14th International Symposium on Turbulence and Shear Flow Phenomena (TSFP14)
Heidelberg, Germany, July 28–31, 2026

DIRECT ESTIMATION OF MULTIFRACTAL DIMENSION OF TURBULENCE AT HIGH REYNOLDS NUMBERS

Susumu Goto
Graduate School of Engineering Science,
the University of Osaka
1-3 Machikaneyama, Toyonaka,
Osaka, 560-8531 Japan
s.goto.es@osaka-u.ac.jp

Daiki Watanabe
Graduate School of Engineering Science,
the University of Osaka
1-3 Machikaneyama, Toyonaka,
Osaka, 560-8531 Japan
d_watanabe@fm.me.es.osaka-u.ac.jp

Tsuyoshi Yoneda
Graduate School of Economics,
Hitotsubashi University
2-1 Naka, Kunitachi, Tokyo 186-8601, Japan
t.yoneda@r.hit-u.ac.jp

ABSTRACT

We propose a method to accurately evaluate the intermittency of developed turbulence and report its application to direct numerical simulation data of high-Reynolds-number (Taylor-length Reynolds number is up to about 1700) turbulence in a periodic cube. Scale-decomposition by a Fourier band-pass filter and objective vortex identification by the low-pressure method are employed to obtain the set of vortex axes at each scale in the inertial range. The set of vortex axes thus obtained is space-filling, with the capacity dimension of 3. However, following the multifractal theory, identifying the set of vortex axes for a given Hölder exponent allows the evaluation of its fractal dimension $D(h)$. The obtained result of $D(h)$ is in good agreement with the one previously estimated in experiments. Thus, the present study establishes a direct estimation method of the multifractal dimension $D(h)$.

Introduction

Kolmogorov (1941)'s similarity hypothesis states that the small-scale statistics of turbulence are determined by the mean energy dissipation rate $\overline{\varepsilon}$ and the fluid's kinematic viscosity ν. This simple similarity hypothesis is powerful enough to predict the statistics of turbulence. For example, the $-5/3$ power law of the energy spectrum derived from the hypothesis is supported by experiments, and there is also the evidence that its coefficient is universal Sreenivasan (1995).

However, since turbulence strongly fluctuates in space and time, the mean energy dissipation rate cannot accurately describe the statistics and we need to consider fluctuations of the dissipation rate ε. It is well known that Landau & Lifshitz (1987) pointed out the non-universality of the temporal fluctuations of ε might lead to the violation of Kolmogorov's similarity hypothesis. Furthermore, there is criticism that energy cascade strongly localizes the energy dissipation rate in space, and thus their spatial fluctuations must be taken into account (Kraichnan, 1974; Frisch, 1995). We sometimes use the p-th order velocity structure functions, $S_p = \overline{(\Delta u(r))^p}$, where $\Delta u(r)$ denotes the longitudinal velocity difference at distance r, to evaluate the spatial intermittency. Then, assuming the scaling law,

$$S_p \sim r^{\zeta_p} \quad \text{(in the inertial range)}, \tag{1}$$

we define the scaling exponent ζ_p for the p-th order velocity structure function. If Kolmogorov's similarity hypothesis holds, the dimensional analysis leads to $S_p \sim \overline{\varepsilon}^{\frac{p}{3}} r^{\frac{p}{3}}$, that is, $\zeta_p = p/3$. However, many experiments showed that $\zeta_p \neq p/3$, especially for large p (Anselmet *et al.*, 1984; Dubrulle, 2019). This means that the Kolmogorov similarity hypothesis requires a correction. This issue of the spatial intermittency (Frisch, 1995) has been extensively studied by meany authors including Kolmogorov (1962) himself.

The present study addresses this classical problem of turbulence with the aid of direct numerical simulations (DNS). Specifically, we aim at proposing a method to accurately evaluate the intermittency of fully developed turbulence by using DNS data at high Reynolds numbers.

