FUNCTIONAL RENORMALIZATION FOR ELASTIC BURGULENCE

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1 Introduction

Elastic turbulence is a chaotic disordered flow state observed in viscoelastic fluids at small, or even vanishing, Reynolds number, where nonlinear elastic stresses contribute to the generation and upkeep of fluctuations. It was first observed experimentally in dilute polymer solutions by Groisman and Steinberg Groisman & Steinberg (2000, 2004), and has been found to be remarkably robust and ubiquitous thereafter. In continuum models such as Oldroyd-B Oldroyd (1950) or FENE-P Petrie & Denn (1976), the velocity field is coupled to an additional stress or conformation field through an objective transport equation. When inertia is also relevant, elastic and inertial nonlinearities act simultaneously. Numerical studies of homogeneous viscoelastic turbulence report modified kinetic and polymeric spectra Berti *et al.* (2007); Rosti *et al.* (2023). Nonetheless, analytical explanations and predictions for these modified scalings remain limited. Moreover, the increased number of dynamical fields significantly complicates the application of established statistical techniques and closures.

The stochastically forced Burgers equation is commonly used as a dimensionally reduced model for Navier-Stokes turbulence, retaining nonlinear advection while dropping the incompressibility constraint. For potential flows it is related to the Kardar-Parisi-Zhang equation for stochastic interface growth Kardar *et al.* (1986); Bec & Khanin (2007). Following this Burgers-type reduction, we consider an elastic Burgers equation with the dynamical variables being the velocity u, and elastic stress σ, while the material parameters are the Reynolds number Re, Weissenberg number W, solvent-viscosity ratio β, and objective-derivative parameter α. The stochastic equations are as follows

$$Re\, D_t u - \nabla\sigma - \beta\Delta u = \xi(x,t), \tag{1}$$

$$W L_u^\alpha \sigma + g(\sigma) - 2(1-\beta)\nabla u = 0, \tag{2}$$

where $\xi(x,t)$ denotes a Gaussian random velocity forcing with white in time and space correlation $\langle \xi(x,t)\xi(x',t')\rangle \sim \delta(t-t')\delta(x-x')$, $D_t = \partial_t + u\nabla$ is the material derivative, and $L_u^\alpha \sigma = D_t\sigma + 2\alpha\sigma\nabla u$ the dimensionally reduced objective derivative where $\alpha = 0$ corresponds to the one-dimensional corotational derivative, whereas $\alpha = \pm 1$ corresponds to the upper and lower convected Oldroyd derivative, respectively. The choices $g(\sigma) = \sigma$ and $g(\sigma) = \sigma^2$ give the Oldroyd-Burgers and Giesekus-Burgers constitutive terms which are of particular interest for the present study.

For stochastic differential equations with Gaussian noise, the Martin-Siggia-Rose-Janssen-de Dominicis construction represents statistical averages by a path integral over field- and response field configurations Martin *et al.* (1973); Janssen (1976); de Dominicis (1976). Schematically,

$$\mathscr{Z}[J,\bar{J}] = \int \mathscr{D}\phi\, \mathscr{D}[i\bar{\phi}]\, \exp\left(-S[\phi,\bar{\phi}] + \int_{x,t} J\phi + \bar{J}\bar{\phi}\right), \tag{3}$$

with $\phi = (u,\sigma)$ and $\bar{\phi} = (\bar{u},\bar{\sigma})$. For the elastic Burgers system above, the corresponding action is

$$S^\alpha = \int_{x,t} \bar{u}\,(Re\, D_t u - \nabla\sigma - \beta\Delta u) + \bar{u}^2 + \bar{\sigma}\,(W L_u^\alpha \sigma + g(\sigma) - 2(1-\beta)\nabla u)\,. \tag{4}$$

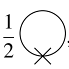
The response fields multiply the equations of motion, and the term $\bar{u}^2$ is the local velocity-forcing contribution. Path integrals of this type are rarely evaluable in closed form, nor is a regular perturbative expansion Peskin (2018) in powers of the Weissenberg number admissible. Instead, we employ the functional renormalization group to extract the scale-invariant behavior of correlation functions near a fixed point of the governing renormalization group flow. The functional renormalization group introduces an infrared regulator term

$$\Delta S_k = \frac{1}{2}\int_{x,x',t} (\boldsymbol{\phi},\bar{\boldsymbol{\phi}})' R_k(x-x')(\boldsymbol{\phi},\bar{\boldsymbol{\phi}})^T, \tag{5}$$

and thus a modified action $S_k = S + \Delta S_k$. Schematically, R_k acts as a scale-dependent mass for modes with momenta $p \lesssim k$, while leaving modes with $p \gg k$ essentially unchanged. The effective average action Γ_k therefore interpolates between the microscopic action at the ultraviolet scale, $\Gamma_{k=\Lambda} \simeq S$, and the full effective action Γ for $k \to 0$. With the full regulated propagator $G_k = (\Gamma_k^{(2)} + R_k)^{-1}$, the Wetterich flow equation reads Wetterich (1993); Dupuis *et al.* (2021)

$$\partial_k \Gamma_k = \frac{1}{2}\,\mathsf{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_k R_k\right] = \frac{1}{2}\,\mathsf{Tr}\, G_k\, \partial_k R_k. \tag{6}$$

Its diagrammatic notation is

$$\partial_k \Gamma_k = \frac{1}{2}$$  , (7)

where the line denotes G_k and the crossed insertion denotes $\partial_k R_k$. The same framework has been applied to Kardar-Parisi-Zhang type problems and to Navier-Stokes turbulence Canet *et al.* (2010, 2011*b*, 2016); Canet (2022). The following sections derive the symmetries and Ward identities of the elastic Burgers action and formulate corresponding closure strategies for the effective action.

2 Extended Symmetries

The physical properties of the model equations, and of the statistical flow equations derived from them, are encoded in their symmetries. The following discussion of extended symmetries and notation follows Olver (1986), whereas the implications on the correlation functions are discussed in depth in Peskin (2018). Recall that every Lie group of a set of Euler-Lagrange equations $\mathsf{E}(\mathscr{L})=0$ for a Lagrangian $\mathscr{L}(\boldsymbol{\phi},..,\partial^n\boldsymbol{\phi})$ yields a conserved charge with an associated Noether current. We consider an infinitesimal variational symmetry in evolutionary form

$$\mathrm{pr}\,\boldsymbol{v}_Q(\mathscr{L})=\mathsf{D}_Q(\mathscr{L})=\mathsf{Div}(\boldsymbol{B}), \tag{8}$$

where D_Q denotes the Fréchet derivative of Q. Note also that the Euler-Lagrange equations are self-adjoint. It thus immediately follows that there is a conserved Noether charge with current $\boldsymbol{B}$. To show the importance of such a transformation in the context of functional renormalization, we consider infinitesimal transformations leaving the functional measure $\mathscr{D}\boldsymbol{\phi}$ invariant and apply it to the generating functional (3)

$$\mathscr{Z}[J]=\int\mathscr{D}\boldsymbol{\phi}\,\exp\left(\int_x-\mathscr{L}+\boldsymbol{J}\boldsymbol{\phi}\right), \tag{9}$$

$$0=\delta_{\boldsymbol{v}_Q}\mathscr{Z}=\int_x\boldsymbol{J}\langle Q\rangle. \tag{10}$$

