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SYMMETRY-BASED PARAMETRIZATION OF THE STABLE ATMOSPHERIC BOUNDARY LAYER

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ABSTRACT

This work concerns stably stratified atmospheric boundary layers (ABL), which form mostly over night due to surface radiative cooling. Downward transport of heat towards the surface produces negative buoyancy flux and suppresses turbulent motions. Weakly stably stratified boundary layers are described under the Monin-Obukhov similarity theory (MOST), in which the characteristic length scale is the Obukhov length L, constructed based on turbulent momentum and heat fluxes. However, as the stratification increases the Monin-Obukhov theory describes boundary layer less well. The present paper focuses on local similarity theories provided by invariant solutions, defined under the Lie–group theory, of a set of governing equations for turbulence statistics. In addition to the local Obukhov length, the derived invariant solutions also depend on non-dimensionalized time. We present experimental verification of the derived formulas and show that they improve parametrization of the strongly stratified ABL.

MONIN-OBUKHOV SIMILARITY THEORY

Similarity theories are a key methodology for analyzing flows in the ABL. They seek universal relationships between different physical variables describing the phenomenon, without explicitly solving the governing equations. The first and most celebrated similarity theory of ABLs was proposed by Monin & Obukhov (1954). Those authors defined the Obukhov length scale

$$L = -\frac{1}{\kappa}\frac{|\overline{uw}_0|^{3/2}}{\overline{wb}_0} = \frac{1}{\kappa}\frac{u_*^2}{b_*} \tag{1}$$

where $\overline{uw}_0$ and $\overline{wb}_0$ are the surface values of momentum and buoyancy fluxes, respectively. The buoyancy, b, is defined as

$$b = g(\theta - \theta_0(z))/\theta_m, \tag{2}$$

where $\theta - \theta_0(z)$ denotes deviation of the potential temperature, θ, from a steady reference state and θ_m is a vertical average.

In MOST, the scales, L, $u_* = |\overline{uw}_0|^{1/2}$, and $u_* b_* = -\overline{wb}_0$, represent the external conditions used for non-dimensionalization. Turbulence statistics are expressed as functions of the stability parameter, $\xi = z/L$. In particular, the non-dimensional mean wind and buoyancy gradients in the stable ABL are parametrized as follows

$$\frac{\kappa z}{u_*}\frac{\mathrm{d}\bar{u}}{\mathrm{d}z} = \phi_m(\xi) = 1 + 5\,\xi, \tag{3}$$

$$\frac{\kappa z}{b_*}\frac{\mathrm{d}\bar{b}}{\mathrm{d}z} = \phi_h(\xi) = 1 + 5\,\xi. \tag{4}$$

Close to the surface and/or at windy conditions ξ is small and $\phi_m \approx \phi_h \approx 1$. This corresponds to the logarithmic solution for the mean wind and mean buoyancy. Under the strong stratification both functions should become linear $\phi_m \approx \phi_h \approx 5z/L$. In the case with weak stability, the MOST is believed to work well within the surface layer, over which the fluxes are approximately constant with height. However, both $\overline{uw}$ and $\overline{wb}$ become functions of height above the surface layer. Nieuswtadt (1954) reformulated the MOST by introducing the local length scale

$$\Lambda = -\frac{1}{\kappa}\frac{\overline{uw}^{3/2}}{\overline{wb}} \tag{5}$$

as a similarity scale. The local scaling is valid only for strong, continuous turbulence. These conditions are not fulfilled in very stable ABLs, which consist of layered structures, representing turbulence intermittency with a ‘sporadic’ character. In such case non-stationarity of the statistics plays an important role and should be accounted for in the parametrization schemes.

GOVERNING EQUATIONS AND INVARIANT SOLUTIONS

We consider flows in the ABL governed by the Navier-Stokes system under the Boussinesq approximation and in the inviscid limit. We perform ensemble averaging of the governing equations, assume horizontal homogeneity and assume that the Coriolis force is balanced by the horizontal pressure gradients. Under those assumptions, the averaged system reads

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \overline{uw}}{\partial z} = 0, \tag{6}$$

$$\frac{\partial \overline{w^2}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \bar{b}, \tag{7}$$

$$\frac{\partial \bar{b}}{\partial t} + \frac{\partial \overline{wb}}{\partial z} = 0, \tag{8}$$

where x is the direction of the mean wind, z is the vertical direction, ρ_0 is a constant mean density, and $\overline{p}$ is the mean pressure.

