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BALANCED NON-EQUILIBRIUM TURBULENCE

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ABSTRACT

The K41 framework remains central to the understanding of turbulent flows. However, in unsteady turbulence, K41's critical equilibrium assumption is expected to hold in an asymptotic manner, as the Reynolds number and wavenumbers tend to infinity, rendering K41 not strictly valid at finite wavenumbers. This work proposes a generalization of K41 for non-equilibrium effects, in decaying turbulence cascades far from initial conditions. The main result is a correction to the -5/3 law for out-of-equilibrium eddies, unrelated to intermittency effects. Experimental and numerical evidence is provided in support of the theoretical results.

Introduction

A landmark of out-of-equilibrium physics remains Kolmogorov's 1941 (K41) theory of turbulence (Kolmogorov, 1941*a*,*c*,*b*), regulating many quantities of engineering interest. A cornerstone assumption of K41 is that of cascade equilibrium (Pope, 2001). Note that the latter is used here in the context of stationarity and is not related to the concept of detailed balance encountered in statistical physics. While cascade equilibrium is exactly fulfilled in statistically static cascades, it is generally expected to be valid only asymptotically in unsteady ones, as turbulent eddies become vanishingly small, and Reynolds number infinite (Pope, 2001; Vassilicos, 2015). Therefore, for such unsteady cascades K41 cannot exactly describe finite, out-of-equilibrium wavenumbers, the physics of which remain obscure.

The recent investigation of Goto & Vassilicos (2016) has indicated that out-of-equilibrium eddies exhibit collective dynamics, at least in the case of freely decaying homogeneous turbulence. In particular, the turbulence cascade was shown to reach a "balanced" non-equilibrium state far from initial conditions, in the sense that the terms of the energy budget equation, expressed in spectral space, scale with each other at non-equilibrium wavenumbers. It is noted that the possibility of an asymptotic approach to a balanced state, far from initial conditions, was initially postulated by George (1989, 1992) for the whole spectral range, while Goto & Vassilicos (2016) observed it, in their direct numerical simulations (DNS), only for eddies comparable to the integral length scale of the cascade.

The above observation of balance thus allows the modelling of the behaviour of the large, out-of-equilibrium scales of the turbulence cascade, at least in the special case of decaying turbulence far from initial and boundary conditions. Using that information, this work proposes an extension of the K41 theory, for out-of-equilibrium effects. The main results are a correction to the -5/3 law and the formulation of an equation which greatly resembles the dissipation equation of the k-epsilon model for decaying homogenous turbulence. The assumptions and conclusions of this work are validated using data from laboratory and numerical experiments. A more thorough presentation of this work can be found in Steiros (2022).

Balanced cascade

A starting point for studies on homogenous decaying turbulence is often the scale-by-scale energy budget

$$\frac{\partial K^{>}(k,t)}{\partial t} = \Pi(k,t) - \varepsilon^{>}(k,t)\,, \tag{1}$$

where $K^{>}(k,t) = \int_k^\infty E(k,t)dk$ and $\varepsilon^{>}(k,t) = \int_k^\infty k^2 E(k,t)dk$ are the high pass filtered turbulence kinetic energy and dissipation rate, respectively, with $E(k,t)$ being the energy spectrum. $\Pi(k,t)$ is the interscale flux from wavenumbers smaller, to wavenumbers larger than k. In the following, when the superscript $>$ is dropped, the integration occurs from 0 to infinity.

Kolmogorov's equilibrium assumption states that for sufficiently large Reynolds numbers and for eddies which are of negligible size compared to the integral length scale $L(t) = \frac{3\pi}{4}\int_0^\infty k^{-1}E(k,t)dk/K(t)$, the cascade is approximately stationary, i.e. $\frac{\partial K^{>}}{\partial t} \approx 0$. If in addition to the above assumption one considers wavenumbers k which are not too large and so that the local dissipation is negligible, then $\varepsilon^{>}(k,t) \approx \varepsilon(t)$. Combining the above two arguments one then obtains

$$\Pi(k,t) \approx \varepsilon(t)\,, \tag{2}$$

which is the well-known repercussion of the equilibrium assumption. This result is foundational, as it directly leads to the -5/3 law for the energy spectrum. Indeed, dimensional analysis arguments lead to $E(k,t) \propto \Pi^{2/3}k^{-5/3}$. Utilization of expression 2 then readily leads to $E(k,t) \propto \varepsilon^{2/3}k^{-5/3}$, i.e. the -5/3 law (Pope, 2001). As the equilibrium assumption (i.e. expression 2) was used in its derivation, we might expect that there will be deviations from the -5/3 law in cases of non-stationarity in the cascade.

