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TURBULENT SCALING LAWS AND SYMMETRY BASED SECOND MOMENT TURBULENCE MODELING

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ABSTRACT

Lie symmetries of the mean momentum equation in the limit of of wall parallel steady turbulent shear flows are derived and discussed in great detail. In comparison to the symmetries of the Navier-Stokes equations, the reduction of the independent variable space breaks some symmetries while simultaneously generating numerous new symmetries. These allow the construction of turbulent scaling laws that are applicable to the turbulent channel's logarithmic and core region. The incorporation of novel symmetries enables the extension of these scaling laws, resulting in superior fits against high Reynolds number channel DNS data.

Steady turbulent shear flow

A very common type of flow is the wall parallel steady turbulent shear flow (WPSSF). It is characterised by only having one component of the mean velocity being unequal to zero. This component is also only depending on one spatial coordinate, i.e.

$$\overline{U}_1 = \overline{U}_1(x_2), \qquad \overline{U}_2 = 0, \qquad \overline{U}_3 = 0. \tag{1}$$

With this, the mean continuity equation is always fulfilled. The above assumptions further imply that there is no x_1 dependence of the Reynolds stresses R_{ij}. However, the flow might be pressure driven so we allow a pressure gradient in this direction.

With this assumption, the Reynolds averaged Navier-Stokes (RANS) equations greatly simplify into the following equations

$$0 = -\frac{\partial \overline{P}}{\partial x_1} - \frac{\partial R_{12}}{\partial x_2} + \nu \frac{\partial^2 \overline{U}_1}{\partial x_2^2}, \tag{2a}$$

$$0 = -\frac{\partial \overline{P}}{\partial x_2} - \frac{\partial R_{22}}{\partial x_2}, \tag{2b}$$

$$0 = \frac{\partial R_{23}}{\partial x_2}, \tag{2c}$$

with ν being the viscosity and $\overline{P}$ the mean pressure. Under these assumptions, the pressure gradient in x_1 direction is not a function of x_2, while $R_{13} = R_{23} = 0$ (Pope, 2000).

The RANS equations can be written either in the more common form based on the velocity fluctuations u_i', in which the Reynolds stress tensor R_{ij} occurs, or in the instantaneous form subsequently denoted by H_{ij}, based on U_i. In the latter, the Reynolds decomposition is not performed and the instantaneous second moment $H_{ij} = R_{ij} + \overline{U}_i\overline{U}_j$ occurs. In the case of steady wall parallel turbulent shear flow, only R_{11} differs from its instantaneous twin and the Reynolds stresses that appear in eq. (2) are equal their the instantaneous formulations, i.e.

$$H_{11} = R_{11} + \overline{U}_1^2, \tag{3a}$$

$$H_{12} = R_{12}, \tag{3b}$$

$$H_{22} = R_{22}, \tag{3c}$$

$$H_{33} = R_{33}. \tag{3d}$$

We now want to find special solutions to the equations eq. (2), called invariant solutions. These are solutions of the equations derived from the Lie symmetries. In order to understand this process, we first give a brief introduction into Lie symmetries, and then more specifically to invariant solutions.

Lie symmetries

A symmetry is a transformation of the independent and dependent variables of a system of differential equations that leaves the system form invariant. Here, the concept of a symmetry as a property of geometrical objects, e.g. the rotational symmetry of a sphere, is extended to differential equations. If the rules of transformation form a Lie Group, they are called Lie Symmetries. See Bluman *et al.* (2010) for a detailed introduction into this topic. A one-parameter Lie point transformation T is defined as

$$T: \quad \mathbf{x}^* = \mathbf{f}(\mathbf{x}, \mathbf{y}; \varepsilon), \quad \mathbf{y}^* = \mathbf{g}(\mathbf{x}, \mathbf{y}; \varepsilon), \tag{4}$$

where $\mathbf{x}$ are the independent and $\mathbf{y}$ the dependent variables. ε is the group parameter with $\varepsilon \in \mathbb{R}$. Additionally, further constraints have to be fulfilled, see Bluman *et al.* (2010). The transformation can then be applied to a differential equation (PDE), e.g. $F(\mathbf{x}, \mathbf{y}, \mathbf{y}^{(1)}, \dots) = 0$. It is a symmetry of this PDE if the form stays invariant under eq. (4), i.e.

$$F(\mathbf{x}, \mathbf{y}, \mathbf{y}^{(1)}, \dots) = 0 \Leftrightarrow F(\mathbf{x}^*, \mathbf{y}^*, \mathbf{y}^{*(1)}, \dots) = 0. \quad (5)$$