To this end, we use datasets of DNS of turbulence of an incompressible fluid in a periodic cube. These datasets were obtained by our DNS (Tsuruhashi *et al.*, 2022), where the Taylor-length based Reynolds number R_λ is moderate ($\lesssim 500$). In addition to these datasets, we also analyze DNS data, which was provided by Ishihara *et al.* (2016), of turbulence at $R_\lambda \approx 1700$.

Methods

We investigate the origin of the spatial intermittency in turbulence through detailed analysis of the DNS data of high-Reynolds-number turbulence. The analysis proceeds in two stages. The first is scale-decomposition. Since developed turbulence consists of vortices of various sizes, its hierarchical structure cannot be captured without proper scale-decomposition. The second is the objective identification of vortices. Visualization using isosurfaces of some quantity is often employed, but the arbitrariness of the threshold makes objective quantification difficult.

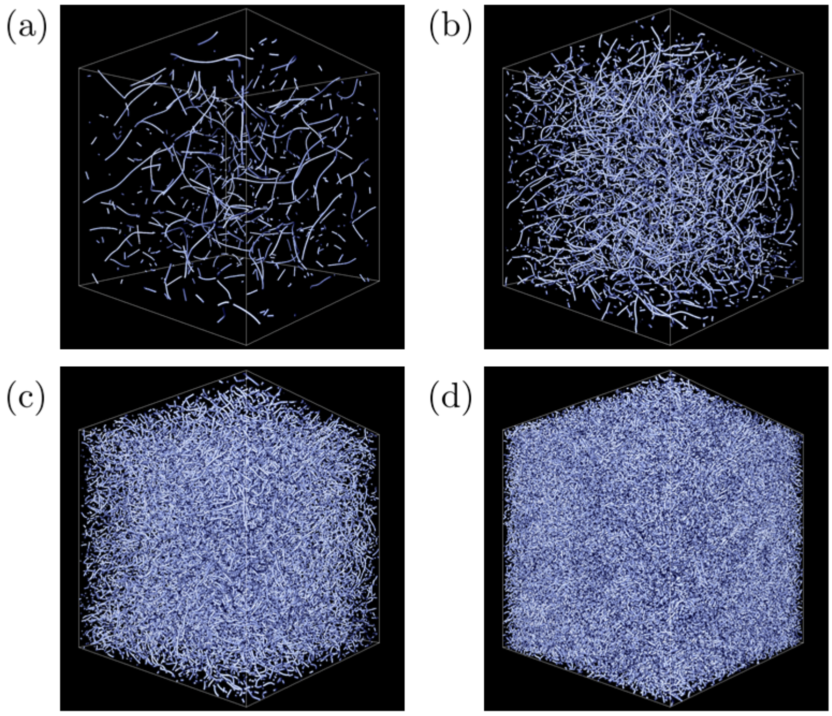

Figure 1. Vortex axes, which are identified by the scale-decomposition and the low-pressure method, in turbulence at $R_\lambda \approx 1700$. The characteristic length scale ℓ_i is (a) 5800η, (b) 2800η, (c) 1400η, (d) 650η. Here, η denotes the Kolmogorov length.

(i) Scale decomposition. As emphasized in our previous studies (Goto, 2008, 2012; Goto *et al.*, 2017; Goto & Motoori, 2024), the enstrophy in developed turbulence always highlights the smallest-scale vortices. This is because when the energy spectrum $E(k)$ is proportional to $k^{-5/3}$, the enstrophy spectrum $k^2E(k)$ scales as $k^{1/3}$, and thus highest-wavenumber (i.e., smallest-scale) structures determine the velocity gradients, and therefore the enstrophy.

Therefore, we use the scale-decomposition to capture the hierarchy of vortices. Since we are dealing with turbulence under periodic boundary conditions, we employ a Fourier band-pass filter on the velocity field for the scale-decomposition. First, we define the representative wavenumber k_i as $k_0\, b^i$, where b is the interval between the bands to be analyzed and can be chosen arbitrarily. Then, the i-th band is defined as $k \in [k_i/\sqrt{a}, k_i\sqrt{a})$. Here, a is a constant determining the band width, and we set $a = 2$ in the present study.

