SYMMETRY-BASED EDDY-VISCOSITY MODELS: A HANDS-ON APPROACH

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ABSTRACT

In the present work, we revisit the classical problem of Reynolds-Averaged Navier–Stokes (RANS) modeling by taking into account additional constraints arising from the Lie-symmetries that govern turbulence. Symmetries are defined as variable transformations that when inserted into an equation leave this equation form invariant. Even though originally a mathematical concept, it turns out that in the context of equations describing physical phenomena, symmetries usually carry physical meaning themselves. Generally speaking, it is desirable to incorporate this physical meaning into a model, so that symmetries can be used to infer constraints on model equations in a mathematically rigorous and algorithmic manner. The symmetries that are important for turbulence fall into two categories: First, classical symmetries, which are observed throughout classical mechanics and are found in the original Navier–Stokes equations as well as any full statistical description of turbulence, and, second, statistical symmetries, which are only observed once a statistical view of turbulence is adopted, and, therefore, cannot be found in the unaveraged Navier–Stokes equations. Since the implications of the classical symmetries are fairly trivial, they are usually accounted for in complete turbulence models. The statistical symmetries, on the other hand, are a relatively recent development, and are generally overlooked in turbulence modeling. The goal of this work is to present the development of a modified version of the k-ε-model, which hopefully illustrates the steps needed to build these additional symmetry constraints into a model. First results show a promising performance of the modified model.

INTRODUCTION

The best known turbulence model for incompressible flow is given by the Navier–Stokes equations

$$\frac{\partial U_i}{\partial x_i} = 0, \quad \frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0 \quad (1)$$

where U_i stands for the velocity, P the pressure divided by the density, t and x_i are temporal and spatial coordinates, respectively, and ν is the kinematic viscosity. However, despite the advent of supercomputers, a full solution of (1) remains too computationally expensive for most real-life applications, and the need for efficient and accurate turbulence models still remains. RANS models occupy an important place in the spectrum of turbulence models, because they are computationally efficient while often being accurate enough. Their starting point are the RANS equations, which can be derived from averaging (1) and read

$$\frac{\partial \bar{U}_i}{\partial x_i} = 0, \quad \frac{\partial \bar{U}_i}{\partial t} + \frac{\partial H_{ij}}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j} \qquad (2)$$

Here, the bar denotes averaging, and $H_{ij} = \overline{U_i U_j}$. The velocity and pressure are usually decomposed into a mean and a fluctuating value (Reynolds, 1895),

$$U_i = \bar{U}_i + u_i, \quad P = \bar{P} + p \qquad (3)$$

leading to

$$\frac{D\bar{U}_i}{Dt} = \frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j} - \frac{\partial R_{ij}}{\partial x_j} \qquad (4)$$

where the new unknown term $R_{ij} = \overline{u_i u_j}$, also known as the Reynolds stress tensor, arises. For this unknown term, an exact equation can be derived, however, this equation again contains unknown correlations. This is known as the famous closure problem of turbulence. In order to obtain a solution to (4), empirical closure relations, i.e. turbulence models, have to be introduced at some level. Presently, we consider the simplest closure of eddy-viscosity models, which rely on inserting a model for the unknown R_{ij} in (4). If we denote this model with

$\tilde{R}_{ij}$ to highlight its approximate nature, (4) as used in eddy-viscosity models reads

$$\frac{D\bar{U}_i}{Dt} = \frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j} - \frac{\partial \tilde{R}_{ij}}{\partial x_j} \quad (5)$$

The simplest model for $\tilde{R}_{ij}$ is given by the Boussinesq approximation (Boussinesq, 1877)

$$\tilde{R}_{ij} = -\nu_t \left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij} \quad (6)$$

where the task of modeling six independent components of the Reynolds stress tensor is reduced to modeling a turbulent (or eddy) viscosity ν_t and the turbulent kinetic energy k.

In this work, we restrict our attention to the classical k-ε-model (Jones & Launder, 1972), though the method can easily be applied to other two-equation models as well. Note, however, that the application to Reynolds-stress models (RSMs) is more complicated, so that we exclude this class of models from the present discussion.