The definition of a Lie symmetry given in eq. (4) is called a global transformation. Lie's first fundamental theorem states that this global form is equivalent to the infinitesimals, also called the local form, that are calculated as follows

$$x_i^* = f_i(\mathbf{x}, \mathbf{y}; \varepsilon) = x_i + \varepsilon \xi_i(\mathbf{x}, \mathbf{y}) + \mathscr{O}(\varepsilon^2), \quad (6a)$$

$$y_k^* = g_k(\mathbf{x}, \mathbf{y}; \varepsilon) = y_k + \varepsilon \eta_k(\mathbf{x}, \mathbf{y}) + \mathscr{O}(\varepsilon^2), \quad (6b)$$

where ξ is the infinitesimal of the independent and η of the dependent variables. Essentially, it is a Taylor expansion of the transformed variable with respect to the group parameter ε. This transformation can be extended, or prolonged, to the partial derivatives of the k-th dependent variables through

$$\begin{aligned} y_{k,i}^* &= y_{k,i} + \varepsilon \eta_{k;i} + \mathscr{O}(\varepsilon^2) \\ &\vdots \\ y_{k,i_1,\dots,i_s}^* &= y_{k,i_1,\dots,i_s} + \varepsilon \eta_{k;i_1,\dots,i_s} + \mathscr{O}(\varepsilon^2), \end{aligned} \quad (7a)$$

with

$$\eta_{k;i_1,i_2,\dots,i_s} = \frac{\mathrm{D}\eta_{k;i_1,i_2,\dots,i_{s-1}}}{\mathrm{D}x_{i_s}} - y_{k,i_1,\dots,i_{s-1},m} \frac{\mathrm{D}\xi_m}{\mathrm{D}x_{i_s}}, \quad (7b)$$

where the total differential operator is defined as

$$\frac{\mathrm{D}}{\mathrm{D}x_i} = \mathrm{D}_i = \frac{\partial}{\partial x_i} + y_{k,i} \frac{\partial}{\partial y_k} + y_{k,i,j} \frac{\partial}{\partial y_{k,j}} + \dots. \quad (8)$$

The local form of the transformation eq. (4) is then given by the infinitesimal generator, prolonged to include derivatives up to order s

$$X^{(s)} = \xi_i \frac{\partial}{\partial x_i} + \eta_k \frac{\partial}{\partial y_k} + \eta_{k;i_1} \frac{\partial}{\partial y_{k,i_1}} + \eta_{k;i_1 i_2} \frac{\partial}{\partial y_{k,i_1,i_2}} + \dots. \quad (9)$$

A PDE $F(\mathbf{x}, \mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(s)}) = 0$ is then invariant under the Lie transformation group eq. (4), now written in infinitesimal form eq. (6), if

$$X^{(s)} F\big|_{F=0} = 0 \quad (10)$$

holds, which is again equivalent to the form invariance under the global transformation of the variables in eq. (5).

Invariant solutions

Turbulent scaling laws can be derived from the symmetries of the given system of equations. The following mathematical descriptions follows Bluman *et al.* (2010).

Given a PDE system $\mathbf{F}(x; y)$

$$F_\sigma(\mathbf{x}, \mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(s)}) = 0, \quad \sigma = 1, \dots, N \quad (11)$$

of N PDEs of order s with n independent variables $\mathbf{x} = (x_1, \dots, x_n)$ and m dependent variables $\mathbf{y} = (y_1, \dots, y_m)$ that has the point symmetry with the infinitesimal generator

$$X = \xi_i(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial x_i} + \eta_k(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial y_k}. \quad (12)$$

Let $\xi(x,y) = (\xi_1(\mathbf{x}, \mathbf{y}), \dots, \xi_n(\mathbf{x}, \mathbf{y}))$ and assume $\xi(x,y) \not\equiv 0$. Then, $y = \theta(x)$, with components $y_\nu = \theta_\nu(x)$, $\nu = 1, \dots, m$, is an *invariant solution* of the PDE system eq. (11) resulting from the point symmetry eq. (12) if and only if

1. $y_\nu = \theta_\nu(x)$ is an invariant surface of the point symmetry eq. (12), i.e. $X(y_\nu - \theta_\nu(x)) = 0\big|_{y=\theta(x)}$.
2. $y = \theta(x)$ is a solution of eq. (11).

This procedure leads to a set of characteristic equations for $y = \theta(x)$ given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}\xi_1(x,y)} = \dots = \frac{\mathrm{d}x_n}{\mathrm{d}\xi_n(x,y)} = \frac{\mathrm{d}y_1}{\mathrm{d}\eta_1(x,y)} = \dots = \frac{\mathrm{d}y_m}{\mathrm{d}\eta_m(x,y)}. \quad (13)$$

Solving this system leads to $n + m - 1$ constants of integration which are invariants of eq. (12).