To account for out-of-equilibrium phenomena one has to find an alternative expression for the energy flux Π, without having to discard the time-dependent term $\frac{\partial K^{>}}{\partial t}$. To do that, we

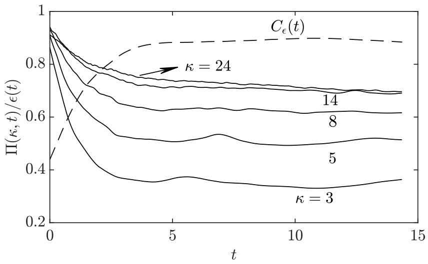

Figure 1. $g \equiv \Pi/\varepsilon$ (solid lines) and C_ε (dashed line) as a function of time, in decaying periodic box turbulence. $t = 0$ marks the onset of decay. Note that $\kappa = kL$.

may use the original idea of George (1989, 1992), who postulated that, far from initial conditions, transient turbulent systems (including the turbulence cascade) will approach a state of "balance" where the dynamic terms of their characteristic budget differential equation should scale with each other.

Under the assumption of balance, the budget equation 1 then leads to $\frac{\partial K^{>}(kL,t)}{\partial t} \propto \Pi(kL,t) \propto \varepsilon^{>}(kL,t)$. For wavenumbers which are not too small, one has $\varepsilon^{>}(k,t) \approx \varepsilon(t)$, and thus balance implies

$$\Pi(k,t) = g(kL)\varepsilon(t)\,. \tag{3}$$

Figure 1 validates equation 3 using decaying periodic box turbulence DNS (see section "Methodology" for details on the dataset). At time $t = 0$ turbulence is left to decay, and after a sufficient time has passed (i.e. so that turbulence is far from its initial conditions) $g \equiv \Pi/\varepsilon$ becomes independent of time for a variety of $\kappa = kL > 1$. Figure 1 also shows that spectral balance coincides with the establishment of Kolmogorov's dissipation scaling $\varepsilon = C_\varepsilon(\frac{2}{3}K)^{3/2}/L$, with C_ε a constant. That is, because combination of equation 3 with the well-known expression for the large scale energy flux $\Pi \propto (\frac{2}{3}K)^{3/2}/L$ directly yields the dissipation scaling.

It can be readily seen that the above self-preserving expression for Π is an (out-of-equilibrium) generalization of Kolmogorov's equilibrium assumption, i.e. equation 2. The arbitrary function $g(kL)$ will be smaller than unity for decaying turbulence, and is expected to approach unity (i.e. equilibrium) as $kL \to \infty$ and $Re \to \infty$, in agreement with Kolmogorov's ideas (Pope, 2001). The main objective of this article is to determine $g(kL)$ for sufficiently large, but smaller-than-infinite wavenumbers, i.e. as equilibrium is being approached but has not yet been reached. Combination of that with the dimensional analysis expression $E(k,t) \propto \Pi^{2/3}k^{-5/3}$ will then yield the out-of-equilibrium correction to the -5/3 law.

Simple transportation of energy

To calculate $g(kL)$ we follow the original idea of Lumley (1992) who postulated that, at sufficiently high Reynolds numbers, a range of scales will form in the cascade, where effects linked to production or dissipation of turbulence kinetic energy will be negligible. Consequently, the energy there is conserved, simply transported downwards in the cascade. We interpret the above statement as the substantial derivative of the energy flux being zero in the self-preserving coordinate system (κ,t), i.e.

$$D\Pi/Dt = \partial\Pi/\partial t + V\partial\Pi/\partial\kappa = 0\,. \tag{4}$$