The classical k-ε-model solves a transport equation for k, which is based on the exact equation that can be derived for the turbulent kinetic energy and reads

$$\frac{Dk}{Dt} = -\tilde{R}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_j} \left(\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right) \quad (7)$$

Herein, the first term on the right-hand side (the production term) appears in closed form, whereas the second one (dissipation) and the last one (diffusion) require modeling. For the diffusion term, a simple gradient-diffusion hypothesis has been introduced in the development of (7), and the dissipation is obtained from the transport equation

$$\frac{D\varepsilon}{Dt} = -C_{\varepsilon,1} \frac{\varepsilon}{k} \tilde{R}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} - C_{\varepsilon,2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j} \left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right) \quad (8)$$

which is entirely empirical and whose structure is based on (7). The constants $\sigma_k, C_{\varepsilon,1}, C_{\varepsilon,2}$ and σ_ε are model parameters, and establishing appropriate values for them is accomplished by calibrating the model against a number of canonical flows. Apart from entering (7), ε also serves as a scale-providing variable that is needed to formulate a dimensionally correct expression for ν_t. The k-ε-model uses

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \quad (9)$$

where the additional model parameter C_μ appears.

A major issue associated with the k-ε-model and similar models is their lack of universality, i.e. when simulating a new flow problem, it is not generally clear if a given model will be sufficiently accurate. In the following, we explore how we can address this issue using symmetry constraints.

This paper is structured as follows: We first give a brief introduction to Lie-symmetries and discuss the relevant symmetries of turbulence. Then, we discuss how conclusions from Lie-symmetry analysis allow enhancing existing turbulence models, and show some results of simple flows obtained with a modified version of the k-ε-model. We end with a short conclusion.

LIE-SYMMETRIES OF TURBULENCE

In the present context, symmetries are defined as transformations that leave a given equation such as (1) form invariant. Note the analogy to geometrical symmetries, which can be defined as geometric transformations (e.g. rotation) that leave geometric objects (e.g. a circle) invariant. The symmetries of the unaveraged Navier–Stokes equations (1) were first calculated by (Bytev, 1972) and, assuming $\nu = 0$, read

$$T_t: \quad t^* = t + a_T, \quad x_i^* = x_i, \quad U_i^* = U_i, \quad P^* = P \quad (10)$$

$$T_{\mathrm{rot}_\alpha}: \quad t^* = t, \quad x_i^* = x_j Q_{ij}^{[\alpha]}, \quad U_i^* = U_j Q_{ij}^{[\alpha]}, \quad P^* = P \quad (11)$$

$$T_{\mathrm{Gal}_i}: \quad t^* = t, \quad x_i^* = x_i + f_{\mathrm{Gal}_i}(t), \quad U_i^* = U_i + f'_{\mathrm{Gal}_i}(t), \quad P^* = P - x_j f''_{\mathrm{Gal}_j}(t) \quad (12)$$

$$T_P: \quad t^* = t, \quad x_i^* = x_i, \quad U_i^* = U_i, \quad P^* = P + f_P(t) \quad (13)$$

$$T_{\mathrm{Sc},I}: \quad t^* = t, \quad x_i^* = x_i e^{a_{\mathrm{Sc},I}}, \quad U_i^* = U_i e^{a_{\mathrm{Sc},I}}, \quad P^* = P e^{2a_{\mathrm{Sc},I}} \quad (14)$$

$$T_{\mathrm{Sc},II}: \quad t^* = t e^{a_{\mathrm{Sc},II}}, \quad x_i^* = x_i, \quad U_i^* = U_i e^{-a_{\mathrm{Sc},II}}, \quad P^* = P e^{-2a_{\mathrm{Sc},II}} \quad (15)$$

where the constant rotational matrices $Q^{[\alpha]}$ are given by

$$Q^{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos a_{\mathrm{rot}_1} & -\sin a_{\mathrm{rot}_1} \\ 0 & \sin a_{\mathrm{rot}_1} & \cos a_{\mathrm{rot}_1} \end{pmatrix} \quad (16)$$

$$Q^{[2]} = \begin{pmatrix} \cos a_{\mathrm{rot}_2} & 0 & -\sin a_{\mathrm{rot}_2} \\ 0 & 1 & 0 \\ \sin a_{\mathrm{rot}_2} & 0 & \cos a_{\mathrm{rot}_2} \end{pmatrix} \quad (17)$$

$$Q^{[3]} = \begin{pmatrix} \cos a_{\mathrm{rot}_3} & \sin a_{\mathrm{rot}_3} & 0 \\ -\sin a_{\mathrm{rot}_3} & \cos a_{\mathrm{rot}_3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (18)$$

respectively. The a_i stand for arbitrary real-valued constants, and $f_P(t)$ and $f_{\mathrm{Gal}_i}(t)$ are free functions of time. Notice that in the case of nonvanishing viscosity, due to a so called symmetry-breaking effect of ν, (14) and (15) combine to

$$T_{\mathrm{Sc,ns}}: \ t^* = t e^{2a_{\mathrm{Sc,ns}}}, \ x_i^* = x_i e^{a_{\mathrm{Sc,ns}}}, \ U_i^* = U_i e^{-a_{\mathrm{Sc,ns}}}, \ P^* = P e^{-2a_{\mathrm{Sc,ns}}} \quad (19)$$

