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Extending the $v^2 - f$ Model to Rough Walls

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ABSTRACT

Incorporating the effects of surface roughness into Reynolds-Averaged Navier-Stokes (RANS) models is a fundamental challenge in turbulence modelling. An established approach involves modifying the Boundary Conditions (BCs) on the wall to simulate the flow behaviour adjacent to a rough surface. The integration of Durbin's (1991) $v^2 - f$ model within the $k - \omega$ framework, despite its potential for model enhancement in terms of wall effects, has not yet explored roughness. This paper introduces a novel BC for the wall-normal stress, coupled with the Wilcox BC. By incorporating this BC, values of the roughness function and friction factor are obtained that align with experimental and numerical studies. The research further delves into the Turbulence Kinetic Energy (TKE) budget balance in the vicinity of rough walls. The proposed BC integrated into the $v^2 - f - k - \omega$ model demonstrates the role of BCs in RANS modelling of rough walls.

INTRODUCTION

Turbulent flows over rough surfaces are common in industry due to surface defects related to machining, erosion, and corrosion (Nikuradse,1950). Predicting such flows is crucial for the aerospace, transportation, and energy sectors. Despite extensive research, modelling turbulent flow over rough surfaces remains a challenge, driving the need for improved turbulence models and BCs to capture the complex physics. Numerical simulations like DNS offer an accurate but computationally expensive approach for predicting turbulence near rough surfaces. Large Eddy Simulation (LES) is more computationally efficient, though still resource-intensive, capturing large-scale turbulent structures while modelling smaller-scale motions. To simulate roughness in DNS or LES, fully resolving the cell size down to the roughness scale is a typical approach. In contrast, RANS models, widely used in industry, provide mean flow fields at lower computational cost but require extensive modelling for closure and cannot directly resolve the flow around roughness elements like DNS or LES.

Incorporating surface roughness into RANS models poses challenges due to the complex near-wall behaviour of the mean flow and turbulence quantities. One popular approach is to use an equivalent sand grain roughness parameter, $K_s^+ = hU_\tau/\nu$ where h is roughness height, U_τ is the friction velocity, and ν is kinematic viscosity, calibrated to represent the effect of roughness on the mean velocity field. This method assumes uniform roughness, with modified BCs implemented on a smooth ground plane. Determining K_s^+ requires experiments or numerical simulations, making it time and resource-intensive. Despite the challenges, K_s^+ remains popular for characterizing roughness effects in industrial flows.

Two-equation models, favoured for their simplicity compared to more complex alternatives like second moment closure, are extensively employed in RANS to incorporate roughness. The $k - \omega$ model and its extensions, preferred over the $k - \varepsilon$ model for their avoidance of near-wall damping functions, introduce a dissipation parameter, $\omega = \varepsilon / C_\mu k$ (Wilcox, 1988). Menter (1993 and 1994) introduced the Shear Stress Transport (SST) $k - \omega$ model, combining $k - \omega$ and $k - \varepsilon$ models to enhance predictions, particularly for boundary layers with adverse pressure gradients, and reduce sensitivity to free stream conditions. Utilizing the SST $k - \omega$ model for turbulent flow over rough surfaces is problematic, as its standard version may inadequately capture the near-wall turbulence. To mitigate this, Hellsten and Lainet (1997) proposed modifying the turbulent viscosity to improve near-wall performance. Various BCs have been suggested to model roughness: Wilcox (2006 and 2008) proposed a wall BC for ω based on sand-grain roughness height, while Knopp et al. (2009) and Aupoix (2015) suggested new values of k and ω at the wall for rough surfaces.

The $v^2 - f$ model, initially proposed by Durbin (1991), incorporates non-local effects of turbulent pressure fields and engages low Reynolds numbers without damping functions. Various enhancements have been made over time. Billard (2012) and Billard and Laurence (2012) introduced a revised version for smooth surfaces, enhancing accuracy and stability by considering v^2/k. Another approach, the $v^2 - f - k - \omega$ model, combines the $v^2 - f$ and $k - \omega$ models to address the numerical instability related to the ε BC. While applied successfully in diverse scenarios, careful validation is required (Davidson et al., 2003). This study evaluates the predictive capability of the $v^2 - f - k - \omega$ model for rough surfaces, including a new wall BC to account for roughness. Comparisons are made to the SST model with Aupoix (2015) and the $k - \varepsilon$ model of Brerton et al. (2021).

