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HOMOGENEOUS SHEAR TURBULENCE: KINETIC ENERGY GROWTH RATE AS A NONLINEAR EIGENVALUE PROBLEM

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ABSTRACT

In homogeneous shear turbulence the turbulent kinetic energy is observed, both experimentally and numerically, to grow exponentially in time, $\mathcal{K}(t) \propto \exp(2\lambda t)$. The ratio of the growth rate λ to the constant shear rate A appears to be a Reynolds-number-independent constant, $\lambda/A \approx 0.057$, yet to date no theory explains the origin of this value.

We present a symmetry-based approach that recasts the growth rate as the eigenvalue of a nonlinear eigenvalue problem (NEVP). Starting from the Euler equations for perturbations on a general linear base flow $\overline{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{x}$, the space-scaling and time-translation symmetries are combined into a two-parameter group whose invariants remove the explicit time dependence and yield a stationary PDE system in which λ, the ratio of the two group parameters, appears as the sought eigenvalue.

From this formulation we derive an explicit expression for λ and prove that it is bounded above and below by the spectral norm of the velocity-gradient tensor, $|\lambda| \leq \frac{2}{5}\|\boldsymbol{A}\|_2$. A characteristics analysis of the linearized two-dimensional problem admits only the trivial solution, demonstrating that the nonlinearity is essential. As the NEVP possesses no energy functional and differs structurally from related problems such as the Gross–Pitaevskii eigenvalue problem, a dedicated finite-element scheme is being developed to compute both the eigenvalue and the eigenfunction.

Introduction

Tavoularis (1985) may have been the first to recognize that turbulent kinetic energy $\mathcal{K}$ in linear shear flows in the final phase exhibits exponential growth, i.e.

$$\mathcal{K}(t) = \frac{1}{2}\overline{u_i' u_i'} \propto \exp(2\lambda t). \tag{1}$$

The physical flow investigated in the following is a shearing base flow $\overline{\boldsymbol{u}} = (Ax_2, 0, 0)^T$ using the constant shear rate A. In this flow, turbulent motion emerges and, after a short initial phase, grows exponentially, as described in (1).

The ratio of λ and the constant shear rate A is observed, both numerically and experimentally, to be a constant for flows at high Reynolds numbers. $\lambda/A \sim 0.2$ seems to be an upper bound in both experimental and numerical investigations (Briard *et al.* 2016). However, to the authors' knowledge, no theory exists that explains the origin and physical reasoning behind the numerical value of $\frac{\lambda}{A} \approx 0.057$.

In the following, we introduce a novel symmetry-based approach and derive a nonlinear eigenvalue problem (NEVP) that describes the growth of perturbations, where the growth rate is the sought eigenvalue. The concept of nonlinear eigenvalue problems is well-established and appears in various fields of physics and engineering, including acoustics, structural mechanics (Güttel & Tisseur 2017) or quantum physics (Henning & Jarlebring 2025). A notable example in the latter field is the Gross-Pitaevskii eigenvalue problem (GPEVP), a nonlinear eigenvalue problem describing ground states in ultracold bosonic gases. Though there is no thematic overlap with this work's eigenvalue problem, there is a structural similarity, as the eigenvalue appears linearly with the nonlinearity appearing in the eigenfunction. Henning & Jarlebring (2025) gives an extensive overview over both solution theory and numerical solution schemes.

Governing equations

We employ the incompressible Euler equations to model the phenomena as we intend to investigate scales way above the Taylor length scale. We apply the Reynolds decomposition

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}' \quad \text{and} \quad p = \bar{p} + p', \tag{2}$$

and assume the perturbations to vanish at infinity, i.e.

$$\boldsymbol{u}' = 0\,, \quad p' = 0 \quad \text{for} \quad |\boldsymbol{x}| \to \infty. \tag{3}$$

We generalize the above shear base flow $\overline{\boldsymbol{u}} = (Ax_2, 0, 0)^T$ into a base flow that is governed by $\bar{\boldsymbol{u}} = \boldsymbol{A} \cdot \boldsymbol{x}$, where $\boldsymbol{A}$ is the constant velocity gradient tensor and $\boldsymbol{x}$ the Cartesian coordinate vector. Purely shearing base flows are given by $(\boldsymbol{A})_{ij} = A\delta_{i1}\delta_{j2}$, where A is the constant shear rate. A general $\boldsymbol{A}$ allows us to also investigate base flows like purely straining flows or flows displaying strain and shear.