The generating functional is invariant, as it does not depend on $\boldsymbol{\phi}$ and $\boldsymbol{v}_Q$ leaves $\boldsymbol{J}$ invariant, as it is in evolutionary form. This can be rewritten in terms of the effective action by using the Legendre relation as

$$0=\int_x\frac{\delta\Gamma}{\delta\langle\phi\rangle}\langle Q\rangle. \tag{11}$$

Taking functional derivatives thereof yields an infinite set of Ward-Takahashi identities, giving exact relations between correlation functions. As a generalization of a true symmetry, a linear-contact Ward identity is defined as the action of a Lie group which acts on the Lagrangian as follows

$$\mathrm{pr}\,\boldsymbol{v}_Q(\mathscr{L}(\boldsymbol{u}))=\mathsf{Div}(\boldsymbol{B})+\gamma(x)\boldsymbol{\phi}, \tag{12}$$

that is leaving the action invariant modulo a linear contact term. Again $\boldsymbol{v}_Q$ is the generator of the group action in evolutionary form. Furthermore, the variation $\mathrm{pr}\,\boldsymbol{v}_Q(\boldsymbol{J}\boldsymbol{\phi})$ shall be linear in the dependent variables, modulo a divergence. The variation of the generating functional then reads

$$\delta_{\boldsymbol{v}_Q}(\mathscr{Z})=0=\int_x\langle\boldsymbol{\phi}\rangle\gamma+\boldsymbol{J}\langle Q\rangle \tag{13}$$

We now focus on how to find such identities. Applying the Euler operator to the invariance condition for the Lagrangian yields

$$\mathsf{E}(\mathrm{pr}\,\boldsymbol{v}_Q(\mathscr{L}))=\mathsf{E}(Q\cdot\mathsf{E}(\mathscr{L}))=\mathrm{pr}\,\boldsymbol{v}_Q(\mathsf{E}(\mathscr{L}))+\mathsf{D}^*_Q(\mathsf{E}(\mathscr{L}))=\gamma(x) \tag{14}$$

where D^*_Q is the adjoint of the Frechet derivative. The previous result follows immediately from

$$\mathsf{E}(\boldsymbol{A}\cdot\boldsymbol{B})=\mathsf{D}^*_A(\boldsymbol{B})+\mathsf{D}^*_B(\boldsymbol{A}) \tag{15}$$

and the fact that the Euler-Lagrange equations are self-adjoint. As a necessary condition for the existence of such a group action, we may restrict ourselves to the subset $\mathsf{E}(\mathscr{L})=0$, such that we obtain

$$\mathrm{pr}\,\boldsymbol{v}_Q(\mathsf{E}(\mathscr{L}))=\gamma \tag{16}$$

which is easily solved using standard methods, thus giving a complete characterisation of all point- or generalized Ward identities. We additionally restrict ourselves to characteristics Q, which are at most linear in the dependent variables. Applying the Euler operator to the source term of the generating functional yields

$$\mathsf{E}(\boldsymbol{J}\cdot\boldsymbol{Q})=\mathsf{D}^*_Q(\boldsymbol{J}) \tag{17}$$

which must be a function solely of x and $\boldsymbol{J}(x)$, if we wish the total variation of $\mathscr{Z}$ to only involve the average fields and no higher-order correlation functions. Assuming a point-transformation $Q^i=(\eta^i(x,\phi)-\xi^j(x,\phi)\phi^i_j)$ yields

$$\mathsf{D}^*_Q(\boldsymbol{J})^i=(-D)_I\left(\frac{\partial Q^k}{\partial\phi^i_I}J^k\right)=\left(\frac{\partial\eta^k}{\partial\phi^i}-\frac{\partial\xi^l}{\partial\phi^i}\phi^k_l\right)J^k+D_j(\xi^jJ^i) \tag{18}$$

where capital letters denote multi-indices and $\phi^i_J=D_J\phi^i$. The previous result can be split at the $\phi^{(1)}$-jet to yield

$$\propto\boldsymbol{\phi}^{(1)}:\ 0=J^k\frac{\partial\xi^j}{\partial\phi^i}-J^i\frac{\partial\xi^j}{\partial\phi^k} \tag{19}$$

which must hold for all $J(x)$ and thus $\xi=\xi(x)$. The terms dependent on ϕ then read

$$\boldsymbol{\phi}:\ f(x,\boldsymbol{J}^{(1)})=\left(\frac{\partial\eta^k}{\partial\phi^i}+\frac{\partial\xi^j}{\partial x^j}\delta^i_k\right)J^k+\xi^j\frac{\partial J^i}{\partial x^j} \tag{20}$$

from which it follows that η can be at most linear in the dependent variables ϕ. Instead of evaluating at $\mathsf{E}(\mathscr{L})=0$ let us explicitly keep an arbitrary source-term and evaluate at $\mathsf{E}(\mathscr{L})-\boldsymbol{J}=0$, resulting in

$$\mathrm{pr}\,\boldsymbol{v}_Q(\mathsf{E}(\mathscr{L}))|_{\mathsf{E}(\mathscr{L})-\boldsymbol{J}=0}=\gamma-\mathsf{D}^*_Q(\boldsymbol{J}) \tag{21}$$

We now introduce an extended generator, treating $\boldsymbol{J}$ as a dependent variable

$$\boldsymbol{v}_{\tilde{Q}}=Q^i\frac{\partial}{\partial u^i}+(\zeta^i-\xi^jJ^i_j)\frac{\partial}{\partial J^i} \tag{22}$$

or equivalently

$$\tilde{\boldsymbol{v}}=\boldsymbol{v}+\zeta^i\frac{\partial}{\partial J^i} \tag{23}$$

We then find the variation of the Lagrangian

$$\begin{aligned}\mathrm{pr}\,\boldsymbol{v}_{\tilde{Q}}(\mathsf{E}(\mathscr{L})-\boldsymbol{J})^i|_{\mathsf{E}(\mathscr{L})-\boldsymbol{J}=0}=\gamma^i-J^k\left(\frac{\partial\eta^k}{\partial\phi^i}+\xi^j_j\delta^i_k\right)\\-\xi^jJ^i_j-\left(\zeta^i-\xi^jJ^i_j\right).\end{aligned} \tag{24}$$

