TURBULENT PUFF RELAMINARIZATION IN PIPE FLOW, SADDLE-NODE BIFURCATION AND AIRY EQUATION

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ABSTRACT

Turbulent puffs in pipe flow remain sustained for an extended period of time before abruptly transitioning to laminar flow through viscous exponential decay. During the statistically stable state, the topology of sectional streamlines reveals the presence of numerous saddle and nodal points in proximity to the wall. The number of saddle and node points decreases as relaminarization begins, and they migrate away from the near-wall region. In the final stage of relaminarization, direct numerical simulations reveal in some cases sequences of saddle–node bifurcations that ultimately lead to the annihilation of saddle-node pairs. The Riccati-based Airy equation governs the spatial-temporal evolution of these pairs.

1 INTRODUCTION

Turbulence is manifested in pipe as localized “puffs”, which are confined “equilibrium” patches surrounded by laminar flow both upstream and downstream (Wygnanski & Champagne, 1973; Wygnanski *et al.*, 1975). Turbulent puffs have been experimentally detected over a broad range of Reynolds numbers, Re=1500–2050 (Hof *et al.*, 2008*a*,*b*). (Here and hereafter $Re = U_m D/\nu$, where U_m is the mean axial velocity, D is the diameter of the pipe, and ν is the kinematic viscosity of the fluid.) The term “equilibrium puffs” implied that they are energetically sustained substances of constant size, i.e., “one which does not grow, split or shrink” (Wygnanski *et al.*, 1975). The conclusion that it “maintains itself indefinitely at around Re = 2200” was premature, given the observation times available in experiments (Wygnanski *et al.*, 1975). Thirty years after turbulent puffs were first found, it was discovered that they are not sustained indefinitely and that their lifetime is finite. Extensive experimental measurements (Hof *et al.*, 2006; Peixinho & Mullin, 2006) and numerical simulations (Willis & Kerswell, 2007; Avila *et al.*, 2011) showed that puffs survive for a long time before abruptly exiting the chaotic state, after which they decay or split as the Reynolds number increases. This may be attributed to a breakdown of the sustaining mechanism, although the underlying cause remains unclear.

To approximate the Reynolds number dependence of the mean lifetime of a puff before sudden decaying or splitting, experimental measurements and direct numerical simulation (DNS) results were approximated by a two-branch $\Lambda-$type superexponential fit, $\tau(Re) = \exp[\exp(aRe+b)]$ (Avila *et al.*, 2011). The intersection of the two branches suggests that Re =2040$\pm$10 is the critical limit below which the mean lifetime of a puff before decaying is less than the mean lifetime before a splitting event. In the Reynolds number range $Re = U_m D/\nu$=1720–2040, the mean lifetime of puffs ranges from a few hundred to $O(10^7)$ advective time units (D/U_m)Avila *et al.* (2011). It is expected that the relaminarization in the final stage occurs through the viscous exponential decay of turbulent energy.

The objective of this study is to monitor turbulent puffs throughout the entire relaminarization process, with particular emphasis on the evolution of flow topology. Visualizing the three-dimensional topology and patterns of unsteady flows is conceptually difficult. A methodology to address this problem in identifying vortex motions in turbulence using the concept of critical points, i.e., the locations where the velocity vanishes, and the streamline slope is indeterminate (Perry & Chong, 1987, 1994). Specifically, the topology of the instantaneous streamline patterns was classified in terms of critical points. In this study, we employed sectional instantaneous streamlines (Bisset *et al.*, 2002) to illustrate the spatiotemporal evolution of organized flow structures in the transition region, that is, near the trailing edge of the puff. The in-plane flow in a pipe is a signature of the disturbed laminar flow, while sectional streamlines provide a two-dimensional cut of a three-dimensional motion through a cross-sectional plane. A detailed examination of the sectional streamline patterns surprisingly uncovered the existence of saddle and nodal points Similar findings have been reported near the turbulent/non-turbulent interface by analyzing the velocity fields of a turbulent wake behind a flat plate in the intermittent zone (Bisset *et al.*, 2002). This study, therefore, examines the temporal evolution of saddle-node pairs during the relaminarization of turbulent puffs in pipe flow.

2 RESULTS

In this article, we report the results of direct numerical simulations (DNS) of laminar-turbulent transitional flow in a pipe. We used the OpenpipeflowWillis (2017) Navier-Stokes solver, the Reynolds number range is $Re = U_m D/\nu$=1880–1920, and the mean axial velocity U_m was fixed. The computational domain length was set to $L = 16\pi D \approx 50D$ to ensure that a puff is surrounded by laminar flow. The flow is periodic in the axial direction, and the velocity field is expanded in a Fourier series in both the azimuthal (θ) and streamwise (z) directions. Further details regarding the computational setup can be found in Khan *et al.* (2024).