Insertion of the various ansatzes into the Euler equations for a rotating frame (with the rotation vector Ω_i and centrifugal forces absorbed in the pressure term) gives:

$$\frac{\partial u_i'}{\partial t} + u_j' \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial A_{ik} x_k}{\partial x_j} + A_{jk} x_k \frac{\partial u_i'}{\partial x_j} + \frac{\partial p'}{\partial x_i} + 2\epsilon_{ijk}\Omega_j u_k' = 0, \quad (4a)$$

$$0 = \frac{\partial u_i'}{\partial x_i}. \quad (4b)$$

We now want to find special solutions to (4), that are invariant under a two-parameter symmetry group containing space-scaling and time-translation. See Bluman *et al.* (2010) for a detailed introduction into this topic.

Derivation of the generalized Nonlinear Eigenvalue Problem

Among other symmetries, the Euler equations remain unchanged under scaling in space: $\hat{x}_i = e^{a_{Sx}} x_i$, and the translation in time: $\hat{t} = t + a_t$ (Pope 2000). When combining these symmetry groups, we obtain a two-parameter symmetry group that transforms the dependent and independent variables as

$$\hat{t} = t + a_t\ , \quad \hat{x}_i = e^{a_{Sx}} x_i\ , \quad \hat{u}_i' = e^{a_{Sx}} u_i'\ , \quad \hat{p}' = e^{2a_{Sx}} p'. \quad (5)$$

Now variables are searched, which are invariant to the symmetry transformation (5). For this, we need the infinitesimal operator of the symmetry group. It can be understood as the first-order Taylor approximation of the symmetry group. For (5) it is given by

$$X = a_{Sx} x_i \frac{\partial}{\partial x_i} + a_t \frac{\partial}{\partial t} + a_{Sx} u_i' \frac{\partial}{\partial u_i'} + 2a_{Sx} p' \frac{\partial}{\partial p'}. \quad (6)$$

The corresponding characteristic system is given by

$$\frac{\mathrm{d}x_i}{a_{Sx} x_i} = \frac{\mathrm{d}t}{a_t} = \frac{\mathrm{d}u_i'}{a_{Sx} u_i'} = \frac{\mathrm{d}p'}{2a_{Sx} p'}. \quad (7)$$

This equation system can now be solved one-by-one by integration, with the integration constants being the invariant variables. For example, we integrate the expression for the independent variables x_i and t, $\frac{\mathrm{d}x_i}{a_{Sx} x_i} = \frac{\mathrm{d}t}{a_t}$, with the constant of integration being the invariant space variable $\tilde{x}$:

$$x_i = \tilde{x}_i e^{\frac{a_{Sx}}{a_t} t} = \tilde{x}_i e^{\lambda t} \quad \Longleftrightarrow \quad \tilde{x}_i = x_i\, e^{-\lambda t}. \quad (8)$$

Here, we introduced $\lambda := \frac{a_{Sx}}{a_t}$ as the ratio of the two group parameters. The new coordinate $\tilde{x}$ is invariant to the combined symmetry (5). This can be checked by applying the transformation (5) to $\tilde{x}_i$ in (8), which leads to $\tilde{x}_i$ itself again.
Similar to the space coordinate, we obtain expressions for the velocity and pressure fields:

$$x_i = \tilde{x}_i e^{\lambda t} \quad , \quad u_i' = \tilde{u}_i'(\tilde{\boldsymbol{x}}) e^{\lambda t} \quad , \quad p' = \tilde{p}'(\tilde{\boldsymbol{x}}) e^{2\lambda t}. \quad (9)$$