Unlike in standard approaches to the parametrization of the ABL, we do not focus on dimensional analyses alone, but investigate mathematical properties of the governing system of equations (6)–(8). Our first approach along this line was to use the technique of nondimensionalization Yano & Wacławczyk (2022). Next, in Yano & Wacławczyk (2023) the method of symmetry transformations was used. Symmetries are defined as transformations of variables which leave this system invariant. These new variables will be denoted by the symbol *. Of particular importance for deriving scaling laws in ABL are the space and time translations $z^* = z + z_0$, $t^* = t + t_0$ and scaling symmetry groups, namely the scaling of space and time

$$t^* = t, \quad z^* = e^{a_z} z, \quad \bar{u}^* = e^{a_z}\bar{u}, \quad \overline{p}^* = e^{2a_z}\overline{p}, \tag{9}$$
$$\overline{uw}^* = e^{2a_z}\overline{uw}, \quad \overline{w^2}^* = e^{2a_z}\overline{w^2}, \quad \overline{wb}^* = \overline{wb}, \quad \bar{b}^* = \bar{b}$$

$$t^* = e^{a_t} t, \quad z^* = z, \quad \bar{u}^* = e^{-a_t}\bar{u}, \quad \overline{p}^* = e^{-2a_t}\overline{p}, \tag{10}$$
$$\overline{uw}^* = e^{-2a_t}\overline{uw}, \quad \overline{w^2}^* = e^{-2a_t}\overline{w^2}, \quad \overline{wb}^* = e^{-a_t}\overline{wb}, \quad \bar{b}^* = \bar{b}$$

and additional scaling group of b in neutral flows (i.e. when temperature can be considered a passive scalar)

$$t^* = t, \qquad z^* = z, \quad \bar{u}^* = \bar{u}, \quad \bar{b}^* = e^{a_b}\bar{b}, \tag{11}$$
$$\overline{uw}^* = \overline{uw}, \quad \overline{w^2}^* = \overline{w^2}, \quad \overline{wb}^* = e^{a_b}\overline{wb}.$$

When the buoyancy plays an active role in the momentum equation (7) the scaling group parameter a_b becomes dependent on the time and space scaling (Yano & Wacławczyk, 2023)

$$a_b = a_z - 2a_t. \tag{12}$$

We also consider the additional statistical scaling (Oberlack & Rosteck, 2010; Oberlack *et al.*, 2022) which has no correspondence among symmetries of the Navier-Stokes equations, however averaged equations (6)–(8) are invariant under this scaling group

$$t^* = t, \quad z^* = z, \quad \bar{u}^* = e^{a_s}\bar{u}, \quad \overline{p}^* = e^{a_s}\overline{p}, \tag{13}$$
$$\overline{uw}^* = e^{a_s}\overline{uw}, \quad \overline{w^2}^* = e^{a_s}\overline{w^2}, \quad \overline{wb}^* = e^{a_s}\overline{wb}, \quad \bar{b}^* = e^{a_s}\bar{b}$$

Wacławczyk *et al.* (2014) related the statistical scaling to the phenomenon of intermittency, understood as alternating occurence of laminar and turbulent flows, where

$$\gamma = e^{a_s} \tag{14}$$

plays the role of the intermittency parameter and (13) represents the conditional statistics for turbulent flow multiplied by the weighting factor γ. Such intermittent flows occur in the very stable ABLs, where turbulent motions are suppressed due to the negative buoyancy flux.