The symmetries (10)–(15) and (19), which we also refer to as classical symmetries, each have an intuitive physical interpretation. The time translation symmetry (10) is connected to the principle that there is no absolute origin in time, so that time shifts cannot have an impact on physical processes. The analogous concept for the spatial coordinates is encoded in (12) if we assume the special case $f_{\mathrm{Gal}_i}(t) = \mathrm{const}$. Equation (11) is the rotational symmetry, which encodes the principle that the orientation of the coordinate system must be arbitrary. This is directly connected to the principle of correct tensor formulation. Note that the similar transformation of time-dependent rotation, i.e. (11) with $a_{\mathrm{rot}_\alpha} = a_{\mathrm{rot}_\alpha}(t)$, is not a symmetry of the Navier–Stokes equations, because inertial effects do play a role here. The generalized Galileian symmetry (12) is also well-known. If we restrict ourselves to $f'_{\mathrm{Gal}_i}(t) = \mathrm{const}$, this symmetry essentially states that a linear movement at a constant velocity cannot affect physical processes, a principle found throughout classical mechanics. For the special case of incompressible flow, we may even allow acceleration, because its effect can be absorbed by the pressure field. Another symmetry that is specific to incompressible flow is given by (13), which states that the absolute value of the pressure does not matter, but only spatial pressure differences. Finally, the scaling symmetries (14) and (15) (or (19) in the viscous case) are connected to the principle of dimensional correctness. This connection becomes clear if one realizes that changing the system of measurement, which must obviously not have any effect on the equations, is nothing but a particular rescaling of the appearing variables.

So far, nothing particularly surprising has been revealed by this symmetry analysis. In fact, the principles encoded by the symmetries (10)–(15) and (19) are essentially equivalent to the principles of invariant modeling (Donaldson & Rosenbaum, 1968). Roughly speaking, two-equation models and more complex ones fulfill all of these symmetries. Only simple special-purpose models, such as the mixing length model (Prandtl, 1925), whose restriction to particular flow types leads to a violation of one or more of the principles listed here, break some of the symmetries discussed so far. In general, it must be emphasized that symmetry-breaking only restricts the universality of a model and says nothing about its usefulness for any one particular flow.

However, additional symmetry constraints can be found by looking at a complete statistical description of turbulence, such as that given by (4) and the infinite hierarchy of equations for the higher moments. Oberlack & Rosteck (2010); Rosteck & Oberlack (2011) found that in this framework, the symmetries

$$
\begin{aligned}
T_{\mathrm{Sc,stat}}:\ & t^* = t,\ x_i^* = x_i,\ \bar{U}_i^* = \bar{U}_i e^{a_{\mathrm{Sc,stat}}},\ \bar{P}^* = \bar{P} e^{a_{\mathrm{Sc,stat}}} \\
& R_{ij}^* = (R_{ij} + \bar{U}_i \bar{U}_j) e^{a_{\mathrm{Sc,stat}}} - \bar{U}_i \bar{U}_j e^{2a_{\mathrm{Sc,stat}}} \qquad (20)\\
T_{\mathrm{Tr,stat,1}}:\ & t^* = t,\ x_i^* = x_i,\ \bar{U}_i^* = \bar{U}_i + a_{\mathrm{Tr,stat},I,i},\ \bar{P}^* = \bar{P} \\
& R_{ij}^* = R_{ij} - \bar{U}_j a_{\mathrm{Tr,stat},I,i} - \bar{U}_i a_{\mathrm{Tr,stat},I,j} \\
& - a_{\mathrm{Tr,stat},I,i} a_{\mathrm{Tr,stat},I,j} \qquad (21)\\
T_{\mathrm{Tr,stat,2}}:\ & t^* = t,\ x_i^* = x_i,\ \bar{U}_i^* = \bar{U}_i,\ \bar{P}^* = \bar{P} \\
& R_{ij}^* = R_{ij} + a_{\mathrm{Tr,stat},II,ij} \qquad (22)
\end{aligned}
$$