The monomials involving derivatives of the dependent variables vanish as before, and if we let

$$\zeta^i(x,\boldsymbol{J})=\gamma^i-J^k\left(\frac{\partial\eta^k}{\partial\phi^i}+\xi^j_j\delta^i_k\right) \tag{25}$$

the right-hand side vanishes. Hence the invariance condition reads

$$\mathrm{pr}\,\boldsymbol{v}_{\tilde{Q}}(\mathsf{E}(\mathscr{L})-\boldsymbol{J})|_{\mathsf{E}(\mathscr{L})=\boldsymbol{J}}=0 \tag{26}$$

$$\tilde{\boldsymbol{v}}=\xi^i\frac{\partial}{\partial x^i}+\eta^i\frac{\partial}{\partial\phi^i}+\left(\gamma^i-J^k\left(\frac{\partial\eta^k}{\partial\phi^i}+\xi^j_j\delta^i_k\right)\right)\frac{\partial}{\partial J^i} \tag{27}$$

and we may find Ward identities through a Lie-group analysis of a modified set of Euler-Lagrange equations with an explicit source term and the generator as above. The construction above is necessary, yet not sufficient. Checking invariance a posteriori remains a necessary step. We now consider viscoelastic Burgers turbulence, governed by equations (1) and (2). Given the response field action (4), we find the following Euler-Lagrange equations in the inertialess $Re=0$ limit

$$\mathsf{E}_{\overline{u}}(\mathscr{L}^\alpha)=2\overline{u}-\nabla\sigma-\beta\Delta u, \tag{28}$$

$$\mathsf{E}_{\overline{\sigma}}(\mathscr{L}^\alpha)=WL^\alpha_u\sigma+g(\sigma)-2(1-\beta)\nabla u, \tag{29}$$

$$\mathsf{E}_{\sigma}(\mathscr{L}^\alpha)=-WL^{-\alpha}_u\overline{\sigma}+W\overline{\sigma}\nabla u+\frac{dg(\sigma)}{d\sigma}\overline{\sigma}+\nabla\overline{u}, \tag{30}$$

$$\mathsf{E}_{u}(\mathscr{L}^\alpha)=-\beta\Delta\overline{u}+2(\beta-1)\nabla\overline{\sigma}+W\left(\overline{\sigma}\nabla\sigma-2\alpha\nabla(\sigma\overline{\sigma})\right). \tag{31}$$

At first, let us consider the elasto-inertial case of $Re\neq0$, where we find the following symmetries and case splits

$$\alpha\neq0:\quad\boldsymbol{X}_1=\partial_t,\quad\boldsymbol{X}_2=\partial_x,\quad\boldsymbol{X}_3=t\partial_x+\partial_u, \tag{32}$$

$$\boldsymbol{X}_4=\frac{t^2}{2}\partial_x+t\partial_u+\frac{Re}{2}\partial_{\overline{u}}, \tag{33}$$

$$\alpha=0,\ g=\sigma:\quad\boldsymbol{X}_5=e^{-\frac{t}{W}}\partial_\sigma. \tag{34}$$

whereas the following extended symmetries, that is, Ward-identity generators, emerge

$$\alpha \neq 0: \quad \boldsymbol{X}_1 = \partial_t, \quad \boldsymbol{X}_2 = f(t)\partial_x + \dot{f}(t)\partial_u, \quad \boldsymbol{X}_3 = f(t)\partial_{\overline{u}}, \tag{35}$$
$$\alpha = 0: \quad \boldsymbol{X}_5 = f(t)\partial_\sigma. \tag{36}$$

where we keep repeated generators that are associated with exact symmetries. We now investigate the purely elastic regime with $Re = 0$, where we then find the exact symmetries

$$\alpha \neq 0: \quad \boldsymbol{X}_1 = \partial_t, \quad \boldsymbol{X}_2 = f(t)\partial_x + \dot{f}(t)\partial_u, \tag{37}$$
$$\alpha = 0: \quad \boldsymbol{X}_5 = e^{-\frac{t}{W}}\partial_\sigma. \tag{38}$$

while the necessary condition for Ward-identity generators yields the following candidates

$$\boldsymbol{X}_1 = \partial_t, \quad \boldsymbol{X}_2 = f(t)\partial_x + \dot{f}(t)\partial_u, \quad \boldsymbol{X}_3 = g(x,t)\partial_{\overline{u}}, \tag{39}$$

$$\begin{aligned} \alpha \neq 0: \quad \boldsymbol{X}_5 = {} & \alpha W \dot{f} x \partial_x - \alpha W f \partial_t + \alpha W \left(\ddot{f}x + 2u\dot{f}\right)\partial_u \\ & + \dot{f}\left(\alpha W \sigma + \beta - 1\right)\partial_\sigma - 2\alpha W \dot{f}\overline{\sigma}\partial_{\overline{\sigma}}, \end{aligned} \tag{40}$$
$$\alpha = 0: \quad \boldsymbol{X}_6 = f(t)\partial_\sigma. \tag{41}$$

The variation of the action can be read off directly for all generators but $\boldsymbol{X}_5$, for which we find the following variation after some manipulations

$$\begin{aligned} \delta_{\boldsymbol{X}_5}(\mathscr{L}) = \Big[& W(2\alpha+1)(\alpha W\sigma + \beta - 1)\ddot{f}(t) \\ & + \left((\alpha W \sigma + \beta - 1)\frac{dg}{d\sigma} - 2\alpha W g\right)\dot{f}(t)\Big]\overline{\sigma}. \end{aligned} \tag{42}$$

The quadratic part of the variation is then set to $\partial_\sigma \partial_{\overline{\sigma}} \delta_{\boldsymbol{X}_5}(\mathscr{L}) = 0$. Hence, the only possible material modes that admit the Ward identity $\boldsymbol{X}_5$ are the Oldroyd and Giesekus models.

$$g(\sigma) = A\sigma^2 + B\sigma, \tag{43}$$
$$f(t) = \exp\left(\frac{(B\alpha W - 2A(\beta-1))t}{(2\alpha+1)W^2\alpha}\right). \tag{44}$$

and the variation of the action becomes

$$\delta_{\boldsymbol{X}_5}(\mathscr{L}) = 2f(t)\overline{\sigma}(\beta-1)\frac{(A(\beta-1) - B\alpha W)(2A(\beta-1) - B\alpha W)}{(2\alpha+1)W^3\alpha^2}. \tag{45}$$

For the full elastic-turbulence equations, the same calculation yields no additional extended symmetries beyond those of the Navier-Stokes equations Canet (2022), even in the Stokesian limit, apart from the non-flowing $\overline{u}$ sector associated with $\boldsymbol{X}_5$. These identities determine the zero-momentum velocity sector, but do not constrain the stress and response-stress sectors. Hence the number of running degrees of freedom is increased, which complicates the construction of closure schemes.