The concrete procedure of the scale-decomposition is as follows. We first Fourier transform $\boldsymbol{u}(\boldsymbol{x})$ obtained by DNS to $\widehat{\boldsymbol{u}}(\boldsymbol{k})$. We set all modes outside the i-th band $k \in [k_i/\sqrt{a}, k_i\sqrt{a})$ as $\boldsymbol{0}$, then perform the inverse Fourier transform to obtain $\boldsymbol{u}_b^{(k_i)}(\boldsymbol{x})$. This can be regarded as the flow field at scale $\ell_i = 2\pi/k_i$.

(ii) Low-pressure method. We objectively identify vortices using the so-called low-pressure method (Miura & Kida, 1997; Kida & Miura, 1998). This method was proposed to identify the axes of the smallest vortices in turbulence. Here, we apply this method to vortices on each scale. More concretely, we solve the Poisson equation using the scale-decomposed velocity field $\boldsymbol{u}_b^{(k_i)}(\boldsymbol{x})$ to obtain the pressure field $p_b^{(k_i)}(\boldsymbol{x})$ on this scale. We then compute the Hessian of this scale-decomposed pressure field and identify vortex axes based on the condition that $p_b^{(k_i)}$ takes a local minimum on the plane spanned by the eigenvectors of the Hessian. A more detailed procedure is described by Tsuruhashi *et al.* (2022).

Results

Figure 1 shows the vortex axes identified for different length scales in the inertial range of turbulence at $R_\lambda \approx 1700$. It presents results for the scales $\ell_i = 5800\eta$, (b) 2800η, (c) 1400η and (d) 650η, where η denotes the Kolmogorov length. The impression from these figures is different from visualizations of the enstrophy in turbulence (or visualizations of scale-decomposed enstrophy). That is, the vortex axes are observed to be space-filling irrespective of ℓ_i.

To evaluate the fractal dimension of these vortex axes, we calculate the total length of the vortex axes for each scale. Results are shown in figure 2(a) for moderate R_λ and (b) for $R_\lambda = 1700$. These figures show that the total length L is always proportional to k_i^2 independently of R_λ. Since the characteristic length ℓ_i is k_i^{-1}, $L \sim \ell_i^{-2}$. Therefore, the number of spheres of size ℓ_i required to cover the set of the vortex axes $L/\ell_i \sim \ell_i^{-3}$. This means that the capacity dimension of the set of the vortex axes is always 3.

These results [figures 1 and 2(a, b)] mean that the vortices identified by the low-pressure method are space-filling and they do not exhibit intermittency. This may be because the low-pressure method identifies vortices regardless of their strength.

Therefore, we focus on the multifractal theory (Parisi & Frisch, 1985) to capture the intermittency. In the theory, the scaling of the velocity field is expressed as $\Delta u \sim r^h$ with the exponent h (the Hölder exponent), and we assume that the set of a given h is fractal with dimension $D(h)$.

On the basis of this idea, we consider the set of vortex axes identified above for a given h and evaluate their total length. As examples, results for $h = 0.32$ and 0.065 are shown in figure 2(c). For $h = 0.32$, the slope is approximately 2, which is similar to the case for the entire vortex axes. However, for $h = 0.065$, the slope is significantly shallower than 2. Therefore, $D(h)$ is approximately 3 for $h = 0.32$, but it is much smaller than 3 for $h = 0.065$.

