell - \partial_j \tau_{ij}^\ell \tag{14}$$

where $\tau_{ij}^\ell = \overline{u_i u_j}^\ell - \overline{u_i}^\ell\, \overline{u_j}^\ell$ is the well-known subgrid-scale (SGS) stress tensor which is the extra term that we obtain when NS is filtered. This tensor plays a significant role in the study of the energy transfer across the scales (Pope, 2000). Our aim is to connect the filtered NS equation, eq. (14), with the mean kinetic-energy decomposition of eq. (1). To this end, we apply the time derivative to both sides of eq. (1). The time derivative on the right-hand side can then be expanded by means of the filtered NS equation itself. In this way, we obtain:

$$\partial_t \langle E_K(t)\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}\theta\, \partial_t \,\langle \partial_j \overline{u}_i^{\sqrt{\theta}} \partial_j \overline{u}_i^{\sqrt{\theta}}\rangle \tag{15}$$
$$= \int_0^\infty \mathrm{d}\theta \left(-\nu \langle \partial_j \partial_k \overline{u}_i^{\sqrt{\theta}}\, \partial_j \partial_k \overline{u}_i^{\sqrt{\theta}}\rangle - \frac{4}{3}\langle \overline{S}_{ij}^{\sqrt{\theta}} \overline{S}_{jk}^{\sqrt{\theta}} \overline{S}_{ki}^{\sqrt{\theta}}\rangle - \langle \Delta \overline{S}_{ij}^{\sqrt{\theta}}\, \tau_{ij}^{\sqrt{\theta}}\rangle \right) \tag{16}$$

where homogeneity or periodic boundary conditions are assumed throughout.[2] Equation (16) provides an exact dynamical counterpart of the kinematic identity in eq. (1). The first term is viscous and, by construction, non-positive. The second term is the *strain self-amplification* contribution, whose role in turbulence theory has received considerable attention in recent years (Tsinober, 2001; Carbone & Bragg, 2020; Johnson, 2020, 2021). The last term is the most obscure as it couples the SGS stress tensor to the filtered second derivative of the strain-rate tensor. Each term is then integrated over the coarse-graining scales. At first sight, eq. (16) appears rather different from the standard unforced energy balance originating from eq. (12)

$$\partial_t \langle E_K(t)\rangle = -\nu \langle \partial_j u_i\, \partial_j u_i\rangle \tag{17}$$

A first term-by-term inspection between eqs. (16) and (17) suggests that the first term on the right-hand side of eq. (16) plays the role[3] of the usual viscous decay term in eq. (17). In consequence, the remaining two terms in eq. (16) therefore balance each other exactly, leading to the identity

$$\int_0^\infty \mathrm{d}\theta \left(-\frac{4}{3}\langle \overline{S}_{ij}^{\sqrt{\theta}} \overline{S}_{jk}^{\sqrt{\theta}} \overline{S}_{ki}^{\sqrt{\theta}}\rangle - \langle \Delta \overline{S}_{ij}^{\sqrt{\theta}}\, \tau_{ij}^{\sqrt{\theta}}\rangle \right) = 0 \tag{18}$$

which is an exact identity related to the gradient production across the scales of Navier–Stokes dynamics, and can be also extended to the case with a non-zero forcing. To gain further insight into eq. (18), one may adopt a Smagorinsky closure (Smagorinsky, 1963), $\tau_{ij}^{\sqrt{\theta}} = -2\left(C_s^2\theta\right)\left(2\overline{S}_{pq}^{\sqrt{\theta}}\,\overline{S}_{pq}^{\sqrt{\theta}}\right)^{1/2}\overline{S}_{ij}^{\sqrt{\theta}}$, from which it follows that

$$\int_0^\infty \mathrm{d}\theta\, \langle \overline{S}_{ij}^{\sqrt{\theta}} \overline{S}_{jk}^{\sqrt{\theta}} \overline{S}_{ki}^{\sqrt{\theta}}\rangle < 0$$

or equivalently by using incompressibility

$$\int_0^\infty \mathrm{d}\theta\, \langle \overline{\lambda}_1^{\sqrt{\theta}} \overline{\lambda}_2^{\sqrt{\theta}} \overline{\lambda}_3^{\sqrt{\theta}}\rangle < 0 \tag{19}$$