The above result of Tavoularis (1) is obviously consistent with (9), i.e., more generally for the Reynolds stress we find $\overline{u_i' u_j'} \propto \exp(2\lambda t)$.
Now, we transform the Euler equations (4) into the new coordinate system. Substitution leads to

$$\left[\tilde{u}_j' + A_{jk}\tilde{x}_k - \lambda \tilde{x}_j\right] \frac{\partial \tilde{u}_i'}{\partial \tilde{x}_j} + \left[A_{ij} + \lambda \delta_{ij}\right] \tilde{u}_j' + \frac{\partial \tilde{p}'}{\partial \tilde{x}_i} + 2\epsilon_{ijk}\tilde{\Omega}_j \tilde{u}_k' = 0, \quad (10a)$$

$$0 = \frac{\partial \tilde{u}_i'}{\partial \tilde{x}_i}, \quad (10b)$$

$$\text{with} \quad \tilde{\boldsymbol{u}}' = 0 \quad \text{and} \quad \tilde{p}' = 0 \quad \text{for} \quad |\tilde{\boldsymbol{x}}| \to \infty. \quad (10c)$$

As can be seen in (10), insertion of the invariant variables into the (otherwise unchanged) Euler system (4) removes t-dependency. Instead, a new eigenvalue λ appeared.
In the following, solution tuples $(\tilde{u}_i'(\tilde{\boldsymbol{x}}), \tilde{p}'(\tilde{\boldsymbol{x}}), \lambda)$ to the partial differential equation (PDE) system are sought. These solutions, by themselves, are stationary, but re-transforming them into physical coordinates makes them time-dependent again: A flow pattern grows exponentially over time, with its velocities and kinetic energy growing as well.

Inspecting (10), the additional scaling symmetry

$$\hat{\tilde{x}}_i' = e^{a}\tilde{x}_i'\ , \quad \hat{\tilde{u}}_i' = e^{a}\tilde{u}_i'\ , \quad \hat{\tilde{p}}' = e^{2a}\tilde{p}'\ , \quad \hat{\lambda} = \lambda \quad (11)$$

can be identified, which has to be broken during numerical simulations as it would appear as a singularity which has to be avoided. Therefore, we introduce an additional normalization condition,

$$\|\tilde{\boldsymbol{u}}'\|_{L^2(\tilde{\Omega})} = \sqrt{\int_{\tilde{\Omega}} \tilde{u}_i' \tilde{u}_i' \mathrm{d}\tilde{x}^3} = \text{const.} \quad (12)$$

In the following we therefore restrict ourselves to solutions $(\tilde{u}_i'(\tilde{\boldsymbol{x}}), \tilde{p}'(\tilde{\boldsymbol{x}}), \lambda)$, which further satisfy (12). This does not meaningfully restrict our solution space, as we are not interested in finding the trivial zero-solution and every other solution can be rescaled to the desired L_2-norm using (11).

Explicit expression of the growth parameter λ

Inspecting (10), there is an obvious direct link between the eigenvalue λ and the velocity perturbation field $\tilde{u}_i'(\tilde{\boldsymbol{x}})$ and an explicit expression for λ may be derived:
The momentum equation (10a) is scalar-multiplied by the velocity fluctuations $\tilde{u}_i'$ and integrated over the whole domain. Employing the divergence theorem and the BC (10c), all boundary integrals drop out and an expression for the eigenvalue λ is determined:

$$\lambda = -\frac{2}{5}\frac{\int_{\tilde{\Omega}} \tilde{u}_i' A_{ij} \tilde{u}_j' \mathrm{d}\tilde{x}^3}{\int_{\tilde{\Omega}} \tilde{u}_i' \tilde{u}_i' \mathrm{d}\tilde{x}^3}. \quad (13)$$

This expression can now be used to obtain the eigenvalue λ belonging to a given eigenfunction $\tilde{\boldsymbol{u}}_i'(\tilde{\boldsymbol{x}})$, or to check results

obtained numerically or analytically.
Furthermore, expression (13) can be used to show that the eigenvalue is bounded from above and below:

$$\begin{aligned}|\lambda| &= \frac{2}{5}\frac{\left|\int_{\tilde\Omega}\tilde u_i' A_{ij}\tilde u_j' \mathrm{d}\tilde x^3\right|}{\int_{\tilde\Omega}\tilde u_i'\tilde u_i'\mathrm{d}\tilde x^3} = \frac{2}{5}\frac{\left|\int_{\tilde\Omega}\tilde{\boldsymbol u}'\cdot(\boldsymbol A\tilde{\boldsymbol u}')\,\mathrm{d}\tilde x^3\right|}{\int_{\tilde\Omega}\tilde{\boldsymbol u}'\cdot\tilde{\boldsymbol u}'\mathrm{d}\tilde x^3}\\ &\le \frac{2}{5}\frac{\int_{\tilde\Omega}|\tilde{\boldsymbol u}'\cdot(\boldsymbol A\tilde{\boldsymbol u}')|\,\mathrm{d}\tilde x^3}{\int_{\tilde\Omega}\tilde{\boldsymbol u}'\cdot\tilde{\boldsymbol u}'\mathrm{d}\tilde x^3} \overset{\bullet}{\le} \frac{2}{5}\frac{\int_{\tilde\Omega}\|\tilde{\boldsymbol u}'\|_2\,\|\boldsymbol A\tilde{\boldsymbol u}'\|_2\,\mathrm{d}\tilde x^3}{\int_{\tilde\Omega}\|\tilde{\boldsymbol u}'\|_2^2\,\mathrm{d}\tilde x^3}\\ &\overset{\diamond}{\le} \frac{2}{5}\frac{\int_{\tilde\Omega}\|\tilde{\boldsymbol u}'\|_2\,\|\boldsymbol A\|_2\,\|\tilde{\boldsymbol u}'\|_2\,\mathrm{d}\tilde x^3}{\int_{\tilde\Omega}\|\tilde{\boldsymbol u}'\|_2^2\,\mathrm{d}\tilde x^3} = \frac{2}{5}\,\|\boldsymbol A\|_2\end{aligned} \tag{14}$$

In the inequality marked with a • sign, the Cauchy-Schwarz-inequality was used. The norm of a matrix-vector product is bounded by the product of the corresponding vector and consistent matrix norm, which is used in the inequality marked ⋄. The eigenvalue is bounded from above and below using the spectral norm of the mean flow velocity gradient tensor: $|\lambda| \le \frac{2}{5}\,\|\boldsymbol A\|_2$, i.e. we have an upper limit for the growth of turbulent kinetic energy of $\mathcal{K}(t) \propto \exp(2\lambda t) \le \exp(4/5\,\|\boldsymbol A\|_2\,t)$.
For the case of pure shear flows, an even tighter upper bound of $|\lambda| \le \frac{1}{5}|A|$ can be found, which matches the experimentally and numerically observed upper bound of $\frac{\lambda}{A}$ quite well (Briard *et al.* 2016).
The formula (13) does not replace the eigenvalue problem (10), since, for example, system rotation has a fundamental influence on the eigenvalue λ via Coriolis forces. However, in (13) all Coriolis forces cancel out. This is a clear indication that (13) is merely a necessary condition, and, hence, the complete NEVP (10) needs to be solved.

Shear flows in a non-rotating two-dimensional frame

In a first step, we have investigated purely shearing flows in a two-dimensional frame, the mean velocity is $\bar{\boldsymbol u} = (Ax_2, 0)^T$. Tuples $(\tilde u_i'(\tilde{\boldsymbol x}), \tilde p'(\tilde{\boldsymbol x}), \lambda)$ are solutions to the NEVP if they solve (10) with $(\boldsymbol A)_{ij} = A\delta_{i1}\delta_{j2}$, for $i, j \in \{1, 2\}$. Interestingly, the case of system rotation – which is of interest in 3D – does not arise in the 2D case, as the Coriolis terms can be absorbed into the pressure. This is most clearly seen by introducing a stream function as defined in equation (16) below, which then yields the modified pressure

$$\hat{\tilde p}' = \tilde p' + 2\,\tilde\Omega\,\tilde\psi', \tag{15}$$

so that system rotation is completely eliminated. This is the classical symmetry of the Euler and Navier-Stokes equations in 2D, which has been preserved here.
Equivalently to the three-dimensional case, we show upper and lower bounds to the eigenvalue: $|\lambda| \le \frac{1}{4}|A|$, i.e. we have an upper limit for the growth of turbulent kinetic energy of $\mathcal{K}(t) \propto \exp(2\lambda t) \le \exp(1/2At)$.
An equivalent formulation of the NEVP to the one in (10) is found by introducing a stream function

$$\tilde u_1' = \frac{\partial\tilde\psi'}{\partial\tilde x_2}, \quad \tilde u_2' = -\frac{\partial\tilde\psi'}{\partial\tilde x_1}. \tag{16}$$