3 Ward Identities for Elastic Burgulence

The previously derived transformations $\boldsymbol{X}_1 = f(t)\partial_x + \dot{f}(t)\partial_u$ and $\boldsymbol{X}_2 = g(x,t)\partial_{\overline{u}}$ lead to the Ward identities

$$\boldsymbol{X}_1: \quad 0 = \int_x \boldsymbol{J}\cdot\nabla\langle\boldsymbol{\phi}\rangle + \partial_t J^u, \tag{46}$$
$$\boldsymbol{X}_2: \quad 0 = \langle 2\overline{u} - (\nabla\sigma + \beta\Delta u)\rangle - J^{\overline{u}}. \tag{47}$$

For the effective theory, these identities state which zero-momentum vertices are fixed by symmetry rather than by the flow. In particular, the second identity implies that the $\overline{u}$ sector of higher-order vertex functions does not flow, while the first determines $(\delta\Gamma^{(k)}/\delta u)(\boldsymbol{p}=0)$. We use the notation $p = (\boldsymbol{p}, \omega)$ for a frequency-momentum variable, following the convention of Peskin (2018). For the constitutive models $g(\sigma) = A\sigma^2 + B\sigma$, the case $\alpha = A = 0$ gives the additional stress-shift identity

$$0 = \int_x (1 - W\partial_t)\langle\overline{\sigma}\rangle - J^\sigma. \tag{48}$$

For $\alpha \neq 0$, using the variation (45) and the function $f(t)$ in (44), and writing $f(t) = \exp(\chi t)$, the corresponding identity becomes

$$\begin{aligned} 0 = \Big\langle \int_{x,t} & \delta_{\boldsymbol{X}_5}(\mathscr{L}) + f\Big(W\alpha(\chi^2 x + 2\chi u)J^u \\ & + \chi(\alpha W\sigma + \beta - 1)J^\sigma - 2\chi\alpha W\overline{\sigma}J^{\overline{\sigma}} \\ & + \boldsymbol{\phi}\cdot\alpha W(\chi x\partial_x - \partial_t)\boldsymbol{J}\Big)\Big\rangle. \end{aligned} \tag{49}$$

At this point, we must acknowledge that $\boldsymbol{X}_5$ will generally by broken when introducing a general regulator R_k into the action and is only recovered for $k \to 0$. Furthermore, regulators that preserve $\boldsymbol{X}_5$ are too restrictive and do not provide a useful infrared regularization of all relevant propagating sectors.

3.1 Scaling Analysis and Critical Exponents

We now consider the scaling behavior near the fixed point. The Ward identities fix part of the scaling structure before any closure is introduced. We introduce rescaled momenta $\boldsymbol{p} \propto k$, frequencies $\omega \propto k^z$ and fields $\phi \propto k^{[\phi]}$. For $A = 0$, $B = 1$ and a general forcing, the Lagrangian reads

$$\begin{aligned} S_k^\alpha = & \int_{x,t} \overline{\sigma}(WL_u^\alpha\sigma + \sigma - 2(1-\beta)\nabla u) + \overline{u}(\nabla\sigma + \beta\Delta u) \\ & - \int_{x',x,t} \overline{u}D_k(x-x')\overline{u}' + \text{Other regulator terms}. \end{aligned} \tag{50}$$

and thus yields the canonical dimensions $[W]_c = -z$, $[\overline{u}]_c = 0$, $[u]_c = z-1$, $[\overline{\sigma}]_c = 1$, $[\sigma]_c = z$, $[D_k]_c = z+2$. Galilean invariance then enforces

$$\boldsymbol{X}_1: \; 0 = \int_x \boldsymbol{J}\cdot\nabla\langle\boldsymbol{\phi}\rangle + \partial_t J^u \to [\boldsymbol{J}] + [\langle\boldsymbol{\phi}\rangle] + 1 = z + [J^u] \tag{51}$$

Using $[\boldsymbol{J}] + [\langle\boldsymbol{\phi}\rangle] = z+1$, which follows from the Legendre relations, gives $[u] = z-1$. The shift in $\overline{u}$, with a scale-dependent forcing promoted to a regulator, leads to

$$\boldsymbol{X}_2: \quad 2\int_{x'}\langle\overline{u}'\rangle D_k = \langle(\nabla\sigma + \beta\Delta u)\rangle - J^{\overline{u}}, \tag{52}$$
$$[\overline{u}] + [D_k] - 1 = [J^{\overline{u}}] = [\sigma] + 1 = [u] + 2. \tag{53}$$

implying the scaling $[\overline{u}] = 0$, $[\sigma] = z$ and $[D_k] = z+2$. The identities for both values of α first of all lead to the scaling $[W] = [t] = -z$. Depending on the case, we then find the relations

$$\alpha = 0: \; [\overline{\sigma}] = [J^\sigma] \to [\overline{\sigma}] = 1 \tag{54}$$
$$\alpha \neq 0: \; [W] + z + 1 = [J^\sigma] = [J^u] - 1 = [\overline{\sigma}] \to [\overline{\sigma}] = 1 \tag{55}$$

With the above scaling relations, let us discuss the scaling of the convective operator

$$\left[\int_{x,t} \nu^{-1}\,\overline{u}u\nabla u\right] = [\nu^{-1}] + [\overline{u}] + 2[u] - z \to [\nu^{-1}] = 2 - z, \tag{56}$$

turns out to be irrelevant for $z > 2$.

3.2 Momentum-Resolving Closure

The most crucial property of Ward identities is that they constrain which parts of the fluctuation-renormalized equations may acquire scale dependence along the flow governed by the exact Wetterich equation (6). We use these constraints to build a closure in which the velocity response sector is kept fixed, while the elastic sector is described by running momentum-dependent functions. The elastic Burgers action can be split into an invariant and a non-invariant part as

$$\begin{aligned} S^\alpha = & \int_{x,t} \overline{u}(\nabla\sigma + \beta\Delta u) - \overline{u}^2 d_k + \overline{\Sigma}((D_t + \chi)\Sigma + 2\alpha\Sigma(\nabla u + \chi)) \\ & + \int_{x,t} \overline{\Sigma}(\alpha g(\sigma) - (2\alpha+1)\chi\Sigma) \end{aligned} \tag{57}$$

with $\overline{\Sigma} \equiv \overline{\sigma}/\alpha$ and $\Sigma = \alpha W\sigma + \beta - 1$. Only the second line is non-invariant under $\boldsymbol{X}_5$, whose variation is consistent with