The solutions $y = \theta(x)$ of particular point symmetries are solutions of the PDE system. Here, the PDE system are the WPSSF equations and the invariant solutions are dubbed turbulent scaling laws.

Lie Symmetries of the WPSSF equations

The equations of WPSSF are a special case of the RANS equations. Some of the symmetries of the latter equations transform to this reduced system, i.e. they are also symmetries of the WPSSF equations. The classical symmetries of RANS, classical in the sense that they are also symmetries of the non-averaged Navier-Stokes equations, are the translations of space and time, rotation, a scaling symmetry (two scaling symmetries for the inviscid case), Galilean transformation and pressure translation. As the space of independent variables is reduced, there no longer is a time translation and a Galilean transformation for the WPSSF equations. Also, the rotational symmetry is broken through the assumptions made for WPSSF. The remaining classical symmetries of the inviscid WPSSF equations are

$$X_{x_i} = \frac{\partial}{\partial x_i}, \quad (14a)$$

$$X_{\overline{P}} = \frac{\partial}{\partial \overline{P}}, \quad (14b)$$

$$X_{Sx} = x_i \frac{\partial}{\partial x_i} + \overline{U}_1 \frac{\partial}{\partial \overline{U}_1} + 2\overline{P} \frac{\partial}{\partial \overline{P}} + 2R_{ij} \frac{\partial}{\partial R_{ij}} + \dots, \quad (14c)$$

$$X_{St} = -\overline{U}_1 \frac{\partial}{\partial \overline{U}_1} - 2\overline{P} \frac{\partial}{\partial \overline{P}} - 2R_{ij} \frac{\partial}{\partial R_{ij}} + \dots, \quad (14d)$$

i.e. translation in space and pressure as well as scaling in space and time. In addition to the classical symmetries, a second set of symmetries exist for the RANS equations, called statistical symmetries. They consist of the statistical scaling and statistical translation symmetry

$$X_{Sc,H} = \overline{U}_1 \frac{\partial}{\partial \overline{U}_1} + \overline{P} \frac{\partial}{\partial \overline{P}} + H_{ij} \frac{\partial}{\partial H_{ij}} + \ldots, \tag{15a}$$

$$X_{\overline{U}_1,H} = \frac{\partial}{\partial \overline{U}_1}, \tag{15b}$$

$$X_{H_{ij}} = \frac{\partial}{\partial H_{ij}}. \tag{15c}$$

Here, the statistical symmetries are presented in the instantaneous H-formulation, as this is a more concise representation. In the R-formulation, for instance, a translation of the mean velocity eq. (15b), also affects the fluctuating second moments. Also, there exists a statistical translation symmetry for every averaged quantity that occurs in the hierarchy of instantaneous momentum equations (Oberlack & Rosteck, 2010). Here, we have listed only two for brevity.

Now, a more detailed calculation of the symmetries of the inviscid WPSSF equation is performed through the computer algebra system Maple (Maplesoft, a division of Waterloo Maple Inc.., 2019). A total of 16 symmetries are found, some of which contain arbitrary functions depending on independent variables. Splitting up these function to recover the classical and statistical symmetries results in 25 symmetries. With eqs. (14) and (15), we have 11 classical and statistical symmetries. Then, the remaining 14 symmetries are three that include arbitrary functions

$$X_1 = f_1(\zeta) \frac{\partial}{\partial \overline{U}_1} \tag{16a}$$

$$X_2 = f_2(\zeta) \frac{\partial}{\partial H_{11}} \tag{16b}$$

$$X_3 = fx_3(\zeta) \frac{\partial}{\partial H_{33}}, \tag{16c}$$

with

$$\zeta = \left\{x_2, \overline{U}_1, H_{11}, H_{12}, H_{22}, H_{33}\right\}, \tag{16d}$$

five symmetries that were first discovered by Rosteck (2013),

$$X_{z1,H} = \frac{\partial}{\partial \overline{U}_1} + 2\overline{U}_1 \frac{\partial}{\partial H_{11}}, \tag{17a}$$

$$X_{z11,H} = x_2 \frac{\partial}{\partial H_{11}}, \tag{17b}$$

$$X_{z12,H} = -x_1 \frac{\partial}{\partial \overline{P}} + x_2 \frac{\partial}{\partial H_{12}}, \tag{17c}$$

$$X_{z22,H} = -x_2 \frac{\partial}{\partial \overline{P}} + x_2 \frac{\partial}{\partial H_{22}}, \tag{17d}$$

$$X_{z33,H} = x_2 \frac{\partial}{\partial H_{33}}, \tag{17e}$$

and a more general formulation of eq. (17d)

$$X_4 = f_4(\zeta) \frac{\partial}{\partial \overline{P}} - f_4(\zeta) \frac{\partial}{\partial H_{22}}. \tag{18}$$

The first of these, eq. (17a) is similar to the statistical translation of the mean velocity eq. (15b). However, while the latter does not transform the instantaneous higher moments, the former does. The remaining z-symmetries are all translations of the second moments that depend on the wall-normal coordinate x_2, with the pressure-gradient in the WPSSF equations also causing a translation for the pressure in two cases.