ROUGH-WALL TURBULENT BOUNDARY LAYER

The fluid kinematic viscosity, υ, dominates the flow physics in the viscous sublayer for a smooth wall, which is characterized by $y^+ = yU_\tau/\upsilon$ values less than 5. The region $10 < y^+ < 100$, referred to as the buffer layer, is responsible for generating most of the turbulence kinetic energy via the non-linear self-sustaining cycle (Jiménez, 2004). Roughness modifies the mean velocity profile close to the wall, which leads to a change in the wall shear stress. The associated downward shift of the velocity profile on a log-law plot using inner coordinates is known as the roughness function, ΔU^+. The mean velocity profile becomes (Wilcox, 2006):

$$U^+ = \frac{1}{\kappa} \ln y^+ + B - \Delta U^+ \quad (\kappa \approx 0.4, B \approx 5.0) \qquad (1)$$

where κ and B are the von Karman constant and intercept for a smooth wall, respectively. A roughness function for sand grain roughness can be expressed as follows for fully rough flow:

$$\Delta U^+ = \frac{1}{\kappa} ln\, K_s^+ + B - A \quad (A \approx 8.5) \tag{2}$$

Here, K_s^+ represents the equivalent sand grain roughness for the dimensionless roughness height, and A is the intercept for a uniform sand-grain surface.

The flow near a smooth or rough wall is characterized by several important parameters, including the friction velocity, $U_\tau = \sqrt{\tau_w/\rho}$, where τ_w and ρ are the local wall shear stress and density, respectively. The friction factor is given by the ratio of the wall shear stress (τ_w) to the dynamic pressure (ρU_b^2) and can be expressed in terms of the friction velocity (U_τ) and bulk flow velocity (U_b) as follows:

$$C_f = 2\frac{\tau_w}{\rho U_b^2} = 2\left(\frac{u_\tau}{U_b}\right)^2 \tag{3}$$

The flow behaviour near rough walls depends on the roughness geometry, which determines the near-wall motions. Krogstad and Efros (2012) highlighted the strong effect of roughness on the wall-normal velocity component due to reduced damping near a rough wall. Despite the importance of the normal Reynolds stress near the wall, it has received less attention over the years, partly due to measurement challenges. In the fully rough regime, roughness directly promotes wall-normal fluctuations, while in the transitionally rough regime, the local inhomogeneity level depends on the roughness geometry, affecting the energy redistribution and Reynolds shear stress generation (Yuan and Piomelli, 2014). Orlandi and Leonardi (2008) established a link between the roughness function and normal Reynolds stress at the crest of the roughness layer. This relationship has led to a new Moody chart. The wall normal stress, with its weak Reynolds number dependence, is especially suitable for roughness characterization (Orlandi, 2013).

As documented in the literature, the wall-normal Reynolds stress $\langle v^2 \rangle$ plays a vital role in the turbulent flow physics near a rough surface. Recognizing this importance, we propose a new BC for $\langle v^2 \rangle$ on a rough wall, employing the $v^2 - f - k - \omega$ formulation. A fundamental question arises: can one represent the flow characteristics within the roughness canopy solely through RANS parameters? Our research aims to address this issue.

MATHEMATICAL MODEL

The aim of this study is to evaluate the performance of the $v^2 - f - k - \omega$ turbulence model for predicting the flow behavior over rough walls. To achieve this, we solve steady state and fully developed channel flow using the Finite Volume Method (FVM). We assume constant properties (ρ and υ) and neglect advection. In this research, we initially consider a smooth channel flow with a specified pressure gradient and Reynolds number. Next, we identify a dimensionless roughness height (K_S^+), that can be visualized as the result of placing a monolayer of uniform sand grains on the channel wall. Within a RANS framework, BCs are used to implement the effect of these sand grains on the near-wall flow. For consistency, we measure the normal distance from the channel wall, mirroring the methodology used for a smooth surface. The grid used 160 non-uniform cells extending from the wall to the center of the channel, with the first node located at $y^+ = 0.1$. The appropriate cell size has been assessed by previous studies and more refinement is not recommended (Aupoix, 2015). The discrete equations were solved iteratively for a normalized convergence criterion of 10^{-6} for U, k, ω, v^2 and f. The mean momentum equation for constant property fully developed turbulent channel flow is given by:

$$0 = \frac{\partial}{\partial y}\left[\nu\left(\frac{\partial U}{\partial y}\right) - \langle uv \rangle\right] - \frac{1}{\rho}\frac{\partial P}{\partial x} \tag{4}$$

The Reynolds shear stress is obtained from an isotropic eddy viscosity model closure:

$$-\langle uv \rangle = \nu_t\left(\frac{\partial U}{\partial y}\right) \tag{5}$$

where ν_t is the eddy viscosity. For constant density, the continuity equation can be used to show that for fully developed flow the mean wall-normal velocity, V, is zero.