(45). To construct a momentum-resolving closure compatible with the Ward identities, we fix the $\overline{u}$ sector as required by $\boldsymbol{X}_3$ in (39) and the non-invariant line, and seek covariant differential operators $\mathscr{D}_t, \mathscr{D}_x$ that map quantities with prescribed transformation properties to quantities with the same properties. In particular, the prolongation of $\boldsymbol{X}_5$ acts as

$$\mathrm{pr}\boldsymbol{X}_5\Phi = \alpha f(t)\chi w\Phi \tag{58}$$

with weight w. A covariant differential operator must preserve this weight. Galilean invariance requires the operators to be built from $D/dt = D_t + uD_x$ and D_x, similar to the KPZ case Canet *et al.* (2011*a*). We use D/dt for the material derivative in order to avoid confusion with the total derivative D_t. The commutator of $\mathrm{pr}\boldsymbol{X}$ and the total derivative D_i reads

$$0 = [\mathrm{pr}\boldsymbol{X}^Q, D_i] = [\mathrm{pr}\boldsymbol{X} - \xi^j D_j, D_i] \tag{59}$$
$$[\mathrm{pr}\boldsymbol{X}, D_i] = [\xi^j D_j, D_i] = -D_i\xi^j D_j \tag{60}$$

hence,

$$\begin{aligned}[\mathrm{pr}\boldsymbol{X}_5, D/dt] &= [\mathrm{pr}\boldsymbol{X}_5, D_t] + u[\mathrm{pr}\boldsymbol{X}_5, D_x] + \boldsymbol{X}_5(u)D_x \\ &= \alpha f\chi D/dt.\end{aligned} \tag{61}$$

$$\mathrm{pr}\boldsymbol{X}_5\frac{D\Phi}{dt} = \frac{D}{dt}(\mathrm{pr}\boldsymbol{X}_5\Phi) + \alpha f\chi\frac{D\Phi}{dt} = \alpha f\chi(w+1)\frac{D\Phi}{dt} + \alpha wf\chi^2\Phi \tag{62}$$

The combination $(D/dt + w\chi)\Phi$ thus transforms with weight $w+1$, which can be compensated through a factor of Σ^{-1} to get a covariant operator. Similarly, we find

$$\mathrm{pr}\boldsymbol{X}D_x\Phi = \alpha\chi f(w-1)D_x\Phi. \tag{63}$$

Hence, a choice of covariant differential operators is

$$\mathscr{D}_t = \frac{1}{\Sigma}\left(\frac{D}{dt} + w\chi\right), \quad \mathscr{D}_x = \Sigma D_x. \tag{64}$$

An arbitrary dependence on invariants such as $\rho = \Sigma^2/\overline{\Sigma}$ could also be introduced. Further invariants can be built by applying $\mathscr{D}_{t/x}$ of appropriate weight either to complete invariants or to their constitutive parts. We use a power-series definition for the operator-valued functions f^i, as introduced in Canet *et al.* (2011*a*):

$$f \equiv \sum_{m,n=0}^{\infty} C_{mn}(-i\mathscr{D}_t)^n(-i\mathscr{D}_x)^m. \tag{65}$$

The functions f^i represent scale-dependent elastic relaxation and stress-velocity couplings, and are taken to be invariant under parity $\boldsymbol{p} \to -\boldsymbol{p}$. The ansatz for the effective action reads

$$\begin{aligned}\Gamma_k = &\int_{x,t} \overline{u}(\nabla\sigma + \beta\Delta u) - \overline{u}^2 d_k + \overline{\Sigma}(\alpha g(\sigma) - (2\alpha+1)\chi\Sigma) \\ &+ \int_{x,t} \overline{\Sigma}\left(f^W(D_t+\chi)\Sigma + f^G\Sigma^2 + 2\alpha f^D\Sigma(\nabla u + \chi)\right).\end{aligned} \tag{66}$$

At finite k, a generic regulator breaks the Ward identity generated by $\boldsymbol{X}_5$, so the exact flow need not remain on the corresponding invariant submanifold. However, since both the microscopic action at $k = \Lambda$ and the physical effective action at $k = 0$ satisfy the identity, we project the truncated flow onto the compatible submanifold. To keep the notation compact, we use

$$f(\Sigma^{-1}(\omega - iw\chi), \Sigma\boldsymbol{p}) \to f(\omega, \boldsymbol{p}) = f(p) \tag{67}$$

for their Fourier transform at constant fields. The mean field is fixed by the equation of motion obtained from $\delta\Gamma_k/\delta\overline{\Sigma} = 0$ at vanishing sources. As usual for the Martin-Siggia-Rose-Janssen-de Dominicis response fields, we take $\overline{\Sigma}^* = 0$ and evaluate the remaining fields at a constant background. This gives

$$0 = \alpha g(\Sigma^*) + \Sigma^*\left(f^W + 2\alpha f^D - \chi(2\alpha+1) + f^G\Sigma^*\right)\Big|_{p=0}. \tag{68}$$

The two-point vertices entering the flow are

$$\begin{aligned}\Gamma_k^{(\overline{\Sigma},\Sigma)} &= \Gamma_{k,\mathrm{fix}}^{(\overline{\Sigma},\Sigma)} + (\chi + i\omega)f^W(p) + 2f^G(p)\Sigma + 2\alpha\chi f^D(p) \\ \Gamma_k^{(\overline{\Sigma},u)} &= 2i\alpha\Sigma\boldsymbol{p}f^D(p),\end{aligned} \tag{69}$$

which can be used to project (6) onto the flow of $f^{W/D/G}$. The field derivatives of the operator-valued functions can then be evaluated through the power series definition of the operators and are given in terms of divided differences. For a monomial let

$$x^n[x_0, x_1, .., x_k] \equiv \sum_{z_0+\ldots+z_k=n-k,\, z_i\geq 0} x_0^{z_0}x_1^{z_1}\ldots x_k^{z_k} \tag{70}$$

with generalization to analytic functions by term-wise application. With $q = (\boldsymbol{q}, \Omega)$, for instance,

$$\frac{\delta f(p)}{\delta u(q)}\Big|_{\mathrm{const.}} \to (\boldsymbol{p}+\boldsymbol{q})f[\omega+\Omega, \omega; \boldsymbol{p}+\boldsymbol{q}], \tag{71}$$
$$\frac{\delta f(p)}{\delta\Sigma(q)}\Big|_{\mathrm{const.}} \to -\frac{\omega+\Omega}{\Sigma}f[\omega, \omega+\Omega; \boldsymbol{p}+\boldsymbol{q}] + \frac{\boldsymbol{p}+\boldsymbol{q}}{\Sigma}f[\omega; \boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}]. \tag{72}$$

Here $f[a,b;\boldsymbol{p}] = (f(a,\boldsymbol{p}) - f(b,\boldsymbol{p}))/(a-b)$ and similarly for momentum arguments. Higher variations are obtained by the same divided-difference rule and inserted into the differentiated Wetterich equationk, which yields a set of closed flow equations.