Next, a symmetry similar to a rotational symmetry exists

$$X_5 = -x_2 \frac{\partial}{\partial x_1} + H_{12} \frac{\partial}{\partial H_{22}}, \tag{19a}$$

which, written in the global form is

$$T_5 : x_1^* = x_1 - \varepsilon x_2,\, H_{22}^* = H_{22} + \varepsilon H_{12}. \tag{19b}$$

This symmetry is similar to the rotational symmetry with respect to the x_3-axis, which no longer exists due to the assumptions made for the WPSSF. The global transformation of the rotational symmetry would also include an infinitesimal transformation of the x_2-axis and of other dependent variables.

Further a scaling symmetry exists that only affects the streamwise direction and the shear-component of the Reynolds stress

$$X_6 = -x_1 \frac{\partial}{\partial x_1} + H_{12} \frac{\partial}{\partial H_{12}}. \tag{20}$$

Three symmetries remain:

$$X_7 = -\left(\overline{P} + H_{22}\right) \frac{\partial}{\partial x_1} + H_{12} \frac{\partial}{\partial x_2}, \tag{21a}$$

$$X_8 = -x_1 x_2 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_2} + x_1 H_{12} \frac{\partial}{\partial \overline{P}} - x_2 H_{12} \frac{\partial}{\partial H_{12}}, \tag{21b}$$

$$\begin{aligned} X_9 = &- x_2(\overline{P} + H_{22}) \frac{\partial}{\partial x_1} + x_2 H_{12} \frac{\partial}{\partial x_2} + \overline{P} H_{12} \frac{\partial}{\partial \overline{P}} \\ &+ H_{12}^2 \frac{\partial}{\partial H_{12}} + H_{12} H_{22} \frac{\partial}{\partial H_{22}}. \end{aligned} \tag{21c}$$

It is hard to grasp the effect these symmetries have when written in their local form. While the first of these is a translation of the two independent variables, the second and third are projective transformations Olver (1995); Bluman *et al.* (2010)

$$\begin{aligned} T_8 : x_1^* = \frac{x_1}{1 - \varepsilon x_2}, x_2^* = \frac{x_2}{1 - \varepsilon x_2}, \overline{P}^* = \overline{P} - H_{12} \frac{\varepsilon x_1}{1 - \varepsilon x_2}, \\ H_{12}^* = \frac{H_{12}}{1 - \varepsilon x_2}, \end{aligned} \tag{22a}$$

$$\begin{aligned} T_9 : x_1^* = x_1 - (\overline{P} + H_{22}) \frac{\varepsilon x_2}{1 - \varepsilon H_{12}}, x_2^* = \frac{x_2}{1 - \varepsilon H_{12}}, \\ \overline{P}^* = \frac{\overline{P}}{1 - \varepsilon H_{12}}, H_{12}^* = \frac{H_{12}}{1 - \varepsilon H_{12}}, H_{22}^* = \frac{H_{22}}{1 - \varepsilon H_{12}}. \end{aligned} \tag{22b}$$

Turbulent scaling laws

Now, with the symmetries of the WPSSF equations, turbulent scaling laws can be computed. Instead of using all the symmetries derived in the previous chapter, only the remaining classical eq. (14), statistical eq. (15) and X_z-symmetries eq. (17) are used. This is due to the fact that the arbitrary functions of some symmetries lead to invariant solutions that have no closed form.

As shown previously, the characteristic system eq. (13) has to be solved in order to compute the scaling laws. We now write the characteristic system with the symmetries mentioned above, i.e. translation in space and pressure, scaling in space and time, statistical translation of mean velocity $\overline{U}_1$, Reynolds stresses R_{ij}, statistical scaling and the additional symmetries eq. (17)

$$\begin{aligned}\frac{\mathrm{d}x_1}{a_{x_1}+a_{Sx}x_1} &= \frac{\mathrm{d}x_2}{a_{x_2}+a_{Sx}x_2} = \frac{\mathrm{d}\overline{U}_1}{(a_{Sx}-a_{St}+a_{Sc})\overline{U}_1+a_{\overline{U}_1}+a_{z1}} \\ &= \frac{\mathrm{d}\overline{P}}{(2a_{Sx}-2a_{St}+a_{Sc})\overline{P}+a_{\overline{P}}-a_{z12}x_1-a_{z22}x_2} \\ &= \frac{\mathrm{d}R_{11}}{(2a_{Sx}-2a_{St}+a_{Sc})R_{11}-a_{Ss}\overline{U}_1^2-2a_{\overline{U}_1}\overline{U}_1+a_{R_{11}}+a_{z11}x_2} \\ &= \frac{\mathrm{d}R_{12}}{(2a_{Sx}-2a_{St}+a_{Sc})R_{12}+a_{R_{12}}+a_{z12}x_2} = \dots,\end{aligned} \tag{23}$$