Cross-differentiation of (10a) for $i = 1$ and $i = 2$ gives

$$0 = \left[\lambda\tilde x_1 - A\tilde x_2 - \frac{\partial\tilde\psi'}{\partial\tilde x_2}\right]\frac{\partial\Delta\tilde\psi'}{\partial\tilde x_1} + \left[\lambda\tilde x_2 + \frac{\partial\tilde\psi'}{\partial\tilde x_1}\right]\frac{\partial\Delta\tilde\psi'}{\partial\tilde x_2}, \tag{17}$$

with the Laplacian operator $\Delta = \frac{\partial^2}{\partial\tilde x_1^2} + \frac{\partial^2}{\partial\tilde x_2^2}$ and the BC $\frac{\partial\tilde\psi'}{\partial\tilde x_i} = 0$ for $|\boldsymbol x| \to \infty$.

Variational analysis of the growth rate eigenvalue

As shown, the growth eigenvalue λ is bounded from above and below by the absolute of the shear rate A. Therefore, the NEVP can be understood as a variational problem: We search for a velocity and pressure field that satisfy the eigenvalue problem (10) and maximize the functional[1]

$$\Lambda(\boldsymbol v(\tilde{\boldsymbol x})) = -\frac{1}{2}A\frac{\int_{\tilde\Omega} v_1 v_2 \mathrm{d}\tilde x^2}{\int_{\tilde\Omega} v_i v_i \mathrm{d}\tilde x^2} = -\frac{1}{2}A\frac{\langle v_1, v_2\rangle_{L^2(\tilde\Omega)}}{\|\boldsymbol v\|^2_{L^2(\tilde\Omega)}}. \tag{18}$$

We search for $\hat{\boldsymbol v}(\tilde{\boldsymbol x}) \in \mathcal{S}$, which is the maximizing vector field, so $\hat{\boldsymbol v}(\tilde{\boldsymbol x}) = \arg\max_{\boldsymbol v\in\mathcal{S}}\Lambda(\boldsymbol v(\tilde{\boldsymbol x}))$ and $\hat\Lambda := \Lambda(\hat{\boldsymbol v})$. $\mathcal{S} \subset \mathcal{H}^1(\Omega)$ is a subset of the Sobolev space of integrable functions with square-integrable gradients. Additionally, its functions have to be incompressible (as in satisfy equation 10b) and vanish at infinity, corresponding to the boundary condition (10c). In the maximizing state, the variational derivative of (18) has to vanish:

$$0 = \left.\frac{\partial\Lambda(\hat{\boldsymbol v} + \varepsilon\boldsymbol s)}{\partial\varepsilon}\right|_{\varepsilon=0} \quad \forall\, \boldsymbol s \in \mathcal{S}. \tag{19}$$

The maximizing space-function $\hat{\boldsymbol v}$ is perturbed by $\varepsilon\boldsymbol s$. In the limit $\varepsilon = 0$, the variational derivative has to vanish for all perturbation vectors $\boldsymbol s \in \mathcal{S}$.
Applying the derivative (19) to (18) gives, rewritten in vectorial notation,

$$0 = \left\langle \boldsymbol s, \begin{pmatrix}\hat v_2 + 4\frac{\hat\Lambda}{A}\hat v_1\\ \hat v_1 + 4\frac{\hat\Lambda}{A}\hat v_2\end{pmatrix}\right\rangle_{L^2(\tilde\Omega)} \quad \forall\, \boldsymbol s \in \mathcal{S}. \tag{20}$$

Finding solutions to (20) does not guarantee that they satisfy the NEVP (10). On the other hand, it is not guaranteed that solutions to the NEVP (10) are (18)-maximizing in the function space $\mathcal{S}$. From a purely physical point of view, however, we do indeed suspect that the largest eigenvalue is dominant, so that we aim to solve (10) simulatnously with a maximization of (18).