3.2.1 Diffusive truncation

Instead of retaining the full frequency and momentum dependence, we consider a purely momentum-resolving *diffusive* closure, dropping the $\mathscr{D}_t$ dependence in the $f's$, denoted by $\tilde{f}$ going forward. The only relevant operator derivatives reduce to

$$\frac{\delta\tilde{f}(p)}{\delta\Sigma(q)}\Big|_{\mathrm{const.}} \to \frac{\boldsymbol{p}+\boldsymbol{q}}{\Sigma}\tilde{f}[\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}], \tag{73}$$
$$\begin{aligned}\frac{\delta^2\tilde{f}(p)}{\delta\Sigma(q)\delta\Sigma(l)}\Big|_{\mathrm{const.}} \to &\frac{(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{l})}{\Sigma^2} \\ &\times\tilde{f}[\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{q}, \boldsymbol{p}+\boldsymbol{q}+\boldsymbol{l}] + (q \leftrightarrow l).\end{aligned} \tag{74}$$

The only remaining non-vanishing vertex functions read

$$\Gamma_{k,\mathrm{flow}}^{(\overline{\Sigma},\Sigma,\Sigma)} = (i\omega + i\Omega + \chi)\frac{\delta\tilde{f}^W(p)}{\delta\Sigma(q)} + 2\alpha\chi\frac{\delta\tilde{f}^D(p)}{\delta\Sigma(q)} \tag{75}$$
$$+ \tilde{f}^G(p) + 2\Sigma\frac{\delta\tilde{f}^G(p)}{\delta\Sigma(q)} + (q \leftrightarrow l), \tag{76}$$
$$\begin{aligned}\Gamma_k^{(\overline{\Sigma},\Sigma,u)} = &-i\boldsymbol{q}\tilde{f}^W(p) + 2i\alpha(\boldsymbol{p}+\boldsymbol{q})\tilde{f}^D(p) \\ &+ 2i\alpha\Sigma(\boldsymbol{p}+\boldsymbol{q})\frac{\delta\tilde{f}^D(p)}{\delta\Sigma(q)},\end{aligned} \tag{77}$$
$$\begin{aligned}\Gamma_k^{(\overline{\Sigma},3\times\Sigma)} = &(i\omega + i\Omega + i\mu + \chi)\frac{\delta^2\tilde{f}^W(p)}{\delta\Sigma(l)\delta\Sigma(q)} + 2\alpha\chi\frac{\delta^2\tilde{f}^D(p)}{\delta\Sigma(l)\delta\Sigma(q)} \\ &+ 2\frac{\delta\tilde{f}^G(p)}{\delta\Sigma(q)} + 2\Sigma\frac{\delta^2\tilde{f}^G(p)}{\delta\Sigma(l)\delta\Sigma(q)}(q \leftrightarrow r) + (l \leftrightarrow r),\end{aligned} \tag{78}$$
$$\begin{aligned}\Gamma_k^{(\overline{\Sigma},\Sigma,\Sigma,u)} = &-i\boldsymbol{l}\frac{\delta\tilde{f}^W(p)}{\delta\Sigma(q)} + i\alpha\Sigma(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{l})\frac{\delta^2\tilde{f}^D(p)}{\delta\Sigma(l)\delta\Sigma(q)} \\ &+ 2i\alpha(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{l})\frac{\delta\tilde{f}^D(p)}{\delta\Sigma(q)} + (q \leftrightarrow l).\end{aligned} \tag{79}$$

The two projected flow equations are obtained by functionally differentiating the Wetterich equation (6) with respect to the response field $\overline{\Sigma}$ and either Σ or u, evaluating the result in constant background fields with $u^* = \overline{u}^* = \overline{\Sigma}^* = 0$ and $\Sigma = \Sigma^*$, and then Fourier transforming. This gives

$$\partial_k\Gamma^{(\overline{\Sigma},\Sigma)} = \frac{1}{2}\mathrm{Tr}\int_q \partial_k R_k G\left(-\Gamma^{(2,\overline{\Sigma},\Sigma)} + 2\Gamma^{(2,\overline{\Sigma})}G\Gamma^{(2,\Sigma)}\right)G, \tag{80}$$
$$\partial_k\Gamma^{(\overline{\Sigma},u)} = \frac{1}{2}\mathrm{Tr}\int_q \partial_k R_k G\left(-\Gamma^{(2,\overline{\Sigma},u)} + 2\Gamma^{(2,\overline{\Sigma})}G\Gamma^{(2,u)}\right)G. \tag{81}$$

Evaluating these expressions at the projection point $\omega = 0$ gives the flow equations for the functions $\tilde{f}^{W/D}$. The matrix contractions are evaluated in the field basis used for the two-point vertices above.

4 Two-Point Statistics

Given our diffusive closure for the flow equation, two-point statistics can be extracted by inverting $\Gamma^{(2)}_{k\to 0}$. Since the complete frequency dependence of the present closure is encoded in two poles located at $\pm\gamma$, that is

$$\gamma(\boldsymbol{p}) = \chi + \frac{\Gamma^{(\bar{\Sigma},\Sigma)}_{\text{fix}} + 2\Sigma f^G + 2\alpha\chi f^D - \frac{2\Sigma}{\beta W} f^D}{f^W}. \tag{82}$$

Assuming $\gamma(p) > 0$, the inverse temporal Fourier transforms of the fixed-point correlation functions are obtained as

$$C_{\Sigma\Sigma} = \frac{4d\alpha^2\Sigma^2[f^D]^2}{\beta^2\boldsymbol{p}^2[f^W]^2\gamma} e^{-\gamma|\tau|}, \tag{83}$$

$$C_{uu} = \frac{2d}{\beta^2\boldsymbol{p}^4}\delta(\tau) + \frac{4d\Sigma f^D}{\beta^3 W\boldsymbol{p}^4 f^W}\left(1 + \frac{\Sigma f^D}{\beta W f^W\gamma}\right) e^{-\gamma|\tau|}, \tag{84}$$

$$C_{u\Sigma} = \frac{2id\alpha\Sigma f^D}{\beta^2\boldsymbol{p}^3 f^W}\left(\operatorname{sgn}(\tau) + 1 + \frac{2\Sigma f^D}{\beta W f^W\gamma}\right) e^{-\gamma|\tau|}. \tag{85}$$

Further, we may conclude that asymptotically $\lim_{\boldsymbol{p}\to\infty} f^D/(\boldsymbol{p}^3 f^W) = 0$ for us to find a decaying spectrum.