where we have introduced the group parameters a_{x1} and a_{x2} for the translation in their respective direction, $a_{\overline{P}}$ for the pressure translation, a_{Sx} and a_{St} as the scalings in space and time and a_{Sc} for the statistical scaling symmetry. Additionally, $a_{\overline{U}_1}$, $a_{R_{11}}$, $a_{R_{12}}$, etc. denote the statistical translation symmetry. Finally, the symmetries eq. (17) are denoted a_{z1}, a_{z11} and so on.

When solving this system, depending on the choice of group parameters, different kind of solutions can be produced. Close to the wall, Kármán (1930) assumed that there is a region in which the flow is entirely determined by the wall-friction velocity u_τ. This external parameter does not allow arbitrary scaling of the mean velocity $\overline{U}_1$. Thus, breaking the transformation of the mean velocity under the three scaling symmetries

$$\overline{U}_1^* = e^{a_{Sx}-a_{St}+a_{Sc}}\overline{U}_1 \tag{24}$$

leading to $a_{Sc} = a_{St} - a_{Sx}$. Now, solving the system eq. (23) for $\overline{U}_1$ leads to

$$\frac{\mathrm{d}\overline{U}_1}{\mathrm{d}x_2} = \frac{a_{\overline{U}_1}+a_{z1}}{a_{x2}+a_{Sx}x_2}, \tag{25}$$

which leads to the well known logarithmic law eq. (26), discussed in the next section. Without the aforementioned restriction for a_{Sc}, a power-law for the mean velocity emerges (see eq. (30) below), which corresponds to the velocity-deficit law valid in the core region of channel flow.

We now first discuss the case of the logarithmic law, valid in the region close to the channel's wall, the logarithmic region. Then, the core region is discussed.

Log region

In this case, we set $a_{Sc} = a_{St} - a_{Sx}$ in order to remove the mean velocity $\overline{U}_1$ from the denominator for the mean velocity in eq. (23). Also, variables are non-dimensionalized with the shear velocity u_τ and the viscous length scale $\delta_\nu = \nu/u_\tau$ and are denoted by $(\cdot)^+$. Then, solving eq. (25) leads to

$$\overline{U}_1^+ = \frac{1}{\kappa}\ln\left(x_2^+ + A_2\right) + B, \tag{26}$$

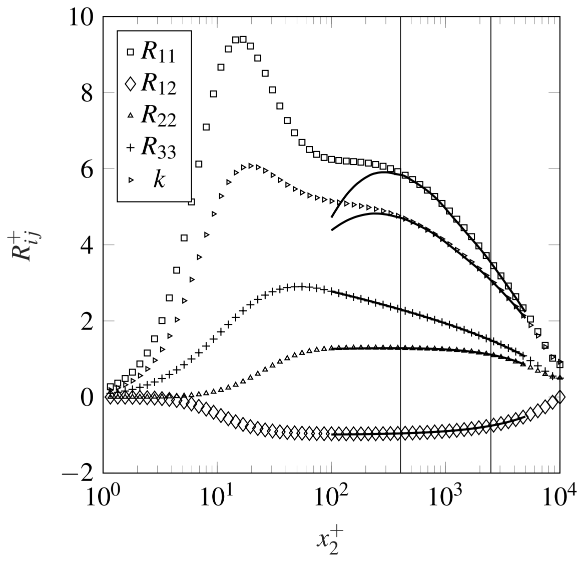

Figure 1. The DNS data of Hoyas *et al.* (2022) and the scaling laws eq. (27). The parameters used are listed in table 1. The vertical lines mark the region in which the scaling laws are fitted. Additionally, the turbulent kinetic energy $k = \frac{1}{2}R_{ii}$ is plotted.

with B being the constant of integration, $\kappa = a_{Sx}u_\tau/(a_{\overline{U}_1} + a_{z1})$ and $A_2 = a_{x2}u_\tau/(a_{Sx}\nu)$. This represents the classical formulation of the log-law, expanded by the translation of the wall-normal coordinate by A_2 due to the translation symmetry eq. (14a).