There are multiple formulations of the $v^2 - f - k - \omega$ model. Here the model of Davidson et al. (2003) was selected since it has shown better performance for some turbulent flows:

$$0 = \frac{\partial}{\partial y}\left[\left(\upsilon + \frac{\upsilon_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + P_k - \beta^* k\omega \tag{6}$$

$$0 = \frac{\partial}{\partial y}\left[\left(\upsilon + \frac{\upsilon_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial y}\right] + \gamma\frac{P_k\omega}{k} - \beta_0\omega^2 + \frac{\sigma_d}{\omega}\frac{\partial k}{\partial y}\frac{\partial \omega}{\partial y} \tag{7}$$

$$0 = \frac{\partial}{\partial y}\left[\left(\upsilon + \frac{\upsilon_t}{\sigma_k}\right)\frac{\partial \langle v^2 \rangle}{\partial y}\right] + kf - 6\langle v^2 \rangle\beta^*\omega \tag{8}$$

$$L^2\frac{\partial^2 f}{\partial y^2} - f = 3\frac{C_1}{T}\left(\frac{\langle v^2 \rangle}{k} - \frac{2}{3}\right) - C_2\frac{P_k}{k} - \frac{1}{T}\left(\frac{\langle v^2 \rangle}{k} - \frac{2}{3}\right) \tag{9}$$

$$\upsilon_t = C_\mu \langle v^2 \rangle T \tag{10}$$

Note that in Eq. 10, $\langle v^2 \rangle$ appears as a velocity scale in the eddy viscosity model. The time (T) and length (L) scales in this model are determined from the following relations:

$$T = max\left(\frac{1}{\beta^*\omega}; C_\xi\sqrt{\frac{\upsilon}{\beta^*\omega k}}\right) \tag{11}$$

$$L = C_L\, max\left(\frac{\sqrt{k}}{\beta^*\omega}; C_\eta\frac{\upsilon^{3/4}}{(\beta^* k\omega)^{1/4}}\right) \tag{12}$$

The values of the model coefficients are given in Table 1:

Table 1. $v^2 - f - k - \omega$ turbulence model constants (Davidson et al. 2003).

$\beta_0 = 0.0708$	$\gamma = \frac{13}{25}$	$\sigma_k = \frac{5}{3}$	$\sigma_\omega = 2.0$	$\sigma_d = \frac{1}{8}$	$C_\xi = 0.001$
$C_L = 0.23$	$C_\eta = 70$	$C_2 = 0.3$	$C_1 = 1.4$	$C_\mu = 0.22$	

BOUNDARY CONDITIONS

At the channel centre, all parameters are subjected to a zero-gradient BC based on symmetry. Traditionally, researchers assign a predetermined value for ω at the first node next to the wall in scenarios involving smooth walls or directly on the wall for rough walls. For smooth walls, the prescribed BC is $\omega_w = 6\upsilon/\beta\, y^2$, the value of β varying depending on the model. Here, y represents the distance from the first node to the wall. To integrate roughness into $k - \omega$ based models, BCs are adjusted to mimic flow behaviours near the rough surface. A common approach involves adjusting the relation for ω_w proportional to the sand grain size (Wilcox, 2006, Wilcox, 2008 and Knopp et al. 2009). The overall effect of increasing sand grain size is to reduce ω_w. In this study, we use the BC of Wilcox (1988) for ω:

$$\omega_w = u_\tau^2 S_R/\upsilon\,; \quad S_R = \begin{cases} \left(\frac{50}{K_S^+}\right)^2 & K_S^+ \leq 25 \\ \frac{100}{K_S^+} & K_S^+ > 25 \end{cases} \tag{13}$$