Equation 4 poses a constraint which can be used to calculate the function g but to do that, we first need an expression for the interscale energy speed $V = d\kappa/dt$. Following the dimensional arguments of Pao (1965), we expect that $dk/dt = C_v\Pi^{1/3}k^{5/3}$, where C_v is a coefficient of proportionality. Using equation 3 and the Kolmogorov dissipation scaling $\varepsilon = C_\varepsilon(\frac{2}{3}K)^{3/2}/L$ we end up with

$$dk/dt = \frac{3}{2}\frac{C_v}{C_\varepsilon^{2/3}}\frac{\varepsilon(t)}{L(t)K(t)}g(\kappa)^{1/3}\kappa^{5/3}\,.$$

The speed of the energy flux across normalized wavenumbers is $\frac{d\kappa}{dt} = L\frac{dk}{dt} + k\frac{dL}{dt}$. It is expected that the second term on the right hand will be negligible compared to the first, i.e. $\frac{d\kappa}{dt} \approx L\frac{dk}{dt}$, as the time scale of the nonlinear energy transport of an individual eddy must be much smaller than that of the dilation of the entire cascade. This postulation was verified using the DNS data, and the term $k\frac{dL}{dt}$ was found to be at least an order of magnitude smaller than $L\frac{dk}{dt}$, after $\kappa = 2$.

Under the above assumption, the spectral velocity becomes

$$V \approx \frac{3}{2}\frac{C_v}{C_\varepsilon^{2/3}}\frac{\varepsilon(t)}{K(t)}g(\kappa)^{1/3}\kappa^{5/3}\,.$$

Injecting the above result to equation 4 and using equation 3 one obtains

$$\frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt} = -\frac{3}{2}\frac{C_v}{C_\varepsilon^{2/3}}\frac{g'}{g^{2/3}}\kappa^{5/3} = -C_0\,, \tag{5}$$

with $g' = dg/d\kappa$. The left hand side of the above equation is a function of time only, while the right hand side a function of the normalized wavenumber κ only. Thus, both sides must equal a constant which is dependent only on initial conditions. In figure 2 we validate the above result with the DNS data. Approximately 3.5 time units after the onset of decay ($t = 0$), both sides of the equation indeed become relatively constant, independent of time or wavenumber.

Dissipation equation of the $k-\varepsilon$ model

The left hand side of equation 5 is

$$\frac{d\varepsilon}{dt} = -C_0\frac{\varepsilon^2}{K}\,, \tag{6}$$

which is the dissipation equation of the (otherwise empirically derived) $k-\varepsilon$ turbulence model, for homogenous decaying turbulence (Pope, 2001). It is noteworthy to mention that integration of the above equation yields a power law decay for the kinetic energy without the need of assuming turbulence invariants (Pope, 2001). If an invariant were to be assumed, it would yield a value for C_0. The DNS dataset indicates a value of $C_0 \approx 1.7$ which is the value obtained from Loitsyanskii's invariant.

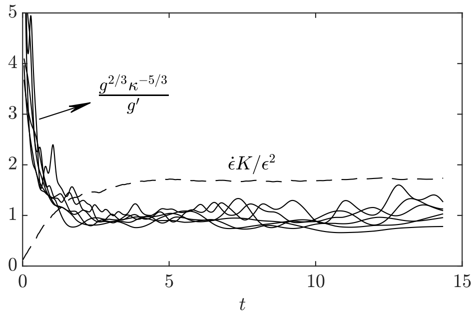

Figure 2. Evolution of the two sides of equation 5 over time, using the DNS dataset (forcing stops at $t = 0$). $g^{2/3}\kappa^{-5/3}/g'$ (solid lines) is plotted for $\kappa = 3,4,5,6,7,8,10$.

-5/3 law correction

Integration of the right hand side of equation 5 yields $g = \left[1 - \frac{C_0 C_\varepsilon^{2/3}}{3C_v}\kappa^{-2/3}\right]^3$ where the constant of integration was calculated using the fact that we expect equilibrium, i.e. $g \to 1$, as $\kappa \to \infty$ and $Re \to \infty$. Using the self preserving expression 3 along with the dimensional analysis result $E(k,t) \propto \Pi^{2/3}k^{-5/3}$ one obtains

$$\frac{E(\kappa,t)}{\varepsilon(t)^{2/3}L(t)^{5/3}} \approx C_k \left[1 - c\kappa^{-2/3}\right]^2 \kappa^{-5/3}, \qquad (7)$$