which have no counterpart in the unaveraged Navier–Stokes equations, appear. Note that these symmetries affect the entire infinite hierarchy of statistical moments, but in order to keep the notation simple, we only write out variables that are important for our present purposes. For more details on these symmetries, we refer to Rosteck (2013). Waclawczyk *et al.* (2014) were able to show that these symmetries, which we refer to as statistical symmetries in the following, also appear in other complete statistical descriptions of turbulence, such as the Lundgren–Monin–Novikov (LMN) hierarchy (Lundgren, 1967) based on probability-density functions (PDFs) and the Hopf functional approach (Hopf, 1952). In the PDF picture, it also becomes possible to develop a physical interpretation of the symmetries (20)–(22), and Waclawczyk *et al.* (2014) were able to establish a connection to intermittency and non-Gaussianity, two frequently observed features of turbulent statistics.

MODIFYING THE k-ε-MODEL USING STATISTICAL SYMMETRY CONSTRAINTS

Whereas the classical symmetries have generally been accounted for in turbulence modeling, the statistical symmetries have so far been overlooked. In fact, it turns out to be difficult to include them into a turbulence model. In the context of eddy-viscosity models, this can be understood by realizing that the exact mean momentum equation (4) only fulfills (21) due to the complicated transformation of the Reynolds stress tensor. Once a model for it is inserted, (21) essentially becomes a translation symmetry in $\bar{U}_i$. In order to fulfill it (without relying on R_{ij}), the velocity must not appear explicitly, but only its derivatives. Clearly, this is violated by the convective term. Similarly, once the exact Reynolds stress tensor is replaced by a model $\tilde{R}_{ij}$, the statistical scaling symmetry (20) is broken by the nonlinearity of the convective term of (4), and, as can be shown, also by the production and dissipation terms of (7) and (8). In a preliminary study, Klingenberg *et al.* (2020) were able to show that in order for reasonable model equations to be possible subject to the constraints introduced by the classical symmetries (10)–(15) (with the last two combining to (19) in the viscous case) and the statistical symmetries (20)–(22), auxiliary velocity and pressure fields, which we call $\hat{U}_i$ and $\hat{P}$, respectively, must be introduced. These fields must behave like the mean velocity (or, respectively, the mean pressure) under all classical symmetries while being invariant under all statistical symmetries. If this auxiliary velocity then replaces the mean velocity in the convective terms of all model equations, the first issue discussed above is solved. In order to also address the second issue associated with the scaling of the source terms, a third scale-providing variable has to be introduced. The model equations for $\hat{U}_i$ and $\hat{P}$ are quite strongly constrained by the assumed symmetry behavior of the two variables, and we use

$$
\frac{\partial \hat{U}_i}{\partial x_i} = 0 \qquad (23)
$$

$$
\begin{aligned}
\frac{\hat{D}\hat{U}_i}{\hat{D}t} &= \frac{\partial \hat{U}_i}{\partial t} + \hat{U}_j \frac{\partial \hat{U}_i}{\partial x_j} = -\frac{\partial \hat{P}}{\partial x_i} \\
&+ \frac{\partial}{\partial x_j}\left((\nu + \iota \nu_t) \left(\frac{\partial \hat{U}_i}{\partial x_j} + \frac{\partial \hat{U}_j}{\partial x_i} \right) \right) - \frac{2}{3} \iota \frac{\hat{\varepsilon}}{\varepsilon} \frac{\partial k}{\partial x_i} \qquad (24)
\end{aligned}
$$

Herein, some freedom exists in the choice of the source terms in that additional source terms would be possible, and the terms containing the model parameter ι are not mandatory from a symmetry perspective. The particular form (23) and (24) is motivated by the assumption that $\bar{U}_i$ and $\hat{U}_i$ should not behave too differently, and, thus, their equations should be structurally similar. The third scale-providing variable $\hat{\varepsilon}$, which appears in the last term of (24), is assumed to behave like ε under all classical symmetries (10)–(15) and (19) and the statistical translation symmetries (21) and (22) while scaling linearly under the statistical scaling symmetry (20). Furthermore, we assume that k behaves like the turbulent kinetic energy under all classical symmetries, is invariant under the statistical translation symmetries and scales linearly under the statistical scaling symmetry. The model variable ε is assumed to behave like the turbulence dissipation rate under all classical symmetries, to be invariant under the statistical translation symmetries and to scale quadratically under the statistical scaling symmetry. Note that these assumptions are made because the behavior of the turbulent kinetic energy and the turbulent dissipation rate under the statistical symmetries is very complicated, and it would not be feasible to incorporate them into the model. At the same time, the main purpose of eddy-viscosity models is to predict the mean velocity as accurately as possible, and the interpretation of k as the turbulent kinetic energy and of ε as the turbulent dissipation rate should not be taken too seriously. In classical turbulence modeling, this insight perhaps most obviously manifests itself in the development of the ε-equation (8), which is entirely empirical, and basing it on the exact equation for the turbulent dissipation rate would make little sense (e.g. Pope, 2000). To summarize, the symmetries (14), (15) and (20) with the auxiliary model variables read