Repeating the same procedure for the second moments results in

$$\begin{aligned}R_{ij}^+ =& C_{ij}\left(x_2^+ + A_2\right)^\omega - B_{ij} + D_{ij}x_2^+ \\ & - (\overline{U}_1^+)^2\delta_{i1}\delta_{j1} - 2E\overline{U}_1^+\delta_{i1}\delta_{j1},\end{aligned} \tag{27}$$

with the exponent $\omega = 1 - a_{St}/a_{Sx}$, C_{ij} being the constants of integration, $E = a_{z1}/(a_{Sx}\omega u_\tau)$, $D_{ij} = a_{zij}\nu/(a_{St}u_\tau^3)$ and $B_{ij} = a_{R_{ij}}/(a_{Sx}\omega u_\tau^2) - D_{ij}A_2/\omega + 2\delta_{i1}\delta_{j1}E/(\omega\kappa)$.

The scaling law for the pressure can then be computed from the WPSSF equation by inserting eq. (27) into the steamwise and wall-normal momentum equations eqs. (2a) and (2b). From this we get the restriction $C_{12} = 0$ which leads to the shear component of the Reynolds stress being a linear function of x_2^+.

To show the quality of the scaling laws, the turbulent channel flow DNS at $Re_\tau = 10^4$ from Hoyas *et al.* (2022) is used to determine the fitting parameters. The resulting fit, together with the DNS data, is shown in fig. 1. Table 1 displays the values for the parameters of the scaling law.

Although the values for D_{ij} seem very small, they are necessary as the linear term in eq. (27) is written with x_2^+, which is of order $\mathscr{O}(x_2^+) = 10^3$ in the logarithmic region. Additional terms are included in the scaling law for H_{11}^+ compared to Oberlack *et al.* (2022). Writing R_{11}^+ from eq. (27) in the instantaneous form results in

$$H_{11}^+ = C_{11}\left(x_2^+ + A_2\right)^\omega - B_{11} + D_{11}x_2^+ - 2E\overline{U}_1^+, \tag{28a}$$

whereas eq. (16) in Oberlack *et al.* (2022) reads

$$H_{11}^+ = C_{11}\left(x_2^+\right)^\omega - B_2, \tag{28b}$$

Table 1. Fitting parameters in the log region using the common values $\omega = 0.1$, $\kappa = 0.3932$, $E = 3.4071$ and $A_2 = 0$. The plot of eq. (27) with these parameters is shown in fig. 1.

	B_{ij}	C_{ij}	D_{ij}
$\overline{U}_1^+$	4.5945	–	–
R_{11}^+	645.4385	648.1605	$-6.1995 \cdot 10^{-4}$
R_{12}^+	−0.9955	0	$9.6238 \cdot 10^{-5}$
R_{22}^+	−0.9241	0.2236	$-1.2025 \cdot 10^{-4}$
R_{33}^+	−5.8632	−1.9449	$-4.7944 \cdot 10^{-5}$

with $B_2 = a_{R_{11}}/(a_{Sx}\omega u_\tau^2)$, in which A_2 was already set to zero. Both E and D_{11} depend on the parameters of the symmetry eq. (17) and their inclusion lead to a better overlap with the DNS data, especially for R_{11}^+.

The plot of the log-law for the mean velocity is not shown as the precision of this turbulent scaling law is known in the literature. Also, the value of ω is the same as in Oberlack *et al.* (2022), with the von Kármán constant κ being nearly the same, see table 1.

Core region

We now discuss the general case in which no assumptions for the group parameters was made. Equation (23) then leads to the ordinary differential equation for the mean velocity

$$\frac{d\overline{U}_1}{dx_2} = \frac{(a_{Sx} - a_{St} + a_{Sc})\overline{U}_1 + a_{\overline{U}_1} + a_{z1}}{a_{x2} + a_{Sx}x_2} = \frac{\sigma_1 \overline{U}_1 + \frac{a_{\overline{U}_1} + a_{z1}}{a_{Sx}}}{x_2 + \frac{a_{x2}}{a_{Sx}}}, \tag{29}$$

with $\sigma_1 = (1 - \frac{a_{St}}{a_{Sx}}) + \frac{a_{Ss}}{a_{Sx}}$, which will be the exponent of the power-law when solving eq. (29). Then, we define a new coordinate system $x_2^\circ = x_2/h$, which is 0 at the center line and 1 at the wall of the channel. All the variables are still written non dimensional through u_τ and h. Also, the scaling laws are often written as deficit laws, which is the subtraction of the center-line value of the variable under investigation, such as the mean velocity or the Reynolds stresses. To differentiate the parameters used in the core region from those of the log region, they are written with a prime.