The last inequality states that the scale-integrated product of the strain-rate eigenvalues is negative. The same sign has been reported in previous studies of fully resolved homogeneous and isotropic (magneto)hydrodynamic turbulence (Capocci *et al.*, 2025*b*; Johnson, 2020, 2021). More generally, the product of the strain-rate tensor eigenvalues plays a central role in flow-topology classification (Chong *et al.*, 1990); hence, eq. (18) may also prove useful in that context. A further simplification occurs in two-dimensional turbulence, where eq. (18) acquires a particularly simple form, since the pointwise identity $S_{ij}S_{jk}S_{ki} = 0$ is readily verified and extends

[2] In deriving eq. (16), the vortex-stretching contribution $\langle \overline{\omega}_i^{\sqrt{\theta}} \overline{S}_{ij}^{\sqrt{\theta}} \overline{\omega}_j^{\sqrt{\theta}}\rangle$ is rewritten in terms of the strain self-amplification by means of the three-field Betchov relation (Betchov, 1956).

[3] A more precise argument is as follows. By incompressibility and homogeneity, the RHS of eq.(17) is written as $\nu\,\langle\omega_i\,\omega_i\rangle$. Likewise, the term $\nu\langle \partial_j \partial_k \overline{u}_i^{\sqrt{\theta}} \partial_j \partial_k \overline{u}_i^{\sqrt{\theta}}\rangle$ can be rewritten in terms of gradients of the filtered vorticity i.e. $\nu\langle \partial_j \overline{\omega}_i^{\sqrt{\theta}} \partial_j \overline{\omega}_i^{\sqrt{\theta}}\rangle$. One may then apply eq. (4) for the sole vorticity field, thereby recovering the viscous term in eq. (17) from the first contribution in eq. (16).

straightforwardly to the filtered strain-rate tensor. The previous relation therefore reduces to

$$\int_0^\infty \mathrm{d}\theta \, \langle \Delta \overline{S}_{ij}^{\sqrt{\theta}} \tau_{ij}^{\sqrt{\theta}} \rangle = 0 \qquad (20)$$

This expression identifies an exact scale-integrated balance involving the coupling between the coarse-grained strain-rate tensor and the subgrid-scale stress. In this respect, it is reminiscent of the integral over wavenumber of the *mode transfer term* (see eq. (6.163) of Pope (2000)). A more detailed investigation will be required to clarify the precise interpretation and implications of this relation in two-dimensional turbulence.

CONCLUSIONS

The present work has extended the exact gradient-based energy identity of incompressible hydrodynamics to two complementary directions. First, we specialised the purely kinematic framework to three-dimensional incompressible MHD, obtaining an exact decomposition of the total energy budget in terms of the invariants of the velocity- and magnetic-field gradient tensors. This formulation yields a physically interpretable decomposition of the energy into extensional, contractile and vortical motion, magnetic stretching and compression, and current-carrying structures across the scales. Applied to data from DNS of homogeneous and isotropic HD and MHD turbulence, the decomposition revealed a more pronounced scale dependence of the individual terms in MHD compared with HD. Specifically, for the MHD case, we observe that vortical motion accounts for about 35% of the total energy, contractile fluid motion for about 19%, extensional fluid motion for about 15%, while the indefinite-type deformations, corresponding to either contraction or extension, contribute only negligibly. Owing to homogeneity and incompressibility, these kinetic contributions are linked by the Betchov relation, which constrains the sum of the contractile and extensional contributions to match the vortical one. The stretching and compression of the magnetic field each account for 7% of the energy budget and remain in approximate equipartition at all scales, while the current-carrying contribution provides about 15%, showing an intriguing small-scale equipartition with the fluid extensional contribution. The magnetic terms are likewise linked by an analogous Betchov-type constraint.

In addition to the MHD extension, we embedded the gradient-based expansion of kinetic energy into the Navier–Stokes equation, thereby deriving an exact identity for the temporal evolution of kinetic energy. This result reveals that the product of the strain-rate tensor eigenvalues has to satisfy a scale-by-scale constraint. In two-dimensional turbulence, this identity simplifies further, yielding an exact scale-by-scale constraint. This expansion can still be generalised by considering triple decomposition of (velocity and magnetic field) gradient tensors (Kolář, 2007; Das & Girimaji, 2020) to further explore the role of gradient dynamics in the energy budget across the scales.