$$\begin{aligned} T_{\mathrm{Sc},I}:\ & t^* = t,\ x_i^* = x_i e^{a_{\mathrm{Sc},I}} \\ & \bar{U}_i^* = \bar{U}_i e^{a_{\mathrm{Sc},I}},\ \bar{P}^* = \bar{P} e^{2a_{\mathrm{Sc},I}} \\ & k^* = k e^{2a_{\mathrm{Sc},I}},\ \varepsilon^* = \varepsilon e^{2a_{\mathrm{Sc},I}},\ \hat{\varepsilon}^* = \hat{\varepsilon} e^{2a_{\mathrm{Sc},I}} \qquad (25) \\ T_{\mathrm{Sc},II}:\ & t^* = t e^{a_{\mathrm{Sc},II}},\ x_i^* = x_i \\ & \bar{U}_i^* = \bar{U}_i e^{-a_{\mathrm{Sc},II}},\ \bar{P}^* = \bar{P} e^{-2a_{\mathrm{Sc},II}} \\ & k^* = k e^{2a_{\mathrm{Sc},II}},\ \varepsilon^* = \varepsilon e^{3a_{\mathrm{Sc},II}},\ \hat{\varepsilon}^* = \hat{\varepsilon} e^{3a_{\mathrm{Sc},II}} \qquad (26) \\ T_{\mathrm{Sc,stat}}:\ & t^* = t,\ x_i^* = x_i,\ \bar{U}_i^* = \bar{U}_i e^{a_{\mathrm{Sc,stat}}},\ \bar{P}^* = \bar{P} e^{a_{\mathrm{Sc,stat}}} \\ & k^* = k e^{a_{\mathrm{Sc,stat}}},\ \varepsilon^* = \varepsilon e^{2a_{\mathrm{Sc,stat}}},\ \hat{\varepsilon}^* = \hat{\varepsilon} e^{a_{\mathrm{Sc,stat}}} \qquad (27) \end{aligned}$$

The other symmetries (10)–(13), (21) and (22) are not repeated here because k, ε and $\hat{\varepsilon}$ are invariant under them.

Using these assumptions, a transport equation for $\hat{\varepsilon}$ can be formulated in analogy to the ε-equation (8)

$$\begin{aligned} \frac{\hat{D}\hat{\varepsilon}}{\hat{D}t} = & -\hat{C}_{\varepsilon,1}\left(\frac{\hat{\varepsilon}^2}{k\varepsilon}\tilde{R}_{ij}\frac{\partial \bar{U}_i}{\partial x_j} + \gamma\frac{\varepsilon}{k}\hat{R}_{ij}\frac{\partial \hat{U}_i}{\partial x_j}\right) - \hat{C}_{\varepsilon,2}\frac{\hat{\varepsilon}^2}{k} \\ & + \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_t}{\hat{\sigma}_\varepsilon}\right)\frac{\partial \hat{\varepsilon}}{\partial x_j}\right) \end{aligned} \qquad (28)$$

where

$$\hat{R}_{ij} = -\nu_t\left(\frac{\partial \hat{U}_i}{\partial x_j} + \frac{\partial \hat{U}_j}{\partial x_i}\right) \qquad (29)$$