Every 50 times steps, the data were collected inside an $8D$-width window centered around the location $z = 0$, where the instantaneous transverse motion energy $e_\perp = \int\int(u_r^2 + u_\theta^2) r dr d\theta$ was found to be the highest; hereafter, z denotes the location in the reference frame linked to the moving puff. The simulations were started from snapshots of a localized turbulent puff and continued until complete relaminarization. In the Openpipeflow code, the heuristic criterion for relaminarization is defined as the instant at which the total energy, after extracting the energies of the axial mean components, drops below 10^{-5}.

The transverse (turbulent) motion velocity field $\mathbf{u}_\perp = (u_r, u_\theta)$ contains the radial (u_r) and azimuthal (u_θ) velocity components. The data were collected inside an $8D$-width window centered at $z = 0$, where the instantaneous transverse motion energy $e_\perp = \int\int(u_r^2 + u_\theta^2) r dr d\theta$ was found to be the highest; hereafter, z denotes the location in the reference frame linked to the moving puff. At $z = -8D$, the laminar parabolic velocity profile remains nearly undisturbed; however, at $z = -4D$, the onset of turbulence was found to be evident from a slight decrease in centerline velocity and the appearance of an inflection point in the velocity profile, consistent with the observations of reference Hof *et al.* (2010). In the following discussion of the results, the laminar-turbulent transition location is referred to as $z = -4D$.

2.1 Relaminarization: Viscous Exponential Decay

Figure 1 (left panel) shows the temporal evolution of the energy of streamwise velocity "perturbations" [1] (e_z) for 22 initial conditions for each of the three Reynolds numbers, Re =1880, 1900 and 1920. The curves visually confirm the abrupt transition to the relaminarization (Avila *et al.*, 2023), as well as the exponential (viscous) decay to laminar flow. The 66 curves are aligned by a decay time, defined as a countdown from t=0, indicating complete relaminarization for each curve. The regime remains statistically steady up at least to 200 D/U_m time units before full relaminarization in all cases, and this regime is therefore referred to as "*statistically stable*" (e.g., at $tU_m/D < -200$). At the final stage of relaminarization ($e_z < 0.01$), e_z, decreases by an order of magnitude every 33 D/U_m time units (Khan *et al.*, 2024).

Figure 1 (right panel) displays a phase space projection of chaotic trajectories ($\mathscr{D}$ vs. e_z). Here, all trajectories, regardless of initial conditions, fall into a chaotic (turbulent) "trap". They remain there for a long, uncertain time, but not forever. All trajectories eventually escape, exhibiting relaminarization. A pattern like this is consistent with a concept of a *chaotic saddle* in phase space. (Avila *et al.*, 2023).

[1] In the Openpipeflow code, the velocity field is decomposed as perturbations from a laminar base parabolic profile. Therefore, the term "turbulent" will henceforth refer to the field after the laminar one has been extracted.

2.2 Saddle-Node Bifurcation

Sectional streamlines (u_r, u_θ) were employed to visualize the local flow structures at a transition region, $z = -4D$, which is the trailing edge of a turbulent puff. The streamlines show the instantaneous flow trajectories of the fluid moving radially and azimuthally at a fixed axial position. Figures 2–3 depict the in-plane flow field before and after the abrupt onset of exponential decay to laminar flow. In addition to streamlines, streamwise streaks are depicted by mapping perturbations (u_z) in the streamwise velocity.

Panels in figure 2 show sectional streamlines at three time instances within the statistically stable stage. The nodal points at the center (0, 0) are the outcome of projecting the three-dimensional flow onto the cross-sectional plane. In the in-plane velocity field, streamlines converge in one direction and diverge in another, forming a saddle (stagnation) point; nodal points are also stagnation points, but the streamlines move either towards the point or away from it (Perry & Chong, 1987).

Before the onset of decay ($t < 1100$), numerous saddle-node pairs, S/N, are observed in the near-wall region. Some persisted for a long time, while others existed only for a short period of time. They appear anywhere in the cross-section, are distributed sporadically, and eventually disappear.