Along with the BC for ω, we propose a BC for the normal stress at the wall using the following formulation:

$$\langle v^2\rangle^+_{wall} = Max\{(0.0388 Re_\tau{}^{0.4517}) ln K_S^+ - (0.1379 Re_\tau{}^{0.4071}), 0\} \qquad K_S^+ < 70 \tag{14}$$

and

$$\langle v^2\rangle^+_{wall} = 5 \times 10^{-4}\, Re_\tau + 1.4495 \qquad K_S^+ \geq 70 \tag{15}$$

where $Re_\tau = U_\tau\, H/\upsilon$ is the friction Reynolds number and H is the channel half-height. Note that Eqs. 14 and 15 are calibrated to predict the correct roughness function and friction factor. This methodology adopts an approach similar to that of Wilcox (2008) and Brereton and Yuan (2018) by assuming zero TKE at the wall. This might appear to be inconsistent with the use of a finite normal stress on the wall. In this sense the flow is approximated as one-dimensional turbulence at the wall, where the roughness elements effectively block velocity fluctuations in the streamwise and spanwise directions. Note that the finite value of $\langle v^2\rangle$ results in a finite value of the eddy viscosity for a rough wall based on the model relation $\upsilon_t = C_\mu \langle v^2\rangle T$. In the proposed model, use of either a zero or finite turbulent kinetic energy at the wall gives the same friction factor and roughness function, and has negligible effect on the flow beyond $y^+ > 10$.

RESULTS

This section presents key results pertaining to the roughness function (ΔU^+), friction factor (C_f), and TKE budget. Figure 1 illustrates the normalized roughness function across various sand grain roughness values, compared with Nikuradse's empirical data and the classic Prandtl-Schlichting sand grain roughness model. The latter is represented by the curve-fit $\Delta U^+ = (1/\kappa)\ln(1 + 0.3K_S^+)$ proposed by White and Majdalani (2006) for all three roughness regimes. Employing the SST $k - \omega$ model with the Aupoix BCs yields results that closely match the experimental data for both transitional and fully rough regimes. Incorporating the new BC into the $v^2 - f$- $- k - \omega$ formulation yields similar roughness function values for transitional and fully rough flows. Assuming zero finite wall normal stress on the wall when using the $v^2 - f - k - \omega$ formulation would lead to inaccuracies in the roughness function estimation, underscoring the significance of the new BC.

Figure 2 presents the friction factor for smooth and rough channel flows. Model results are compared with the results of Brerton et al. (2021) and the Colebrook equation. The hydraulic diameter concept was used to calculate the friction factor for a turbulent channel flow based on the Colebrook relation, which was developed for circular pipe flow. However, near the wall, where viscous effects dominate, the velocity gradients, shear stresses, and viscous interactions in a pipe and channel flow are similar. Consequently, the hydraulic diameter approach can be effectively used in analyzing frictional losses for both pipe and channel flows (White and Majdalani, 2006). Brerton et al. (2021) evaluated the friction factor using a specific wall-function incorporated into the $k - \varepsilon$ model for fully developed turbulent channel flow. The new model gives results close to the Colebrook relation. The $v^2 - f - k - \omega$ model can predict the constant friction factor region within the fully rough regime even for higher Reynolds numbers.

Figure 3 presents the TKE budget for smooth and rough channel flows. The equivalant sand grain roughness is $K_S^+ = 200$ for the rough channel, and the friction Reynolds number is $Re_\tau = 2000$ for both flows. In this figure, D, P and ε represent the diffusion, production and dissipation of TKE, respectively. Dissipation is obtained from the TKE and specific dissipation. These terms can be found on the right hand side of Eq. 6. For the smooth case, dissipation is balanced with the production away from the wall. Near the wall, diffusion redistributes some of TKE as production diminishes (Kim et al., 1987). The diffusion is zero at the wall which is different from the DNS results. This can be attributed to a problematic feature of the $k - \omega$ model. Specific dissipation is mathematically a singular point at the wall and it goes to infinity as TKE reduces to zero on the wall. As we decrease the cell size near the smooth wall, we get closer to this singular point and the BC adjusted for ω at the first node sits above the values from DNS studies. In case of a rough wall, Figure 3 shows that production is again balanced with the production away from the wall. Due to the finite value of $\langle \boldsymbol{v^2}\rangle$ at the wall (to sustain turbulence and υ_t on the wall), production remains finite. Dissipation goes to zero at the wall based on the prediction for ω, so diffusion acts to balance the finite production. Beyond $y^+ > 10$, diffusion is insignificant for both smooth and rough channel flows. Away from the wall, production and dissipation for both channel flows behave very similar which is consistant the Townsend's (1977) outer-layer similarity hypothesis. This hypothesis suggests that turbulent flows over smooth and rough walls behave similarly away from the surface at very high Reynolds numbers, provided there is significant difference in scale between the typical roughness