For the mean velocity, solving eq. (29) results in the deficit-law

$$\overline{U}_1^{+cl} - \overline{U}_1^+ = C_1' \left(x_2^\circ + A_2'\right)^{\sigma_1} + \tilde{B}', \tag{30}$$

with $A_2' = a_{x2}/(a_{Sx}h)$, $\tilde{B}' = \overline{U}_1^{+cl} - B'$, $B' = -(a_{\overline{U}_1} + a_{z1})/(a_{Sx}\sigma_1 u_\tau)$ and the constant of integration C_1'.

The Reynolds stresses scale with $\sigma_2 = 2(1 - \frac{a_{St}}{a_{Sx}}) + \frac{a_{Ss}}{a_{Sx}}$

$$\begin{aligned} R_{ij}^{+cl} - R_{ij}^+ = &C_{ij}' \left(x_2^\circ + A_2'\right)^{\sigma_2} + \tilde{B}_{ij}' + D_{ij}' x_2^\circ \\ &+ \overline{U}_1^{+2}\delta_{i1}\delta_{j1} + \frac{2E_2'}{(\sigma_2 - \sigma_1)}\overline{U}_1^+ \delta_{i1}\delta_{j1}, \end{aligned} \tag{31}$$

with C_{ij}' again being the constants of integration, $E_2' = a_{z1}/(a_{Sx}u_\tau)$, $D_{ij}' = a_{zij}h/(a_{Sx}(\sigma_2 - 1)u_\tau^2)$ and $\tilde{B}_{ij}' = R_{ij}^{+cl} + B_{ij}'$ with $B_{ij}' = a_{R_{ij}}/(a_{Sx}\sigma_2 u_\tau^2) + D_{ij}'A_2'/\sigma_2 - 2\delta_{i1}\delta_{j1}E_2'B'\sigma_1/(\sigma_2(\sigma_2 - \sigma_1))$.

These scaling laws hold only for the general case of $\sigma_1 \neq \sigma_2$. This can be seen from the singularity in eq. (31), as well as in B_{11}'. There is no change in the mean velocity for $\sigma_1 = \sigma_2$, but the $i = j = 1$ Reynolds stress component then also contains a logarithmic term

$$\begin{aligned} R_{11}^{+cl} - R_{11}^+ = &\left(x_2^\circ + A_2'\right)^{\sigma_1} \left[C_{11}'' + 2C_1'E_2' \ln\left(x_2^\circ + A_2'\right)\right] + \tilde{B}_{11}'' \\ &+ D_{11}' x_2^\circ + \overline{U}_1^{+2} - 2B'\overline{U}_1^+, \end{aligned} \tag{32}$$

with the same definitions for E_2' and B' as before. C_{11}'' is a different constant of integration and $\tilde{B}_{11}''$ also differs. The logarithmic terms comes only for $E_2' \neq 0$ into effect. For $E_2' = a_{z1} = 0$, eq. (32) is identical to eq. (31) and thus the consideration of the two cases $\sigma_1 = \sigma_2$ and $\sigma_1 \neq \sigma_2$ is only relevant when the symmetry eq. (17a) is taken into account.

Now we compare eq. (31) in its instantaneous form

$$\begin{aligned} H_{11}^{+cl} - H_{11}^+ = &C_{11}' \left(x_2^\circ + A_2'\right)^{\sigma_2} + B_{11}' + H_{11}^{+cl} \\ &+ D_{11}' x_2^\circ + 2\frac{E_2'}{(\sigma_2 - \sigma_1)}\overline{U}_1^+ \end{aligned} \tag{33a}$$

to the deficit law given in Oberlack *et al.* (2022) (eq. 19 therein)

$$H_{11}^{+cl} - H_{11}^+ = C_{11}'(x_2^\circ)^{\sigma_2}, \tag{33b}$$

where A_2' was set to zero and the deficit law at $x_2^\circ = 0$ was also set to zero through $a_{R_{11}}/(a_{Sx}\sigma_2 u_\tau^2) = -H_{11}^{+cl}$. Applying the same assumptions to eq. (33a) yields

$$\begin{aligned} H_{11}^{+cl} - H_{11}^+ = &C_{11}' \left(x_2^\circ\right)^{\sigma_2} + D_{11} x_2^\circ \\ &+ 2\frac{E_2'}{(\sigma_2 - \sigma_1)}\left(\overline{U}_1^+ - B'\frac{\sigma_1}{\sigma_2}\right), \end{aligned} \tag{34}$$

which is eq. (33b) extended through a term linear in x_2° due to the X_{z11}-symmetry eq. (17b) and with an additional term that contains the mean velocity, occurring only for $a_{z1} \neq 0$. In order for the deficit law eq. (34) to still be zero at $x_2^\circ = 0$, a different $a_{R_{11}}$ has to be chosen.