with $c = \frac{C_0 C_\varepsilon^{2/3}}{3C_v}$. Determination of the constant c requires knowledge of C_ε, C_v and C_0. However, only the latter can be calculated analytically, by assuming a turbulence invariant. Thus, C_v and C_ε need to be determined from observations. Our DNS results (i.e. figures 1 and 2) suggest $C_\varepsilon \approx 0.9$, $C_v \approx 1$ and $C_0 \approx 1.7$ for the DNS, which lead to $c = 0.53$. To calculate the out-of-equilibrium correction of the -5/3 law for the one-dimensional spectrum E_{11} (useful for experimental hot-wire measurements), we may utilize the following expression which links the two types of spectra (see Comte-Bellot & Corrsin (1971))

$$E(\kappa,t) = \frac{1}{2}\kappa^3 \frac{\partial}{\partial\kappa}\left[\frac{1}{\kappa}\frac{\partial}{\partial\kappa}E_{11}(\kappa,t)\right].$$

A double integration then yields

$$\frac{E_{11}}{\varepsilon^{2/3}L^{5/3}} = \frac{18}{55}C_k\kappa^{-5/3}\left[1 - 1.209c\kappa^{-2/3} + 0.407c^2\kappa^{-4/3}\right]. \qquad (8)$$

In both equations 7 and 8 a power-law out-of-equilibrium correction (brackets) has been superimposed on the usual -5/3 law.

Figure 3 validates the above conclusions using appropriately compensated spectra obtained from decaying turbulence DNS. Since turbulence is decaying, its large scales will be out-of-equilibrium, and the -5/3 will not be exact. Indeed, figure 3 shows that the compensated spectrum becomes flat only when utilizing the out-of-equilibrium expression 7. At very high normalized wavenumbers, close to 50, there is a tendency for the spectrum to approach the -5/3 law. This is the point where Kolmogorov's equilibrium can be thought approximately valid. Indeed, at this normalized wavenumber expression 7 produces almost identical results to the -5/3 slope, indicating that out-of-equilibrium effects have started to become negligible.

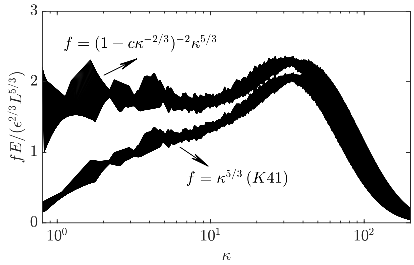

Figure 3. Validation of expression 7 using DNS. The compensated spectra become flat using the out-of-equilibrium correction, and not the classical K41 formulation.

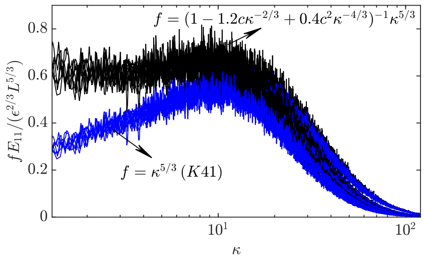

Figure 4. Validation of expression 8 using hot wire anemometry. The compensated spectra become flat using the out-of-equilibrium correction, and not the classical K41 formulation.

Figure 4 validates expression 8 using hot wire measurements of grid turbulence at various stages downstream of the grid (see section "Methodology" for details). Much similar to the three-dimensional spectra of the DNS, the experimental one-dimensional spectra require the novel out-of-equilibrium correction to the -5/3 law to flatten. As wavenumbers grow, the new correction tends to become indistinguishable to the -5/3 law, a sign that out-of-equilibrium effects have become negligible.

Methodology

For validation purposes, a data-set of periodic-box decaying turbulence is used, the details of which are presented in Goto & Vassilicos (2016). A forcing $f = (-\sin(k_f x)\cos(k_f y), \cos(k_f x)\sin(k_f y), 0)$ with $k_f = 4$ is imposed on the Navier-Stokes equations, and is turned off at $t = 0$ when dissipation is maximum, allowing the turbulence to decay. The simulation size was $N^3 = 2048^3$. The spatial resolution $k_{max}\eta$ was slightly larger than one at t_0, while $k_{max}\eta$ increased during decay. The Taylor scale Reynolds number was naturally decaying, but was roughly equal to 120.