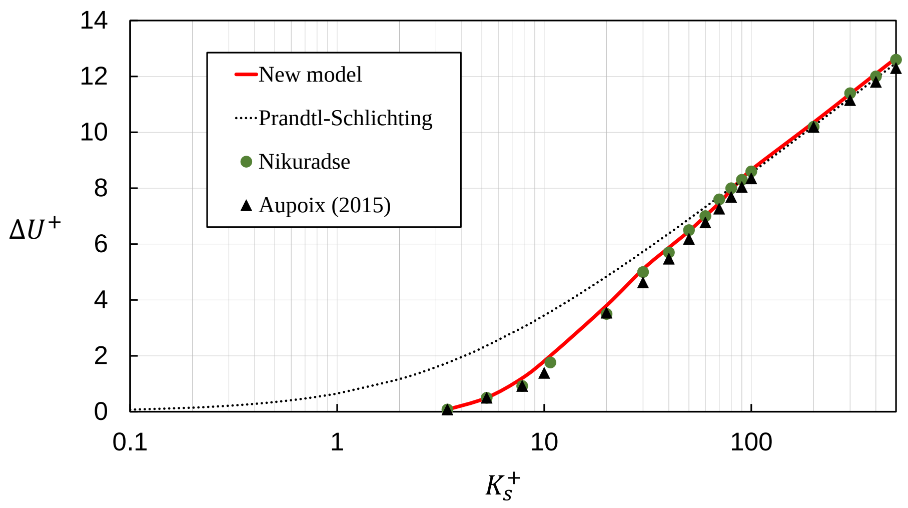

Figure 1. Comparison of the normalized roughness function to experimental results and the prediction of Aupoix (2015).

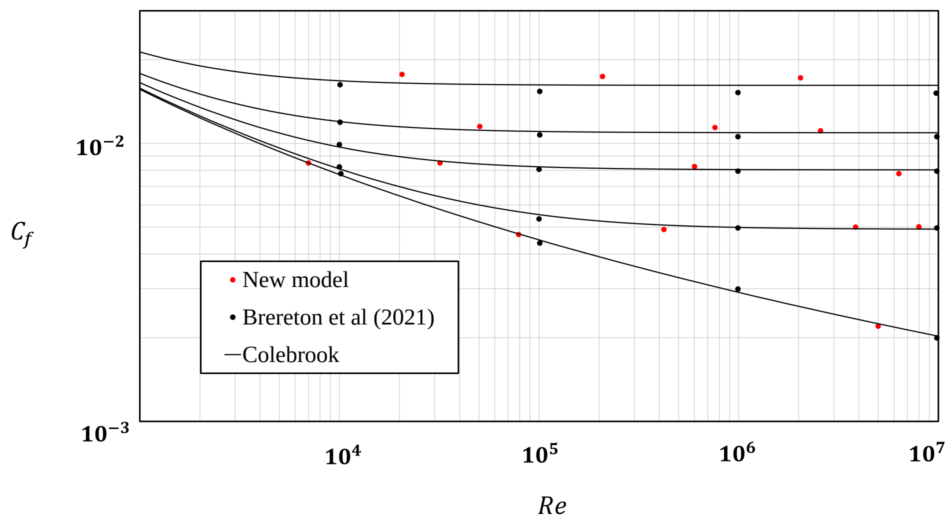

Figure 2. Friction factor for smooth and rough channel flows.

height (h) and the outer length scale of the flow (H, channel half-height).

Figure 4 presents the turbulent eddy viscosity normalized by the molecular kinematic viscosity for different sand grain roughness values. Orlandi (2013) indicated that, within a certain distance from the wall, the eddy viscosity remains relatively constant. This behavior in the present results can be attributed to the limiter imposed on the length scale of the model. The eddy length scales cannot grow beyond a specific size near the roughness elements, which is contingent upon the size and arrangement. Orlandi (2019) established a relationship between the eddy viscosity and the normal Reynolds stress on the crest plane, both of which are linked to the equivalent sand grain size. These relationships further highlight the importance of the wall normal Reynolds stress component in reproducing the flow behaviour over rough surfaces. Figure 4 indicates that the normalized turbulent viscosity on the wall has reaches near 10 from 0.01 when the sand grain size increases from 20 to 100. Such a significant increase in the normalized turbulent viscosity is unrealistic. Note that for a fixed value of pressure gradient and friction velocity in a channel flow, the effect of increasing K_S^+ is to decrease the bulk velocity. The eddy viscosity is artificially elevated to effectively reproduce the roughness function and friction factor. This underscores a limitation inherent to these models: while they can successfully predict the roughness function and friction factor, they do so at the expense of accuracy in capturing the actual physics near the roughness elements. This trade-off highlights the need for further model development to better reconcile the balance between capturing the bulk flow characteristics and correctly representing the near-wall physics.