Like in the logarithmic region, the scaling laws are fitted against the DNS data of Hoyas *et al.* (2022) in the region $0 \leq x_2^\circ \leq 0.7$, shown in fig. 2. The parameters, listed in table 2 differ from those of Oberlack *et al.* (2022). Whereas the latter authors found that $\sigma_1 \approx \sigma_2 \approx 1.95$, we find that $2\sigma_1 \approx \sigma_2$.

Writing the scaling laws in the core region as deficits suggests displaying, and therefore fitting, the Reynolds stresses in a double logarithmic representation. However, there are a few issues with this approach: first, doing so over represents the center of the core region, roughly $0 \leq x_2^\circ \leq 0.1$, and secondly, a good fit in a double logarithmic representation can only be achieved if $R_{ij}^+|_{x_2^\circ=0} = R_{ij}^{+cl}$. While a good fit is undoubtedly aligned with the DNS's center-line values of the Reynolds stresses, it is conceivable that an even more optimal fit could be achieved by allowing for a discrepancy between R_{ij}^+ and R_{ij}^{+cl}. For the mean velocity deficit, an optimal fit was achieved for $\tilde{B}' = 0$. Similarly, table 2 shows that $\tilde{B}_{ij}'$ is nearly

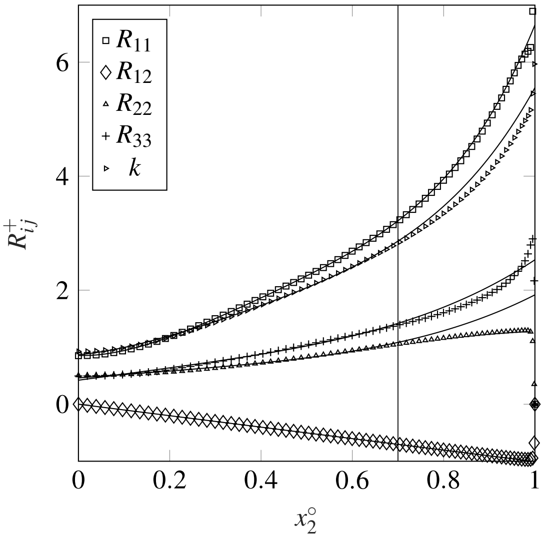

Figure 2. The DNS data of Hoyas *et al.* (2022) and the scaling laws eq. (31). The parameters used are listed in table 2. The vertical line marks the right border of the region in which the scaling laws are fitted. Additionally, the turbulent kinetic energy $k = \frac{1}{2}R_{ii}$ is plotted.

Table 2. Fitting parameters in the core region with the common values $\sigma_1 = 1.8197$, $\sigma_2 = 4.0503$ and $E_2' = -60.9131$, A_2' and $\tilde{B}'$ set to zero. The plot of eq. (31) with these parameters is shown in fig. 2.

	$\tilde{B}_{ij}'$	C_{ij}'	D_{ij}'
$\overline{U}_1^{+}$	–	5.7639	–
R_{11}^{+}	745.1922	−31.0101	0.3620
R_{12}^{+}	$1.9254 \cdot 10^{-4}$	–	1.0054
R_{22}^{+}	0.0448	−0.8545	−0.6030
R_{33}^{+}	0.0743	−1.0355	−1.0764

zero for all components with the exception of R_{11}^{+}. This is due to the fact that in eq. (31) additional constant terms appear due to $\overline{U}_1^{+}|_{x_2^{\circ}=0} \neq 0$. Evaluating the R_{11}^{+} deficit at $x_2^{\circ} = 0$ yields $\tilde{B}_{11}' + 2E_2'B'/(\sigma_2 - \sigma_1) + B'^2 = -0.0043$, thus nearly zero.

Conclusion

The derived Lie symmetries of the wall parallel steady turbulent shear flow equations have been used to construct turbulent scaling laws that hold in the channel flow's logarithmic and center region. These scaling laws are special solutions of the equations that describe this flow and it has been shown that they are an excellent model for high Reynolds number turbulent channel flow.

We wish to include these scaling laws into a turbulence model. In order for the model to be compatible with the laws derived herein, it needs to have the same symmetries as those used to compute the laws. Then, the model can be calibrated through the fitting parameters tables 1 and 2.

To date, there is no compatible turbulence model. While commonly used two-equation turbulence models fulfill all the classical symmetries, they are not invariant under the statistical symmetries and even have more symmetries than the RANS equations (Oberlack, 2001). A second moment closure turbulence model is invariant under all classical symmetries and does not contain additional ones, but is still not invariant under the statistical symmetries. To address this issue, Klingenberg *et al.* (2020) developed the minimum framework of a second moment turbulence model that is invariant under both classical and statistical symmetries. However, challenges such as the modeling of the return-to-isotropy, production and dissipation still remain and are under current research.

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