We study the organized flow structures, focusing on the near-wall streaks and the in-plane (u_r, u_θ) sectional streamlines at the laminar-turbulent upstream transition region, $z = -4D$. Figure 3 offers an additional perspective on the flow structure following the onset of transition to laminar flow. Analysis of the in-plane sectional streamline patterns revealed the presence of saddle and nodal points in the flow field. Similar findings have been reported near the turbulent/non-turbulent interface analyzing the sectional streamlines topology of the velocity fields of a turbulent wake behind a flat plate in the intermittent zone (Bisset *et al.*, 2002). From figure 3, we identify saddle (S_4) and nodal (N_4) points that are a sample of a saddle-node pair (S_4/N_4). They become closer over time, until they collide and disappear in the panel $t = 1112$.

The described behavior resembles a saddle-node bifurcation developing over time, culminating in annihilation. The topological collapse is not sudden; it develops gradually, suggesting a mechanism like critical slowing down near saddle-node bifurcations of dynamical systems. In figure 4a, we show the distance between a saddle point and a nodal point, identified as a pair, as a function of the square root of the DNS time of the corresponding snapshot. Straight lines indicate the *square-root scaling law*, a *very general feature of systems that are close to a saddle-node bifurcation* (Strogatz, 2018): $r_{SN} \propto \sqrt{\tau^* - \tau}$. This provides evidence that the disappearance of turbulence-supporting structures occurs via a bifurcation mechanism, rather than being purely stochastic.

An ordinary differential equation $dr/dt = \mu - r^2$ ($\mu > 0$ is a constant) is a classic example a system with the saddle-node bifurcation. It has two fixed points: $r_1 = \sqrt{\mu}$ (stable) and $r_2 = -\sqrt{\mu}$ (unstable). As μ goes down, the two fixed points get closer until they hit each other at $\mu = 0$ and disappear, indicating annihilation. In the context of puff relaminarization in this study, the square-root scaling of the distance between the saddle and node points, $r_{SN} \propto \sqrt{\tau}$, means that a S/N pair moves along a parabola, as illustrated in figure 4b, where the r-location of the S/N pairs of snapshots of different times are projected onto a (r, τ) parabolic curve.

Based on the parabola in figure 4b, we propose a simple illustrative model of the saddle-node bifurcation:

$$\frac{dr}{dt} = t - r^2, \qquad (1)$$

where r is the r-coordinate of the saddle and node points. Equation (1) is the Riccati equation. The Riccati equation can be transformed into a second-order linear differential equation, known as the Airy equation. Indeed, substituting $r = \dot{u}/u$ into (1) yields the second order Airy equation (Vallee & Soares, 2010) for $u(t)$: $\ddot{u} - tu = 0$. Therefore, the Airy equation, which arises in the analysis of dynamics near critical points in physics phenomena (Vallee & Soares, 2010), leaves its signature by describing the saddle-node bifurcation that is observed during the relaminarization of a turbulent puff.

3 CONCLUDING REMARKS

To demonstrate the relaminarization of turbulent puffs in pipe flow, sectional streamlines at the upstream laminar–turbulent interface are employed. Sectional streamlines reveal saddle and nodal points that develop (and vanish) irregularly close to the wall during the statistically stable turbulent (chaotic) regime, showing no deterministic pattern over time. The nonlinear dynamics governed by the Navier-Stokes equations are reflected in these features. Upon the onset of relaminarization, the number of saddle and node points decreases. They gradually migrate toward the flow core, leaving a near-wall region that has become laminar, as evidenced by the mapping of streamwise velocity perturbations. Small turbulent eddies decay at a significantly faster rate than large-scale streamwise structures in the statistically stable regime, which is highly anisotropic. This eliminates the streamwise vorticity, thereby disrupting the self-sustaining mechanism by which large-scale structures extract energy from the mean shear flow.

During the final stage of relaminarization, the energy of the streamwise velocity perturbations is less than 0.001, whereas the total energy associated with transverse (in-plane) motion is at least three orders of magnitude lower. The saddle-node pairs that are still present at this stage are indicative of the regime developing toward the final laminar state. The saddle-node bifurcation is one possible explanation for their ultimate disappearance. In these cases, it was shown that the Riccati-based Airy equation can be used to describe the spatial and temporal evolution of saddle-node pairs.