Here, the additional model parameters $\hat{C}_{\varepsilon,1}, \gamma, \hat{C}_{\varepsilon,2}$ and $\hat{\sigma}_\varepsilon$ appear. Note that the γ-term is not strictly necessary from a symmetry perspective, but was found to improve the numerical stability of the model. The model equations for k and ε can now be modified according to the statistical scaling symmetry, leading to

$$\frac{\hat{D}k}{\hat{D}t} = -\frac{\hat{\varepsilon}}{\varepsilon}\tilde{R}_{ij}\frac{\partial \bar{U}_i}{\partial x_j} - \hat{\varepsilon} + \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) \qquad (30)$$

$$\begin{aligned} \frac{\hat{D}\varepsilon}{\hat{D}t} = & -C_{\varepsilon,1}\frac{\hat{\varepsilon}}{k}\tilde{R}_{ij}\frac{\partial \bar{U}_i}{\partial x_j} - C_{\varepsilon,2}\frac{\varepsilon\hat{\varepsilon}}{k} \\ & + \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right) \end{aligned} \qquad (31)$$

The equations (1), (6), (9), (23), (24), (28), (30) and (31) and (5) with $\hat{D}\bar{U}_i/\hat{D}t$ instead of $D\bar{U}_i/Dt$ then form a turbulence model that is invariant under all classical and statistical symmetries. For more details on the derivation of the model equations, we refer to Klingenberg & Oberlack (2022).

RESULTS OF THE MODIFIED k-ε-MODEL

The model development presented in the previous section is largely based on mathematical arguments. In order to assess the performance of the resulting model in practice and to establish appropriate values for the appearing model constants, it has to be applied to a range of canonical flow cases. The steps for the model calibration partly overlap with those used for the classical k-ε-model, but, as it turns out, the modified model can incorporate a wider variety of flows. More details on the model calibration are given in Klingenberg & Oberlack (2022).

First, the case of homogeneous turbulence, where all spatial gradients vanish, is invoked. Due the resulting simplification of the model equations, an analytical solution for the temporal evolution of k can be found, which reads

$$k(t) \propto t^{\frac{1}{1-\hat{C}_{\varepsilon,2}}} \qquad (32)$$

A comparison with experimental data leads to the choice $\hat{C}_{\varepsilon,2} = 1.92$, which is analogous to the constraint arising from homogeneous turbulence for the classical k-ε-model.

A slightly more complicated case arises from homogeneous shear turbulence, i.e. the case where one component of the velocity gradient, say $\partial\bar{U}_1/\partial x_2$, takes a constant nonzero value, but all other spatial gradients vanish as before. Under these circumstances, turbulence is sustained, and experiments show that the ratio k/ε becomes constant for large times, and the ratio of production and dissipation also approaches a constant value determined by $C_{\varepsilon,1}$. Since the evolution equation of k/ε is exactly the same as for the classical k-ε-model, the same conclusion of $C_{\varepsilon,1} = 1.44$ can be obtained for the modified model.

Furthermore, the modified model should be capable of accurately predicting the logarithmic law of the wall. Constraints on the model parameters can be inferred by inserting the classical ansatz

$$\begin{aligned} \bar{U}_1 = \frac{u_\tau}{\kappa}\log x_2 + B,\ \hat{U}_1 = \hat{C}\log x_2 + \hat{B} \\ k = C_k x_2^{n_k},\ \varepsilon = C_\varepsilon x_2^{n_\varepsilon},\ \hat{\varepsilon} = C_{\hat{\varepsilon}} x_2^{n_{\hat{\varepsilon}}} \end{aligned}$$

into the model equations (5), (28), (30) and (31). The requirement that all x_2-terms must cancel then leads to the result $n_k = 0,\ n_\varepsilon = n_{\hat{\varepsilon}} = -1$, which is also observed in the classical k-ε-model. In analogy to the classical model, the k-equation in conjunction with experimental results implies $C_\mu = 0.09$. Furthermore, assuming that we can neglect the small γ-term, the equations for ε and $\hat{\varepsilon}$ yield the constraint

$$\frac{C_{\varepsilon,1} - C_{\varepsilon,2}}{\hat{C}_{\varepsilon,1} - \hat{C}_{\varepsilon,2}} = \frac{\hat{\sigma}_\varepsilon}{\sigma_\varepsilon} \qquad (33)$$

Note that it is possible to take the γ-term into account, but then, due to the more complicated form of the equations, a simple constraint on the model parameters cannot be inferred.