We derived solutions of the governing system which remain invariant under the given set of scaling transformations (Wacławczyk *et al.*, 2023). The key point is that under the Lie symmetry analysis, velocity, buoyancy and fluxes do not scale independently, but are related with each other through the scaling parameters $\beta = a_t/a_z$ and $\chi = a_s/a_z$. When buoyancy plays an active role, the derived invariant solutions take the following form

$$t - t_0 = X_t\,|z - z_0|^{\beta} \tag{15}$$
$$\bar{u} - u_0 = C_u(X_t)\,|z - z_0|^{1-\beta+\chi}, \tag{16}$$
$$\bar{b} - b_0 = C_b(X_t)\,|z - z_0|^{1-2\beta+\chi}, \tag{17}$$
$$\overline{uw} - uw_0 = C_1(X_t)\,|z - z_0|^{2-2\beta+\chi}, \tag{18}$$
$$\overline{w^2} - w_0^2 = C_2(X_t)\,|z - z_0|^{2-2\beta+\chi}, \tag{19}$$
$$\overline{wb} - wb_0 = C_3(X_t)\,|z - z_0|^{2-3\beta+\chi}. \tag{20}$$

The system is expected to approach these solutions, far enough from boundaries and when initial conditions has been forgotten.

NON-DIMENSIONAL WIND VELOCITY AND BUOYANCY GRADIENTS

In the outer part of the ABL statistics of turbulence are affected by the boundary layer heigth h. Hence, we assume that $z_0 = h = const$ in Eqs. (15)–(20). Moreover, we take $u_0 = \bar{u}(h)$ and $b_0 = \bar{b}(h)$ as external velocity and buoyancy scales. Nieuswtadt (1954) predicted that at $z = h$ the fluxes are close to zero. For this reason we assume $uw_0 = 0$, $w_0^2 = 0$ and $wb_0 = 0$. Then, solutions (15)–(20), when written in terms of ϕ_m and ϕ_h read

$$\phi_m = \frac{\kappa z}{\sqrt{|\overline{uw}|}}\frac{d\bar{u}}{dz} = \frac{z}{\Lambda}\left(1 - \frac{z}{h}\right)^{\chi} F\left(\frac{u_0}{h}\tilde{X}_t\right), \tag{21}$$

$$\phi_h = \kappa z \frac{\sqrt{|\overline{uw}|}}{-\overline{wb}}\frac{d\bar{b}}{dz} = \frac{z}{\Lambda}\left(1 - \frac{z}{h}\right)^{\chi} H\left(\frac{u_0}{h}\tilde{X}_t\right). \tag{22}$$

where $\tilde{X}_t = (t - t_0)(1 - z/h)^{-\beta}$. The ratio u_0/h was introduced for dimensional consistency. Close to the surface the ratio $z/h \ll 1$ and $\Lambda \approx L$. If, additionally, dependence on time is neglected, Eqs. (21) and (22) reduce to

$$\phi_m \propto \frac{z}{L} = \xi, \quad \phi_h \propto \frac{z}{L} = \xi, \tag{23}$$

which is a limit of Eqs. (3) and (4) at strong stratifications.

Further from the surface if, $\chi \neq 0$ and/or the non-dimensional functions depend on $\tilde{X}_t$, the heigth of the boundary layer h will affect the scaling, as the second length scale apart from Λ. According to predictions (18) and (19) the ratio of momentum fluxes and the variance of vertical fluctuations should not depend on χ but only on $\tilde{X}_t$

$$\frac{\overline{uw}}{\overline{w^2}} = G\left(\frac{u_0}{h}\tilde{X}_t\right). \tag{24}$$

Another variable which does not depend on χ is the turbulent turbulent Prandtl number

$$Pr_t = \frac{\phi_h}{\phi_m} = \frac{H}{F} = K\left(\frac{u_0}{h}\tilde{X}_t\right) \neq const. \qquad (25)$$

When the transiency effects become important Pr_t will not be constant. This result is in contrast to the predictions of the standard MOST with the functions (3) and (4), where $Pr_t = \phi_h/\phi_m = 1$.