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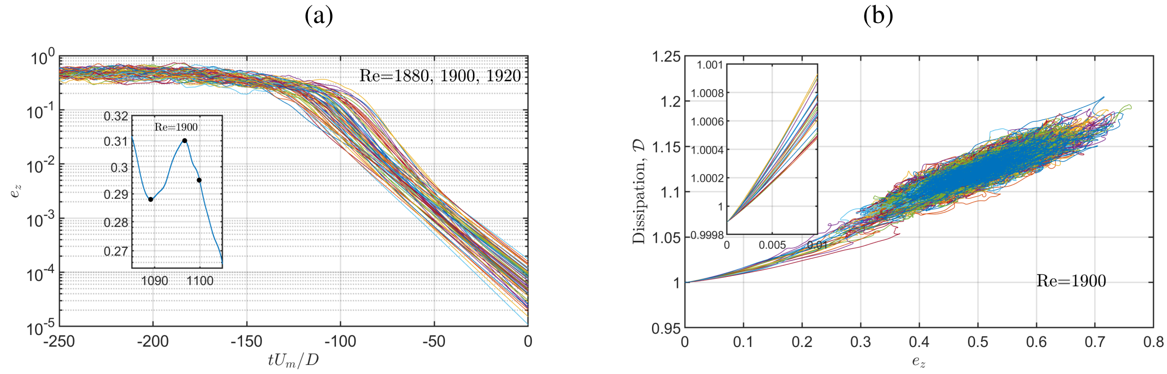

Figure 1. (a) The last 250 advective time units (D/U_m) before complete relaminarization at t=0; 22 initial conditions for each Re. (b) Typical for all Reynolds numbers, phase-space projection of chaotic trajectories (including the relaminarization stage) of the total energy of streamwise velocity fluctuations, $e_z = \int\int\int u_z^2 r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$, and the total energy dissipation, $\mathscr{D}$, computed for different initial conditions.

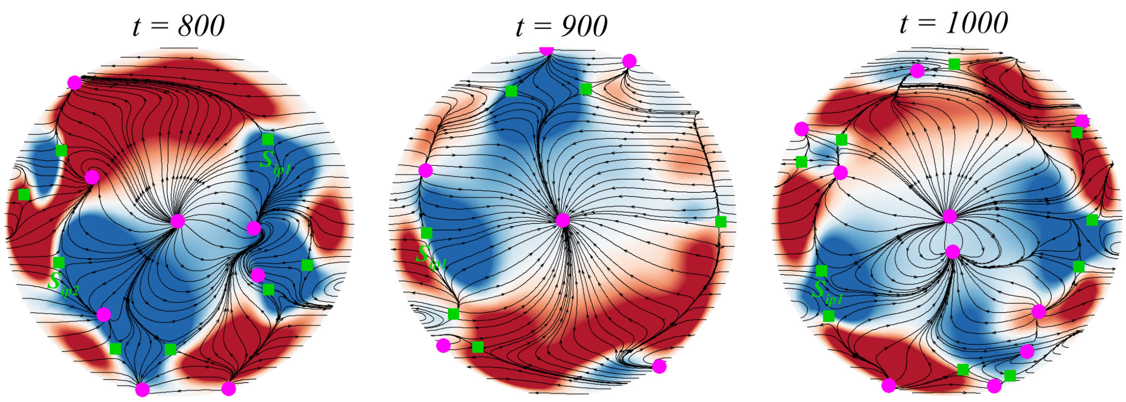

Figure 2. $Re = 1900$, $z = -4D$. Typical sectional streamlines (u_r, u_θ), along with color mappings of the streamwise velocity perturbations (u_z); time (t) in D/U_m units; squares (green) and circles (purple) denote saddle and nodal points, respectively.