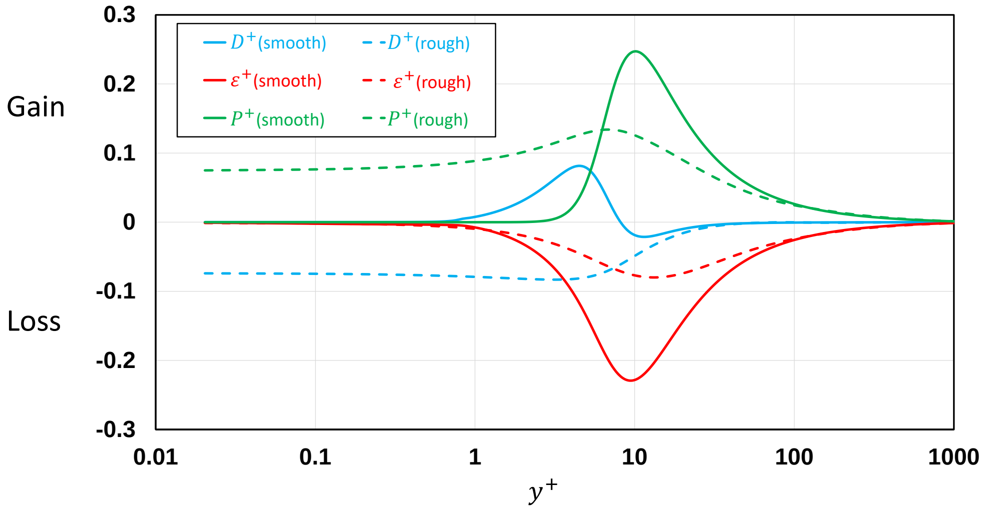

Figure 3. TKE budget for smooth and rough ($K_S^+ = 200$) channel flows.

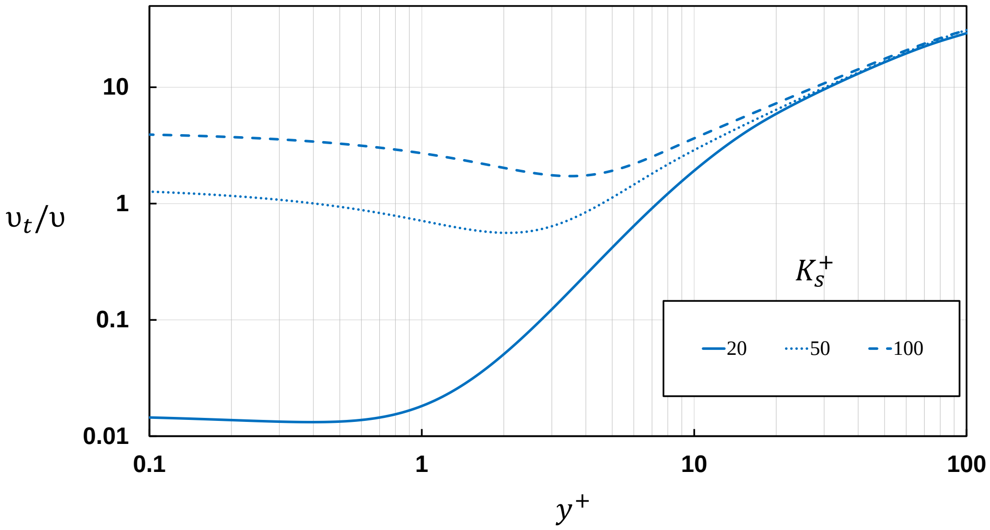

Figure 4. Normalized turbulent eddy viscosity near rough wall for various sand grain roughness.

CONCLUSION

The study introduces a novel approach for predicting turbulent channel flows over rough surfaces using the $v^2 - f - k - \omega$ turbulence model. Employing a finite value of $\langle v^2 \rangle$ at the wall to predict roughness effects, the model demonstrates promising performance in capturing the mean flow behavior, aligning well with experimental and numerical data. The model correctly predicts important parameters such as the roughness function and friction factor, compared to empirical correlations and experimental results. Balancing the trade-off between capturing the bulk flow characteristics and near-wall physics remains a challenge, highlighting the need for further model refinement.

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