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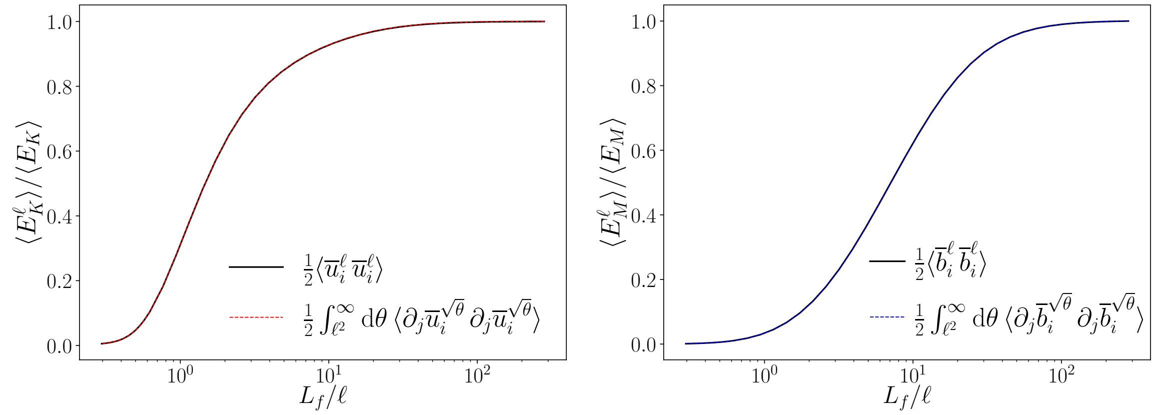

Figure 2. Resolved-scale mean kinetic energy (left) and magnetic energy (right), normalised by their respective unfiltered values, as functions of L_f/ℓ calculated via eqs. (2) and (9) and via the standard definitions $\langle E_K^\ell \rangle = \langle \overline{u}_i^\ell \overline{u}_i^\ell \rangle / 2$ and $\langle E_M^\ell \rangle = \langle \overline{b}_i^\ell \overline{b}_i^\ell \rangle / 2$ respectively. The two corresponding curves clearly overlap.

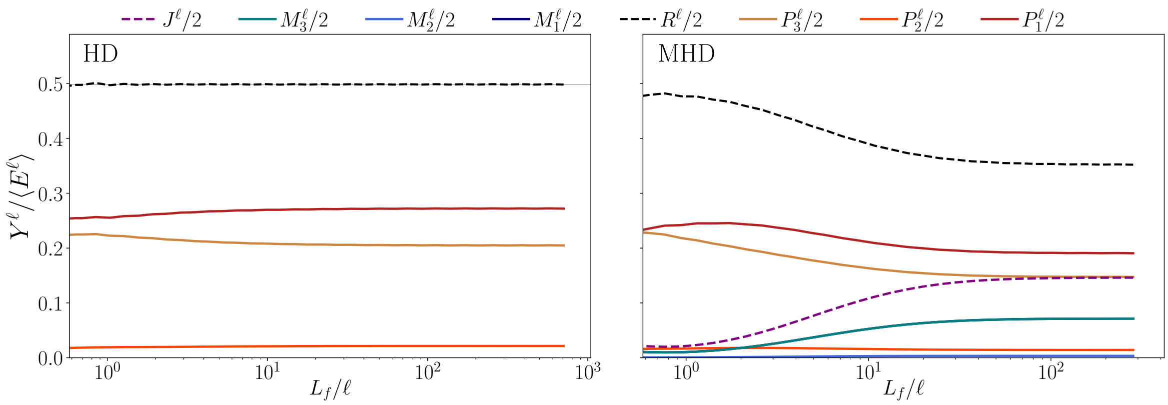

Figure 3. Quantification of the terms originating from the gradient-based expansion of the total energy. Left panel: HD; right panel: MHD. Each contribution in eqs. (10) and (11) is normalised by the corresponding mean resolved-scale energy and shown as a function of L_f/ℓ, where Y^ℓ denotes the specific contribution. For HD, only eq. (10) is relevant, since the total energy consists solely of kinetic energy.

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