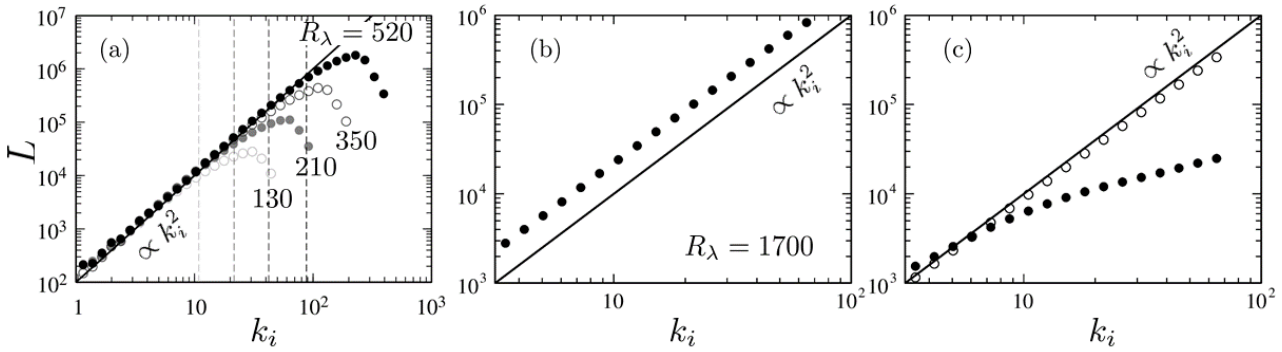

Figure 2. (a, b) The total length L of the vortex axes as functions of the wavenumber k_i of the band. It is seen that $L \sim k_i^2$ irrespective of the Reynolds number. (c) The total length of the vortex axes conditioned by the Hölder exponent: open circles, $h = 0.32$; closed circles, $h = 0.065$. $R_\lambda = 1700$.

Thus, we can precisely determine the fractal dimension $D(h)$ for each h. The results are shown in figure 3(a). The result is consistent with previous findings (Meneveau & Sreenivasan, 1987, 1991; Sreenivasan, 1991; Dubrulle, 2019; Nguyen *et al.*, 2019), and the result for ζ_p [figure 3(b)], which is obtained using the Legendre transform $\zeta_p = \min_h (3 - D(h) + ph)$, also agrees well with the existing experimental data.

Conclusions

We have proposed a method to accurately quantify the spatial intermittency of turbulence on the basis of vortical structures. By combining the Fourier band-pass filter with the low-pressure method, we have objectively identified the hierarchy of vortex axes in turbulence at high Reynolds numbers up to $R_\lambda = 1700$ (figure 1). The total length of the identified vortex axes is proportional to the square inverse of the characteristic length ℓ_i [figures 2(a, b)]. This implies that the vortex axes are space-filling. In contrast, the total length of the vortex axes conditioned by the Hölder exponent h varies with h [figure 2(c)]. Furthermore, the dimension $D(h)$ evaluated from the conditioned total length of vortex axes is consistent with the multifractal picture (figure 3). The proposed method for the direct estimation of $D(h)$ enables us to discuss the origin of the spatial intermittency in terms of coherent structures in turbulence. Details will be discussed in the symposium.

Acknowledgments

The authors thank Prof. Takashi Ishihara for providing us with the DNS data. This study was partly supported by JSPS Grants-in-Aid for Scientific Research (24H00186, 25K01158, and 25K01158) and Grant-in-Aid for Transformative Research Areas (JP26H00389). The numerical analyses were conducted by using the computational resources of the supercomputer Fugaku through the HPCI System Research Projects (hp240278 and hp260139) and the Plasma Simulator under theauspices of the NIFS Collaboration Research Programs (NIFS24KISC007 and NIFS26KISC034).

REFERENCES

Anselmet, F., Gagne, Y., Hopfinger, E.J. & Antonia, R.A. 1984 High-order velocity structure functions in turbulent shear flow. *J. Fluid Mech.* **140**, 63–89.

Dubrulle, Bérengère 2019 Beyond Kolmogorov cascades. *Journal of Fluid Mechanics* **867**, 1–63.

Frisch, U. 1995 *Turbulence, The Legacy of A. N. Kolmogorov*. Cambridge Univ. Press.

Goto, S. 2008 A physical mechanism of the energy cascade in homogeneous isotropic turbulence. *J. Fluid Mech.* **605**, 355–366.

Goto, S. 2012 Coherent structures and energy cascade in homogeneous turbulence. *Prog. Theo. Phys. Suppl.* **195**, 139–156.

Goto, S. & Motoori, Y. 2024 Hierarchy of coherent vortices in developed turbulence. *Rev. Mod. Plasma Phys.* **8**, 23.