Investigation of the linearized problem using characteristics

To obtain insights into the solutions to (17), equation (17) is linearized by neglecting the non-linear terms. This turns the NEVP into a linear eigenvalue problem with the eigenvalue λ and eigenfunction $\Delta\tilde\psi'$. This equation can be solved using the method of characteristics, resulting in its solution

$$\tilde\Delta\tilde\psi' = F\left(\exp\left(\frac{\lambda\tilde x_1}{A\tilde x_2}\right)\tilde x_2\right), \tag{21}$$

where F is a free function, which is to be determined using the problem's boundary conditions. An exemplary set of characteristics is displayed in Figure 1 for an arbitrary choice of $\frac{\lambda}{A} = 0.1$. As can be seen, all characteristics intersect in the

[1] This is the two-dimensional version of (13).

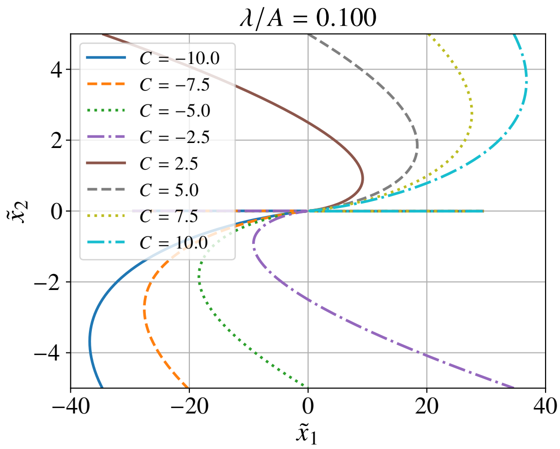

Figure 1. Characteristics of the linearized NEVP for different values of $C = \exp\left(\frac{\lambda\tilde{x}_1}{A\tilde{x}_2}\right)\tilde{x}_2$

origin and extend to "infinity".

The Laplacian of the stream function $\Delta\tilde{\psi}' = \frac{\partial\tilde{u}_1'}{\partial\tilde{x}_2} - \frac{\partial\tilde{u}_2'}{\partial\tilde{x}_1}$ can be understood as the vorticity of the velocity field. We therefore assume $\Delta\tilde{\psi}' = 0$ at the boundary at infinity, resulting in $\Delta\tilde{\psi}' \equiv 0$ everywhere, as $\Delta\tilde{\psi}'$ has to be constant along each characteristic and each one, again, reaches into "infinity". If we were to let go of that assumption of vanishing $\Delta\tilde{\psi}'$ in the boundary, we would have an intersection of non-uniform-valued characteristics in the origin, resulting in a non-physical non-continuous vorticity.

Hence, this first linear analysis shows that the linearized eigenvalue problem is insufficient for describing the growth of turbulent kinetic energy. Consequently, the entire NEVP has to be solved numerically.

Applicability of pre-existing (numerical) solution strategies

As the NEVP (10) is structurally similar to the GPEVP - the eigenvalue appears linearly, with the eigenfunction appearing nonlinearly and being under a norm condition (like equation 12), we turned to Henning & Jarlebring (2025) for an overview over numerical methods.

Here, however, critical differences arose: The solutions to the GPEVP in its PDE-form (similarly to the momentum equation 10a) can be reformulated as extremizers to an energy functional. This functional serves as a versatile basis for numerical schemes. Using the methods outlined by Bluman *et al.* (2010), we were able to show that (10), or equivalently (17), does not possess such a corresponding energy functional.

Another difference is that in the GPEVP, the eigenvalue appears multiplied by the eigenfunction. In contrast, the eigenvalue is multiplied by a derivative operator of the eigenfunction in both (10) and (17). A result of this difference is that the numerical schemes reviewed in Henning & Jarlebring (2025) are not applicable to this work's NEVP. To the authors' knowledge, no pre-existing solution strategies for nonlinear eigenvalue problems can be re-used here.

Numerical analysis

In this work, an approach using the Finite Element Method (FEM) was used to treat the NEVP (10) numerically. For this, the open source FE framework FEniCS (Alnæs *et al.* 2015) was chosen, as it already comes with a vast amount of (non-)linear PDE- and linear eigenvalue-solvers, but, due to its open nature, gives the freedom for adaption to the novel PDE system (10).