4.0.1 Dimensionless form

In order to extract statistical properties of (1) and (2) at the scale invariant fixed point, we introduce dimensionless variables, denoted by hats, as

$$f^i_k(\Sigma^*\boldsymbol{p}) = X_i^{1/2}\hat{f}^i_k(\hat{\boldsymbol{p}}), \qquad \Sigma^*\boldsymbol{p} = \hat{\Sigma}^* k Z_\Sigma^{1/2}\hat{\boldsymbol{p}}, \tag{86}$$

$$\phi^i = \hat{\phi}^i Z_i^{1/2}, \qquad \omega = \hat{\omega}\Omega_k. \tag{87}$$

Hence, with $k\partial_k = \partial_s$, the flow equation for the dimensionless functions reads

$$\partial_s\hat{f}^i = -\frac{[X_i]}{2}\hat{f}^i + \left(1 + \zeta_\Sigma + \frac{1}{2}\partial_s\ln Z_\Sigma\right)\hat{\boldsymbol{p}}\partial_{\hat{\boldsymbol{p}}}\hat{f}^i + \mathscr{I}^i(\hat{\boldsymbol{p}}), \tag{88}$$

where $[X_i] = \partial_s\ln X_i$ and

$$\zeta_\Sigma = \partial_s\ln\hat{\Sigma}^*. \tag{89}$$

The term proportional to ζ_Σ is the scaling contribution generated by the running argument $\Sigma^*\boldsymbol{p}$. The background is kept dynamical and is expected to flow to a nonzero constant, thus $\zeta_\Sigma \to 0$.

The Ward-identity-protected sector fixes all normalizations except the response-stress field-strength renormalization and the dynamical scaling Ω_k. For a general choice of d_k the protected normalizations give

$$Z_{\bar{u}} d_k = k\Omega_k, \qquad kZ_{\bar{u}}^{1/2}Z_u^{1/2} = \Omega_k, \tag{90}$$

$$Z_{\bar{u}}^{1/2}Z_\sigma^{1/2} = \Omega_k, \qquad Z_\Sigma^{1/2}\Omega_k = Z_\sigma^{1/2}. \tag{91}$$

The remaining velocity normalization is fixed by the physical identification $\chi \sim \nabla u$. Since $\chi = \hat{\chi}\Omega_k$, this implies $kZ_u^{1/2} = \Omega_k$. Inserting this into $kZ_{\bar{u}}^{1/2}Z_u^{1/2} = \Omega_k$ gives $Z_{\bar{u}} = 1$. Consequently $d_k = k\Omega_k$, and the stress-gradient normalization gives $Z_\sigma^{1/2} = \Omega_k$. Together with $Z_\Sigma^{1/2}\Omega_k = Z_\sigma^{1/2}$ this yields $Z_\Sigma = 1$.

The running functions renormalize as

$$X_W = Z_{\bar{\Sigma}}^{-1}Z_\Sigma^{-1}k^2, \qquad X_D = Z_{\bar{\Sigma}}^{-1}Z_u^{-1}\Omega_k^2. \tag{92}$$

Using $Z_\Sigma = 1$ and $Z_u^{1/2} = \Omega_k/k$, both reduce to $X_W = X_D = k^2 Z_{\bar{\Sigma}}^{-1}$. Hence $[X_W] = [X_D] = 2 - \eta_{\bar{\Sigma}}$, with $\eta_{\bar{\Sigma}} = \partial_s\ln Z_{\bar{\Sigma}}$. In the reduced convention (88) becomes

$$\partial_s\hat{f}^i_k = -\frac{1}{2}\left(2 - \eta_{\bar{\Sigma}}\right)\hat{f}^i_k + (1 + \zeta_\Sigma)\hat{\boldsymbol{p}}\partial_{\hat{\boldsymbol{p}}}\hat{f}^i_k + \mathscr{I}^i(\hat{\boldsymbol{p}}). \tag{93}$$

The remaining field-strength renormalization is then fixed by introducing an additional scalar coupling, determining the change of the $\Gamma_{k,\text{fixed}}$ contribution

$$\partial_s\lambda_k = \frac{\lambda_k}{2}(\eta_{\bar{\Sigma}} - 2) \tag{94}$$

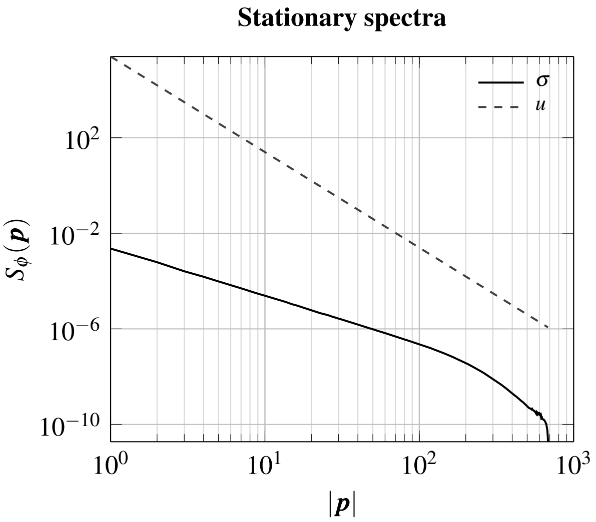


Figure 1. Stationary spectra. The left panel shows the power-spectra of $|\sigma|^2$, $|\psi|^2$ and $|u|^2$, whereas the right panel shows the corresponding local logarithmic slopes.

which we assume to remain constant. The dynamical scaling is then determined by the pole-location $\gamma(\boldsymbol{p}_c)$ at at the reference momentum $\boldsymbol{p}_c$. The complete reduced set of equations is therefore (**??**) and

$$0 = \zeta_\Sigma - \partial_s\ln\hat{\Sigma}^* \tag{95}$$

$$0 = \lambda_k\left(\alpha g(\hat{\Sigma}^*) - \hat{\chi}\hat{\Sigma}^*(2\alpha+1)\right) + \hat{\chi}\,(2\alpha+1)\,\hat{\Sigma}^* + (\hat{\Sigma}^*)^2\hat{f}^G(0). \tag{96}$$

The flow equation for $\hat{f}^G$ carries and additional contribution $-z\hat{f}^G$ when compared to (93). The regulator in use is

$$R_k(p) = \begin{bmatrix} 0 & -\beta\boldsymbol{p}^2 r & 0 & -i\operatorname{sign}(\Sigma^*)\boldsymbol{p}^3k^{-1}Z_{\bar{\Sigma}}^{-1/2}r \\ & 0 & i\dfrac{\boldsymbol{p}^3k^{-2}}{\alpha W_k}r & -i\boldsymbol{p}^3k^{-2}Z_{\bar{\Sigma}}^{-1/2}r \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}. \tag{97}$$

The result is a closed set of flow equations for the dimensionless functions $\hat{f}^i_k$, the background $\hat{\Sigma}^*$, and the running scalings z and $\eta_{\bar{\Sigma}}$.