Goto, S., Saito, Y. & Kawahara, G. 2017 Hierarchy of antiparallel vortex tubes in spatially periodic turbulence at high reynolds numbers. *Phys. Rev. Fluids* **2**, 064603.

Ishihara, T., Morishita, K., Yokokawa, M., Uno, A. & Kaneda, Y. 2016 Energy spectrum in high-resolution direct numerical simulations of turbulence. *Phys. Rev. Fluids* **1**, 082403(R).

Kida, S. & Miura, H. 1998 Swirl condition in low-pressure vortex. *J. Phys. Soc. Jpn.* **67**, 2166–2169.

Kolmogorov, A. N. 1941 The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. *Dokl. Akad. Nauk SSSR* **30**, 301–305, English translation in Proc. Roy. Soc. London, Ser. A **434**, 9–13 (1991).

Kolmogorov, A. N. 1962 A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high reynolds number. *J. Fluid Mech.* **13**, 82–85.

Kraichnan, R. H. 1974 On kolmogorov's inertial-range theories. *J. Fluid Mech.* **62**, 305–330.

Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics, (second edition)*. Pergamon, Oxford.

Meneveau, Charles & Sreenivasan, K. R. 1987 The multifractal spectrum of the dissipation field in turbulent flows. *Nuclear Physics B (Proceedings Supplements)* **2**, 49–76.

Meneveau, C. M. & Sreenivasan, K. R. 1991 The multifractal nature of turbulent energy dissipation. *J. Fluid Mech.* **224**, 429–484.

Miura, H. & Kida, S. 1997 Identification of tubular vortices in turbulence. *J. Phys. Soc. Jpn.* **66**, 1331–1334.

Nguyen, F., Laval, J.-P., Kestener, P., Cheskidov, A. & Shvydkoy, R. 2019 Local estimates of Hölder exponents in turbulent vector fields. *Physical Review E* **99** (5), 053114.

Parisi, G. & Frisch, U. 1985 On the singularity structure of fully developed turbulence. In *Turbulence and Predictability in Geophysical Fluid Dynamics* (ed. M. Ghil, R. Benzi

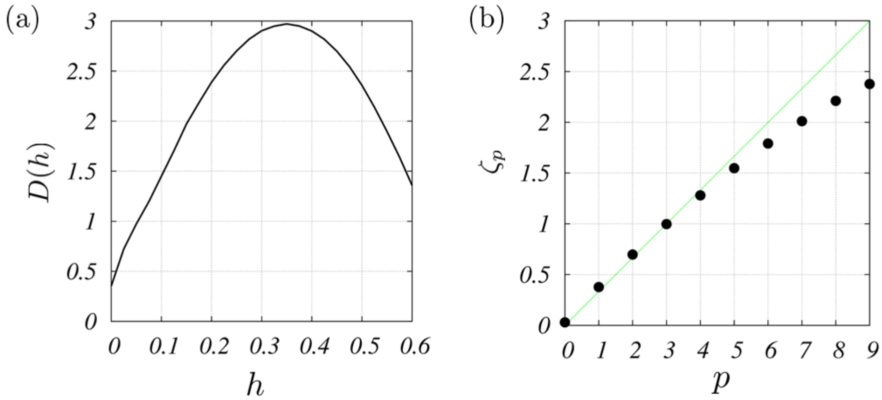

Figure 3. (a) The multifractal dimension $D(h)$ directly estimated by the proposed method. (b) The scaling exponent ζ_p evaluated by the Legendre transform of $D(h)$. Solid line indicates $\zeta_p = p/3$.

& G. Parisi), pp. 84–87. North-Holland.

Sreenivasan, K. R. 1991 Fractals and multifractals in fluid turbulence. *Annual Review of Fluid Mechanics* **23**, 539–600.

Sreenivasan, K. R. 1995 On the universality of the kolmogorov constant. *Phys. Fluids* **7**, 2778–2784.

Tsuruhashi, T., Goto, S., Oka, S. & Yoneda, T. 2022 Self-similar hierarchy of coherent tubular vortices in turbulence. *Phil. Trans. Royal Soc. A* **380**, 20210053.