The experiments were conducted in the wind tunnel of the Lille Fluid Mechanics Laboratory (LMFL) which has a test section of 1×2 m^2 and a length of 21 m. Hot wire measurements were conducted using a single Pt-W 5 micron wire, 3 mm long, with a 1 mm sensing element, driven by a TSI IFA300 anemometer at 50 kHz acquisition frequency with a low-pass filter at 20 kHz. Two planar grids were tested with different bar thicknesses, but identical blockage (30 %). Three inlet velocities were tested 4, 5 and 7 ms^{-1}, and all mea-

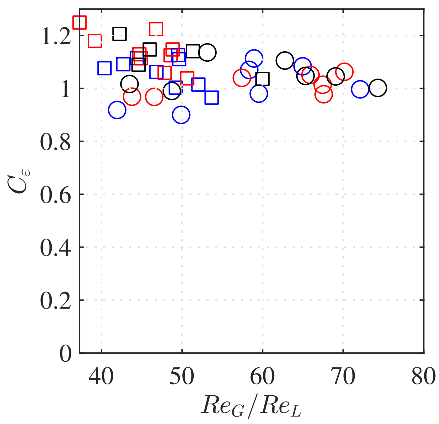

Figure 5. Measured dissipation coefficient for the two grids tested (circles and squares) for inlet velocities of 7, 5 and 4 ms^{-1} (blue, red and black colours).

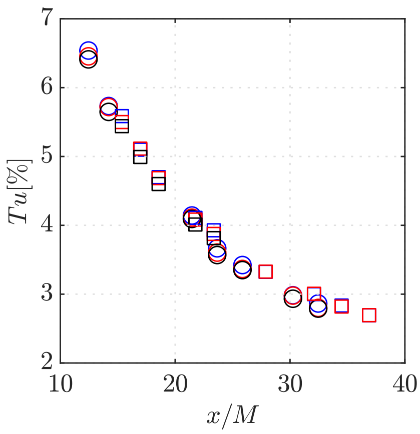

Figure 6. Measured turbulence intensities for the two grids tested (circles and squares) for inlet velocities of 7, 5 and 4 ms^{-1} (blue, red and black colours).

surements were conducted at the region of the flow where $C_\varepsilon = const$, i.e. sufficiently far from the grid (in our case this occurred approximately ten mesh-sizes downstream of the grid). The Taylor scale Reynolds number ranged from 75 to 150, depending on the case. All tested cases demonstrated a qualitatively similar behaviour to the one shown in figure 4. Figures 5 and 6 show the measured dissipation coefficient and turbulence intensities for the various cases tested. It is noted that for the calculation of c in expression 8 the current value of $C_\varepsilon \approx 1$ was used, while C_v was assumed unity, as in the DNS. C_0 was taken to be 1.83, which corresponds to Saffman's invariant, using the fact that this invariant has been often linked to grid turbulence (Sinhuber *et al.*, 2015).

Concluding discussion

This work generalizes the K41 framework for unsteady cascades far from initial conditions, on the basis of two assumptions. First, that far from initial conditions the cascade exhibits a self-preserving approach to equilibrium, and second that quasi-isotropic wavenumbers are characterized by a simple transportation of energy. Both of these assumptions are supportd by periodic box DNS. The main result of the above framework is a correction to the $-5/3$ law for non-equilibrium and quasi-isotropic wavenumbers, in cascades far from initial conditions.

We end this work with a quick comparison of the above result with the K62 refined theory of Kolmogorov (Kolmogorov, 1962) and the spectrum of Pao (1965), both of which provide corrections to the K41 framework. K62 accounts for the spatial fluctuations of the small scales in ε. Its physical argument is therefore different from the current theory which assumes a "balanced" non-equilibrium. Indeed, K62 produces a drastically different correction to the -5/3 law, than the current theory. Pao's correction (Pao, 1965) on the other hand is based on the speed V that energy is transferred across wavenumbers, but yields corrections only for the dissipative scales of turbulence, leaving larger eddies, and thus the -5/3 law, unaffected. These corrections thus differ from the one proposed in this work and in fact we cannot think of any reason why all three corrections cannot apply simultaneously.

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