At this point in the discussion, we deviate from the line of argumentation used for calibrating the classical k-ε-model for the first time, because we now consider a newly found core-region scaling law (Oberlack *et al.*, 2022). This scaling law matches direct numerical simulation (DNS) data in a large central region of channel flow. Inserting it for the mean velocity along with a power-law ansatz for the other variables

$$\hat{U}_1 = \hat{C}_1 x_2^{\hat{\sigma}_1} + \hat{C}, \ \bar{U}_1 = C_1 x_2^{\sigma_1} + C \\ k = C_k x_2^{n_k}, \ \varepsilon = C_\varepsilon x_2^{n_\varepsilon}, \ \hat{\varepsilon} = C_{\hat{\varepsilon}} x_2^{n_{\hat{\varepsilon}}} \tag{34}$$

leads to the exponents being restricted to $n_k = 1$, $n_\varepsilon = \sigma_1$, $n_{\hat{\varepsilon}} = -\sigma_1 + 1$ and $\hat{\sigma}_1 = 0$. DNS data suggests an asymptotic value of $\sigma_1 = 2$ for large Reynolds numbers, which is also predicted by the modified model if the model parameters fulfill the constraint arising from the log law (33). It is remarkable that no additional constraint arises from this flow, which highlights that more symmetries in a model increase its universality. The classical k-ε-model, on the other hand, does not perform well for this calibration case, because it predicts a value of $\sigma_1 = 1/2$, which contradicts numerical evidence.

Another flow that is not generally taken into account for the calibration of the k-ε-model is shear-free one-dimensional turbulence. Here, we assume that the velocity gradient vanishes, but allow a gradient of the scalar variables in one direction, which leads to a balance of dissipation and diffusion. This flow is also practically interesting, because it serves as a simple model for a quiet body of water in which turbulence is introduced by surface waves (Umlauf *et al.*, 2003). The ordinary differential equation (ODE) system arising from inserting these simplifications into the model equations is difficult to solve analytically, but experimental evidence suggests the power-law ansatz (Umlauf *et al.*, 2003)

$$k = C_k x_2^{n_k}, \ \varepsilon = C_\varepsilon x_2^{n_\varepsilon}, \ \hat{\varepsilon} = C_{\hat{\varepsilon}} x_2^{n_{\hat{\varepsilon}}} \tag{35}$$

The resulting nonlinear system of algebraic equations is solved numerically, and, for the choice of model parameters given by (43), the exponent n_k is determined to be -3.7, which is reasonably close to the experimentally observed range of $-1.7\ldots-3.0$ (Nokes, 1988). On the other hand, the classical k-ε-model with standard model parameters predicts $n_k \approx -4.97$, and changing the model parameters to yield more accurate results immediately leads to an inaccurate performance in the log region. The modified model, on the other hand, can produce reasonable results in both cases.

As a final test case, we consider the self-similar plane turbulent jet issuing into x_1-direction. Inserting the classical similarity ansatz

$$\bar{U}_1 = \frac{\tilde{u}}{x_1^{\frac{1}{2}}}, \ \hat{U}_1 = \frac{\tilde{\hat{u}}}{x_1^{\frac{1}{2}}}, \ \hat{U}_2 = \frac{\tilde{\hat{v}}}{x_1^{\frac{1}{2}}} \\ k = \frac{\tilde{k}}{x_1}, \ \varepsilon = \frac{\tilde{\varepsilon}}{x_1^{\frac{5}{2}}}, \ \hat{\varepsilon} = \frac{\tilde{\hat{\varepsilon}}}{x_1^{\frac{5}{2}}} \tag{36}$$

into the model equations (5), (23), (24), (28), (30) and (31) leads to the ODE system

$$\tilde{\hat{v}}' = \frac{1}{2}\tilde{\hat{u}} + \eta\tilde{\hat{u}}' \tag{37}$$

$$\tilde{\hat{u}}\left(\frac{1}{2}\tilde{\hat{u}} + \eta\tilde{\hat{u}}'\right) + \tilde{\hat{v}}\tilde{\hat{u}}' = \iota C_\mu \left(\frac{\tilde{k}^2}{\tilde{\varepsilon}}\tilde{\hat{u}}'\right)' \tag{38}$$

$$\tilde{\hat{u}}\left(\frac{1}{2}\tilde{u} + \eta\tilde{u}'\right) + \tilde{\hat{v}}\tilde{u}' = C_\mu \left(\frac{\tilde{k}^2}{\tilde{\varepsilon}}\tilde{u}'\right)' \tag{39}$$

$$\tilde{\hat{u}}\left(\tilde{k} + \eta\tilde{k}'\right) + \tilde{\hat{v}}\tilde{k}' = C_\mu \frac{\tilde{k}^2\tilde{\hat{\varepsilon}}}{\tilde{\varepsilon}^2}\tilde{u}'^2 - \tilde{\hat{\varepsilon}} + \frac{C_\mu}{\sigma_k}\left(\frac{\tilde{k}^2}{\tilde{\varepsilon}}\tilde{k}'\right)' \tag{40}$$