5 Stochastic Numerical Experiment

To validate the theory presented herein, we solved equations (1) and (2) numerically in the inertialess limit. The velocity equation is then solved algebraically in Fourier space. The stress equation is advanced pseudo-spectrally on a periodic domain and the stochastic velocity increment is treated as a white-in-time impulse. To improve numerical stability, the evolution equation is formulated in terms of the logarithm

$$\psi = \ln\left(1 + \frac{\alpha W}{1-\beta}\sigma\right), \tag{98}$$

which ensures positivity of the conformation explicit. The material parameter used herein are $W = 1$, $\beta = 1/2$, $\alpha = 1$, $A = 1/4$, $B = 1$ and the forcing strength was chosen as $1/10$. The numerical grid is of size $N = 2048$ with time-step size $\Delta t = 10^{-6}$ and statistics extracted over the interval $t \in [1/2, 16]$ with ensemble-averaging over 256 runs. The resulting spectra show a clear power law $u \sim \boldsymbol{p}^{-2}$, which is a direct consequence of the application of the inverse Stokes operator, and the fact that both the noise and stress gradient balance $\sigma_x \sim \xi \sim \boldsymbol{p}^0$, as can be seen in figure 4. We also observe that there exists a

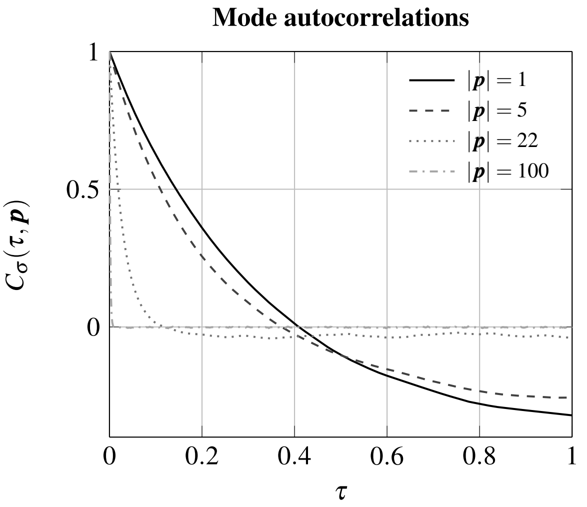


Figure 2. Temporal autocorrelations of selected stress modes $C_\sigma(\tau, |\boldsymbol{p}|) = \langle \sigma(|\boldsymbol{p}|, t)\sigma(|\boldsymbol{p}|, t+\tau)\rangle$.

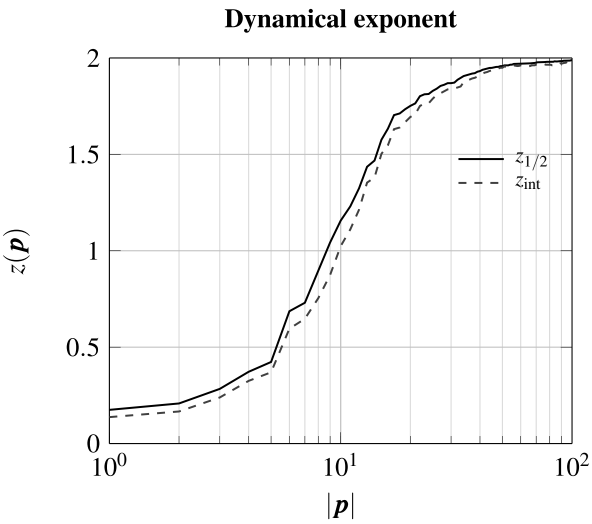


Figure 3. Dynamical exponent extracted from the decay time of $C_\sigma(\tau, |\boldsymbol{p}|)$ through $\tau \sim |\boldsymbol{p}|^{-z}$.

regime where the scaling predictions of the Ward identity analysis $u \sim \boldsymbol{p}\sigma$ holds. The dynamical exponent is extracted from the stress autocorrelation C_σ, shown in figure 2. In figure 3 we observe a peak of $z \approx 2$, which is the regime where the inertial operator is marginal. Whence, the scale where the Stokesian fixed point should live. Finally, the balance of the terms in the stress equation, shown in fiugre 4, confirm that the dynamics are not dominated by linear Itô drift and relaxation. Rather, we are in an advection dominated regime with the stretching term $2\alpha u_x \sigma$ taking a subleading role.

6 Conclusion

We formulated a stochastically forced elastic Burgers equation as a reduced setting for studying elastic contributions to Burgers-type turbulence with functional methods. Starting from the corresponding Martin-Siggia-Rose-Janssen-de Dominicis action, derived from the governing stochastic differential equation, we derived the extended symmetry generators and the associated Ward identities by introducing an algorithmic method based on Lie-group methods. The resulting constraints trivially identify the response-velocity sector as protected and show that the stress sector remains considerably less constrained than in the Navier-Stokes or standard Burgers cases. Particularly noteworthy is the emergence of non-trivial Ward identity in the stokesian limit, which cannot be obtained by gauging existing symmetries as in the KPZ or Navier Stokes setting Canet (2022).

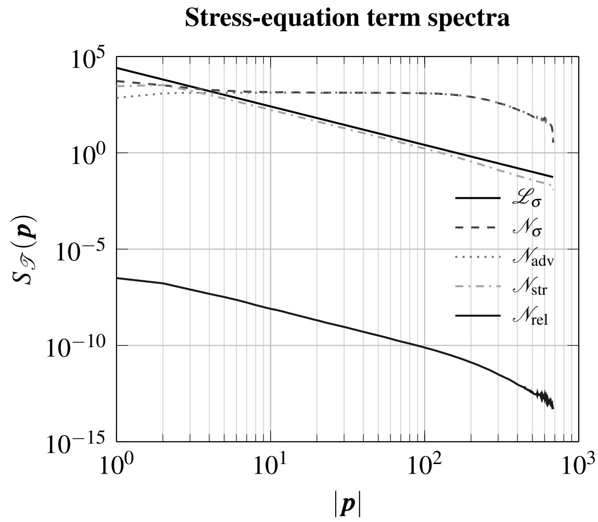


Figure 4. Spectral stress-equation balance. $\mathscr{L}_\sigma$ is the linear contribution induced by the Itô drift and nonlinear contribution $\mathscr{N}_\sigma$ become comparable over large parts of the momentum range.

This Ward identity yields both considerable insights into the scaling behavior near the renormalization group fixed point and restricts closure schemes.

The scaling analysis indicates that the inertial contribution is irrelevant at the elastic fixed point considered here, which motivates the inertialess formulation used in the subsequent closure. We then constructed a symmetry-covariant, momentum-resolving ansatz for the effective action and reduced it to a diffusive truncation suitable for numerical implementation.

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