A Taylor-Hood element setup (Taylor & Hood 1973), containing a quadratic velocity and linear pressure element, was chosen for numerical stability's sake.

As shown using the method of characteristics, the linearized version of (17) is solved by the zero-solution. Inspecting (17) one can see that the linear problem dominates the problem in a large part of the physical domain: Apart from the region around the two lines $0 = \lambda\tilde{x}_1 - A\tilde{x}_2$ and $0 = \lambda\tilde{x}_2$, the nonlinear terms in the square brackets of (17) are vastly outweighed due to the physical coordinates growing with the domain size. Therefore, the solutions to the original NEVP (10) are mostly zero, apart from those two lines in the $\tilde{x}_1$-$\tilde{x}_2$-space. Preliminary numerical solutions confirm this argument.

The numerical calculations are conducted on a square, truncated domain. Here, the domain size is chosen to be big enough that the solution fields already have fallen off significantly when approaching the zero-valued domain boundaries, in order to minimize the truncation error. To improve numerical accuracy, a non-homogeneous mesh is chosen, with a fine resolution around the origin (where the nonlinear effects of the NEVP play a major role), coarsening exponentially when approaching the boundaries.

During first computations, numerical instabilities have been observed. They can be explained by the absence of dissipative terms in the NEVP (10) and also by the convective terms growing unbounded with the domain size. Here, different remedial strategies have been implemented. Firstly, a Streamline Upwind/Petrov-Galerkin (SUPG), see Brooks & Hughes (1982), has been implemented. Secondly, a dissipative term has been added to (10), in which the newly introduced viscosity parameter ν can be used to stabilize the simulation. The viscosity parameter can be lowered in the following, iteratively keeping the results as initial guesses. Using this strategy, the original NEVP is solved while keeping the advantageous viscosity-related stabilization.

Conclusion and outlook

We have introduced a symmetry-based framework that reinterprets the exponential growth of turbulent kinetic energy in homogeneous shear turbulence as a nonlinear eigenvalue problem. By combining the space-scaling and time-translation symmetries of the Euler equations for perturbations on a general linear base flow into a two-parameter group, the explicit time dependence is removed and the growth rate λ emerges naturally as the eigenvalue of a stationary PDE system. The framework is general: an arbitrary constant velocity-gradient tensor $\boldsymbol{A}$ and system rotation can be accommodated, so that pure shear, pure strain and combined strain–shear flows all fall within the same formulation.

Two analytical results were established. First, an explicit expression (13) relates the eigenvalue to its eigenfunction and yields rigorous bounds, $|\lambda| \leq \frac{2}{5}\|\boldsymbol{A}\|_2$ in three dimensions and $|\lambda| \leq \frac{1}{4}|A|$ for two-dimensional shear, and hence an a-priori upper bound on the kinetic-energy growth. Second, a characteristics analysis of the linearized two-dimensional problem admits only the trivial solution, proving that the observed growth is an inherently nonlinear phenomenon and that the full NEVP must be solved.

Because the NEVP possesses no underlying energy functional and, unlike the Gross–Pitaevskii eigenvalue problem, carries the eigenvalue on a derivative of the eigenfunction, established numerical strategies are not directly transferable. As a first step towards a numerical treatment we have set up a finite-element discretization in FEniCS using Taylor–Hood elements. Numerical instabilities, attributable to the absence of dissipative terms and to the convective terms growing with the domain size, were encountered; SUPG stabilization and an artificial-viscosity term whose magnitude can be reduced iteratively were introduced to address them. Preliminary computations are consistent with the analytical expectation that the solutions concentrate along the lines $\lambda\tilde{x}_1 = A\tilde{x}_2$ and $\tilde{x}_2 = 0$, away from which the linear terms dominate.

The central open question – a first-principles prediction of the value $\lambda/A \approx 0.057$ – remains the goal of ongoing work. Obtaining robustly converged solutions of the full three-dimensional, optionally rotating, NEVP and evaluating the explicit expression (13) should provide the sought theoretical estimate and allow the influence of rotation and mean-flow strain on the growth rate to be assessed.

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