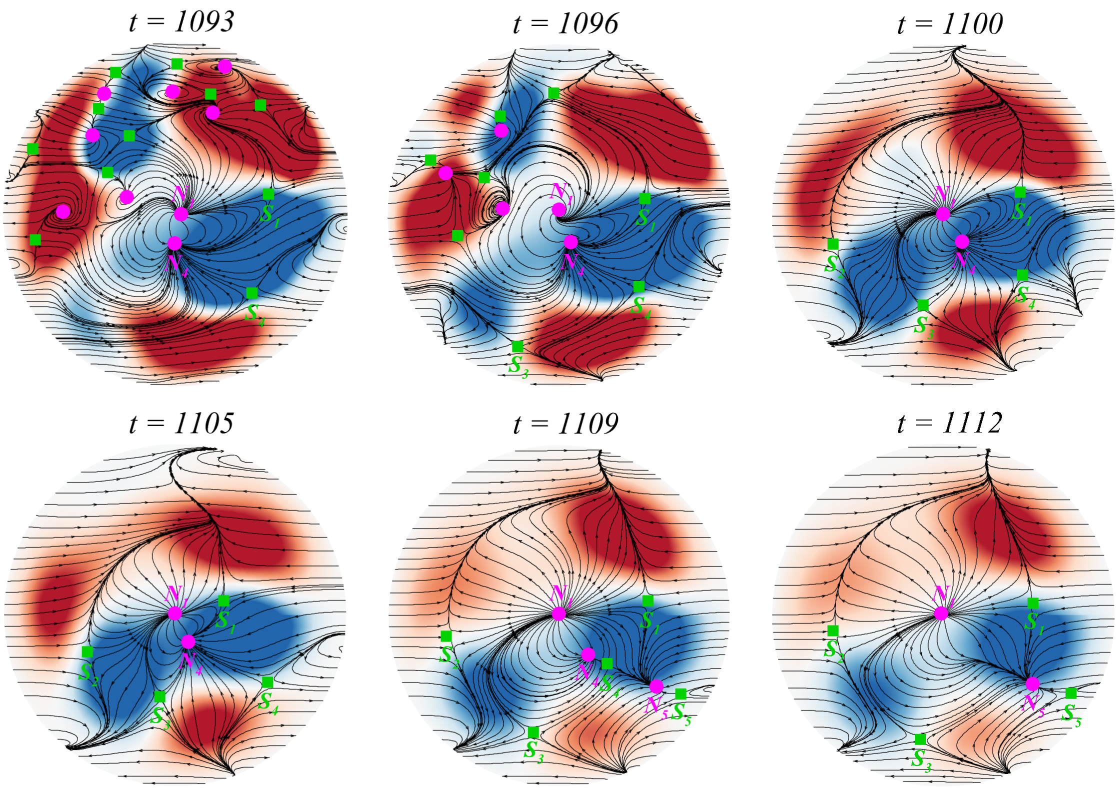

Figure 3. $Re = 1900$, $z = -4D$. Sectional streamlines (u_r, u_θ), along with color mappings of the streamwise velocity perturbations (u_z) are shown at different time instants during the decay stage; squares (green) and circles (purple) denote saddle and nodal points, respectively, S_i and N_i denote pairs of saddle and nodal points. The onset of relaminarization at $t = 1097$, complete relaminarization at t = 1220.

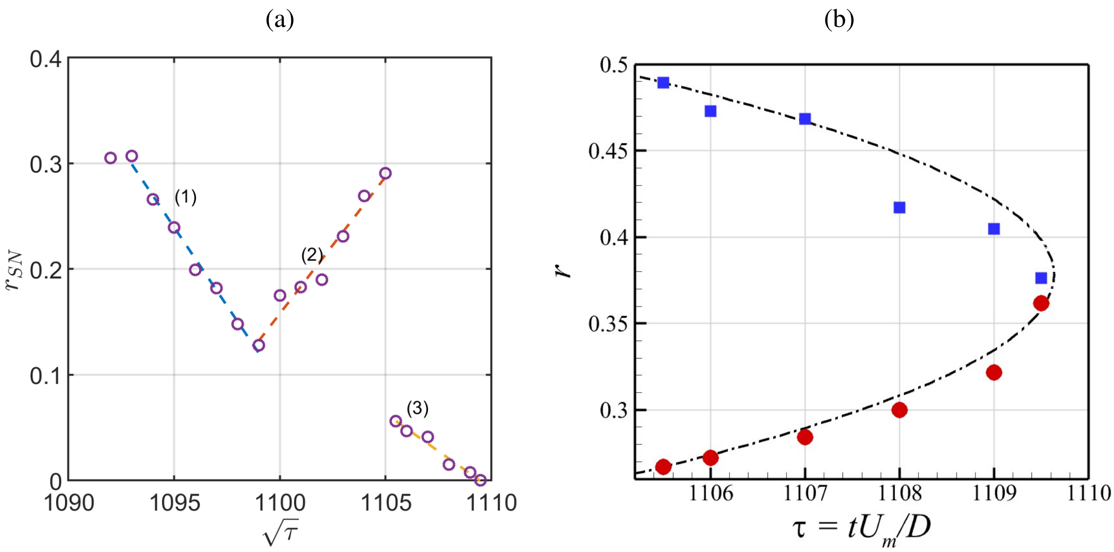

Figure 4. $Re = 1900$, $z = -4D$. The saddle-node annihilation of the S_4/N_4 pair shown in figure 3. (a) The square-root scaling, $r_{SN} \propto (\tau_* - \tau)^{1/2}$; r_{SN} is a saddle-node distance. Three distinct stages: approach (1), repulsion (2) and annihilation at the final stage (3). (b) The S_4/N_4 pair moves along a parabola $t_a - t \propto (r - r_a)^2$; r is the r-coordinate of a saddle and a node point, t_a and r_a are the time and the coordinate, respectively, during the annihilation stage.