In Wacławczyk *et al.* (2023) it was assumed that Eq. (24) can be inverted, such that the turbulent Prandtl number becomes a function of the ratio $\overline{uw}/\overline{w^2}$ instead of $\tilde{X}_t$

$$Pr_t = \frac{\phi_h}{\phi_m} = K'\left(\frac{\overline{uw}}{\overline{w^2}}\right). \qquad (26)$$

As an important direct consequence from Eqs. (21) and (22), we also derive the Richardson number:

$$Ri = \frac{\mathrm{d}\bar{b}}{\mathrm{d}z}\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}z}\right)^{-2} = \left(1-\frac{z}{h}\right)^{-\chi}\frac{H}{F^2}, \qquad (27)$$

It can also be observed that non-zero intermittency scaling $\chi = a_s/a_z \neq 0$ introduces dependence of the Richardson number Ri on height in Eq. (27), whereas the classical MOST predicts a constant Ri at large stratifications.

To represent ϕ_m and ϕ_h in terms of the Richardson and Prandtl numbers, Eq. (27) should be solved for $(1-z/h)^\chi$. After introducing the result into Eqs. (21) and (22) we obtain

$$\phi_m = \frac{z}{\Lambda}\frac{1}{Ri}Pr_t\left(\frac{\overline{uw}}{\overline{w^2}}\right), \qquad (28)$$

$$\phi_h = \frac{z}{\Lambda}\frac{1}{Ri}Pr_t^2\left(\frac{\overline{uw}}{\overline{w^2}}\right). \qquad (29)$$

Eqs. (28), (29) reduce to (3) and (4) in the surface layer, where $\Lambda \approx L$ and under the assumption $Pr_t = 1$. When the conditions are close to neutral, $\xi \propto Ri$, such that $\phi_m \approx \phi_h \approx 1$. For strong stratifications the MO theory predicts $Ri = 0.2 = const$. Then $\phi_m = \phi_h \approx 5z/\Lambda \approx 5\xi$ in the surface layer.

RESULTS

In our analysis we use measurement data from the Surface Heat Budget of the Arctic Ocean (SHEBA) experiment (Persson *et al.*, 2002). The campaign took place from Oct 1997 to Oct 1998 on board of a Canadian icebreaker. Turbulent fluxes and mean meteorological data were collected at five levels on a 20m tower. Turbulent covariances available in the database were calculated with the 1–h averaging window. The data available in the open database (https://data.eol.ucar.edu/project/SHEBA) were post–processed as outlined in Grachev *et al.* (2005). In particular, the low–frequency components of covariances were removed to filter–out the effect of gravity waves.

The measurement carried out on the Arctic offers several advantages over those on the mid-latitudes. During the polar night, a long–lasting stable atmospheric boundary layer can be quasi stationary, such that $h \approx const$. Moreover, a surface covered with snow and ice is usually flat, uniform, and with no large-scale slopes. Thus, no influence of katabatic flows needs to be considered. SHEBA database is widely used until now in many studies devoted to the stable ABL. In spite of many advantages, data suffered from unavoidable measurement uncertianties and convergence errors due to finite time of averaging. Moreover, due to instrumental issues part of the data is missing. As an illustration, figure 1 presents the sample time series of the turbulent Prandtl number and the ratio $\overline{uw}/\overline{w^2}$ measured at approximately 8m above the ice level. The scatter of results is considerable, however, the lines seem to be correlated. In such case transiency of Pr_t can be parametrized by considering Pr_t a function of $\overline{uw}/\overline{w^2}$ as assumed in Eq. (26).

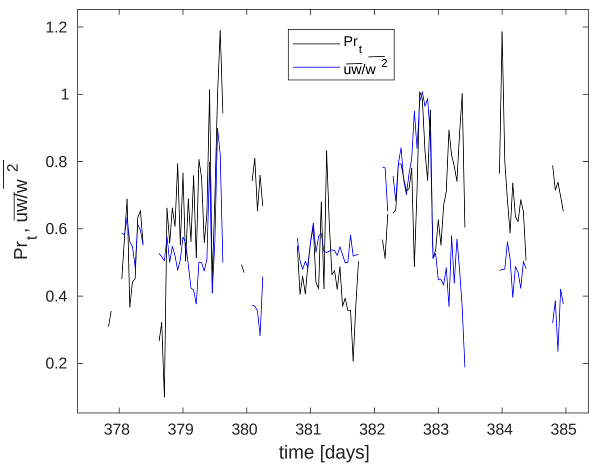

Figure 1. Time series of Pr_t and $\overline{uw}/\overline{w^2}$ measured at approximately 8m above the ice level.