$$\tilde{\hat{u}}\left(\frac{5}{2}\tilde{\varepsilon} + \eta\tilde{\varepsilon}'\right) + \tilde{\hat{v}}\tilde{\varepsilon}' = C_\mu C_{\varepsilon,1}\frac{\tilde{k}\tilde{\hat{\varepsilon}}}{\tilde{\varepsilon}}\tilde{u}'^2 - C_{\varepsilon,2}\frac{\tilde{\varepsilon}\tilde{\hat{\varepsilon}}}{\tilde{k}} + \frac{C_\mu}{\sigma_\varepsilon}\left(\frac{\tilde{k}^2}{\tilde{\varepsilon}}\tilde{\varepsilon}'\right)' \tag{41}$$

$$\tilde{\hat{u}}\left(\frac{5}{2}\tilde{\hat{\varepsilon}} + \eta\tilde{\hat{\varepsilon}}'\right) + \tilde{\hat{v}}\tilde{\hat{\varepsilon}}' = C_\mu \hat{C}_{\varepsilon,1}\left(\frac{\tilde{k}\tilde{\hat{\varepsilon}}^2}{\tilde{\varepsilon}^2}\tilde{u}'^2 + \gamma\tilde{k}\tilde{\hat{u}}'^2\right) - \hat{C}_{\varepsilon,2}\frac{\tilde{\hat{\varepsilon}}^2}{\tilde{k}} + \frac{C_\mu}{\hat{\sigma}_\varepsilon}\left(\frac{\tilde{k}^2}{\tilde{\varepsilon}}\tilde{\hat{\varepsilon}}'\right)' \tag{42}$$

Herein, primes denote derivation with respect to the similarity variable $\eta = x_2/x_1$.

It was found that choosing

$$\sigma_k = 0.91, \ \iota = 2.0 \\ C_{\varepsilon,1} = 1.44, \ C_{\varepsilon,2} = 1.92, \ \sigma_\varepsilon = 1.3 \\ \hat{C}_{\varepsilon,1} = 1.42, \ \gamma = 0.05, \ \hat{C}_{\varepsilon,2} = 1.92, \ \hat{\sigma}_\varepsilon = 1.25 \tag{43}$$

for the model parameters not only fulfills all constraints developed above, but also yields very good agreement for the plane jet. The numerical results for the mean velocity are shown in Fig. 1. Evidently, the modified model performs slightly better than the classical model. In particular, it is interesting that the performance at the intermittent edge is slightly superior, which might be related to the fulfilling of the statistical symmetry (20), since this symmetry has also been connected to intermittency. However, more research is needed before this can be asserted with confidence.

CONCLUSION

In the present work, symmetry-based modifications are applied to the classical k-ε-model. The resulting model is tested against a range of canonical flow cases and found to be more universal than the classical k-ε-model. To some degree, this increased generality can be expected simply because of the additional model parameters appearing in the modified model, but it must be stressed that the additional statistical symmetries fulfilled by the model also contribute to the increased universality. This is perhaps most clearly observed for the core-region scaling law, which the modified model predicts accurately as long as the model parameters are calibrated against the log law. This result is promising, because it could be an indication that the modified model performs well in other flows it was not originally calibrated against, which is something that the classical k-ε-model and similar models tend to struggle with.

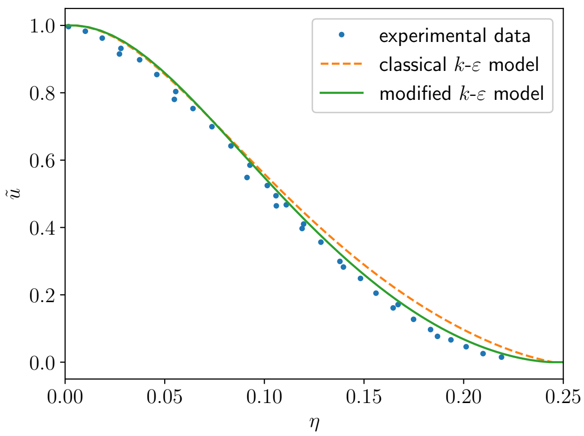

Figure 1: Plane jet experimental data (Wygnanski & Fiedler, 1969) compared with classical and modified, statistically invariant k-ε-model

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