It was found in Wacławczyk *et al.* (2023) that for SHEBA data the best power-law fit of function K' in Eq. (26) is

$$K' = 1.1(\overline{uw}/\overline{w^2})^{0.7}. \qquad (30)$$

However, this approximation works well at strong stratifications. In weakly stratified or neutral ABL the relative errors of $\overline{wb}$ and $\mathrm{d}\bar{b}/\mathrm{d}z$ measurements are very large. As a result the scatter of results for Ri and Pr_t is considerable. For this reason, treating $\xi = z/L$ as a measure of stratification, we filtered out data with $\xi < 0.2$ which are expected to follow the MOST quite well. We plot the remaining data of the turbulent Prandtl number as a function of $\overline{uw}/\overline{w^2}$ in figure 2, together with the fit (30). As predicted by Eq. (26), Pr_t is not constant. It clearly decreases with decreasing $\overline{uw}/\overline{w^2}$.

Figures 3 and 4 present the non-dimensional mean wind and mean buoyancy gradients ϕ_m and ϕ_h. They are plotted as functions of ξ/Ri, as suggested by formulas (28) and (29) under the assumption $Pr_t = 1$. For some of the data points this assumption is justified. However, data with large ξ/Ri (which correlates with large stratifications) increasingly deviate from linear functions. Moreover, data measured at weak stratifications follow rather different power-laws, namely

$$\phi_m \propto (\xi/Ri)^{1/3}, \quad \phi_h \propto (\xi/Ri)^{-1} \qquad (31)$$

By substituting definitions of ξ, Ri, ϕ_m and ϕ_h it can be shown that these formulas reduce to $\mathrm{d}\bar{u}/\mathrm{d}z \propto u_*/z$ and $\mathrm{d}\bar{b}/\mathrm{d}z \propto b_*/z$,

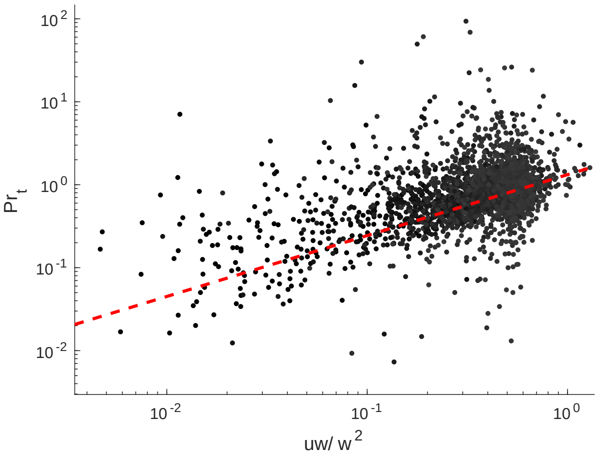

Figure 2. Turbulent Prandtl number as a function of $\overline{uw}/\overline{w^2}$ for data with $\xi > 0.2$. Red line is the best-fit power-law function (30).

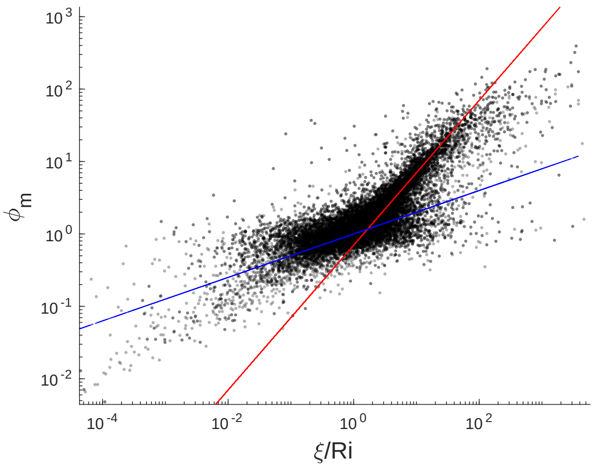

Figure 3. Non-dimensional wind velocity gradient as a function of the parameter ξ/Ri.

that is, to logarithmic solutions. It is expected that at weak stratifications $\xi \propto Ri$. However, this condition is not fulfilled exactly due to large relative errors of $\overline{wb}$ and $\mathrm{d}\bar{b}/\mathrm{d}z$, resulting in large estimation uncertainties of Ri and L. As a result ϕ_m and ϕ_h do not follow the formulas (28) and (29).

CONCLUSIONS

This work shows that new, interesting results can be derived when considering symmetries of the equations governing flows in the stable ABL. They are used to derive invariants, i.e. functions which do not change their forms after a transformation of variables. The invariants play an important role in the description and parametrization of turbulent flows. In the case of stably stratified ABL, the condition (12) becomes crucial. It represents the mutual dependence of velocity and temperature fields, when temperature plays an active role in the dynamics

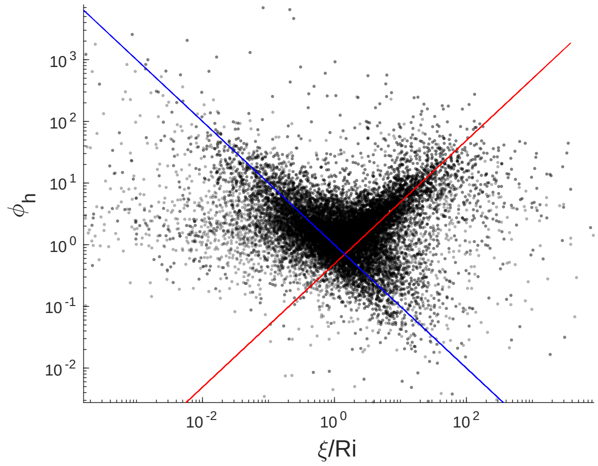

Figure 4. Non-dimensional buoyancy gradient as a function of the parameter ξ/Ri.

of ABL. This condition leads to the linear solution in the stable ABL, predicted by the MOST, see Yano & Wacławczyk (2023) and Eq. (23).

However, as discussed by Grachev *et al.* (2005), at large stratifications the measurement data increasingly deviate from the MOST predictions. In particular, the scaling of the non-dimensional functions ϕ_m and ϕ_h is closer to $\sim \xi^{0.3}$. This is explained as the influence of external (or global) intermittency in the stable ABL caused by local collapses of turbulence. In this work we account for the intermittency by using the statistical scaling group (13). This scaling introduces dependence of ϕ_m and ϕ_h on the height of the boundary layer h in Eqs. (21) and (22). Dependence on h also enters through the variable $\tilde{X}_t$. It follows that the local Obukhov length Λ is not the only length-scale which parametrizes ABL at large stratifications. On the other hand, the turbulent Prandtl number Pr_t in Eq. (25) and the ratio $\overline{uw}/\overline{w^2}$ in Eq. (24) do not depend on χ but only on $\tilde{X}_t$.

The derived dependence on non-dimensionalized time is intriguing, as the non-stationarity becomes important especially at large stratifications due to intermittent structure of ABL. Using the SHEBA data we presented sample time series of $\overline{uw}/\overline{w^2}$ and Pr_t with a visible correlation between the two variables, in spite of considerable scatter of results due to measurement errors and insufficient convergence of statistics. The result suggest that the non-stationarity can be parametrized by presenting Pr_t as a function of $\overline{uw}/\overline{w^2}$. Veryfying the derived formulas with other databases and/or against results of numerical experiments is a promising direction for a further study.

In future studies we also plan to account for the horizontal transports and the effect of Coriolis force. The presence of Coriolis force will possibly modify symmetries of the considered equations. Taking it into account may improve predictions at very large stratifications, where the ABL height is relatively small, and the statistics of the whole ABL are influenced by the Earth's rotation.

Hopefully, derived invariant functions will improve parametrizations of the stable atmospheric boundary layers and provide the basis for turbulence closures which account for the intermittent structure of ABL.

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