12th International Symposium on Turbulence and Shear Flow Phenomena (TSFP12)
Osaka, Japan (Online), July 19-22, 2022

FOURIER-AVERAGED NAVIER-STOKES ANALYSIS OF PERIODIC WAKES: A NEW TECHNIQUE

Benjamin R.S. Freeman[1], Robert J. Martinuzzi[2], Arman Hemmati[3]
[1,3] Dept. of Mechanical Engineering, University of Alberta, Edmonton, AB, Canada T6G 2R3
[2]Dept. of Mechanical and Manuf. Engineering, University of Calgary, Calgary, AB, Canada T2N 1N4
[1]brfreema@ualberta.ca, [2] rmartinu@ucalgary.ca, [3]arman.hemmati@ualberta.ca

ABSTRACT

Periodic and quasi-periodic fluid motion is observed in engineering and flow applications, which involve a range of time and length scales. These coherent and regularly recurring structures profoundly affect momentum transport and turbulence energy budget. Hence, understanding the underlying mechanisms and flow dynamics associated with these structures are important for characterizing the shear flow phenomena. In particular, characterizing these periodic processes is important for guiding the design and control of fluid systems. Here, a new mathematical technique is proposed to assess and quantify the interactions of periodically occurring structures in an unsteady turbulent flow, hereon referred to as the "Fourier-averaged Navier-Stokes" or FANS. In this method, Fourier decomposition in time domain is applied to the Navier-Stokes equations and particular attention is given to the resulting nonlinear momentum flux terms. Examining the contributions of each decomposed momentum component provided physical insights into the interactions between structures across multiple time-scales. A general overview of the technique is provided with a summary derivation of the method. Illustrative case studies highlight the application and benefits of FANS to periodic flows.

INTRODUCTION

Several techniques have been developed with the aim to deepen our understanding of unsteady turbulent wakes exhibiting strong temporal periodicity. These flows are important to study due to their prevalence and complicated physics. Methods such as reduced-order models (ROM) or stability analysis have been widely and successfully used to investigate such flows. The conventional phase-averaging procedure (Eriksen & Krogstad, 2017) is limited to a single dominant frequency, and thus it is not suitable, by construction, to analyze complex interactions of different modes of motion. The many variants of Proper Orthogonal Decomposition (POD) and Dynamic Mode Decomposition (DMD) aim to find optimal reconstructions of the flow kinematics. Thus, they are useful in examining the flow in a model-free context (Chen *et al.*, 2012; Noack *et al.*, 2016). However, POD can fail to produce coherent results for flows with multiple important frequencies. These methods also do not provide a simple approach to compare important interactions between modes. Meanwhile, linear stability analysis tackles the onset and continuation of perturbations through a linearized analysis of the Navier-Stokes equations, which prevents it from describing nonlinear processes (Beneddine, 2017). Hence, none of these analyses are capable of providing a description of the forces arising from different flow structures over multiple timescales, particularly regarding interaction between these structures. Several flow assessment techniques have been introduced over the past decade, which connect mathematical observations with physical processes in such flows. Two notable methods are the Spectral Proper Orthogonal Decomposition (SPOD) and Bispectral Mode Decomposition (BMD). SPOD was coined by Picard & Delville (2000) to refer to a specific method of Lumley (Lumley, 1970). This method involves the determination of modes that vary in both space and time, granting it greater flexibility than classic POD (Towne *et al.*, 2018). Recently Towne *et al.* (2018) formally established a connection between SPOD and resolvent analysis, in that the two will be identical for white-noise (i.e. uncorrelated in space and time) forcing of the linearized Navier-Stokes equations used in resolvent analysis. This establishes a strong connection between the findings of SPOD and the linearized equations, which in turns suggests that SPOD modes have physical correspondence to important flow mechanisms. However, the connection relies on a white-noise assumption and the use of linearized equations in resolvent analysis, which clouds the overall scope of application when investigating interactions between these structures using SPOD.

The BMD suggested by Schmidt (2020) uses statistical methods to find optimal representations of Fourier modes. In this case, 'optimal' modes ($\hat{q}(f)$) from a quantity of interest ($q(t)$) are those that maximize triadic interactions in the flow, as defined through

$$b(f_1, f_2) = \int_V \hat{q}(f_1)\hat{q}(f_2)\hat{q}(f_1+f_2)\mathrm{d}V.$$

Here, BMD attempts to find correlations between the "input" of a triadic interaction, that is $q(f_1)q(f_2)$, and the resulting "output", $q(f_1+f_2)$. As a result, BMD may be effective at finding modes and regions of the flow that are highly involved in nonlinear interactions, but ignores the specific transfers of momentum that are involved in these processes.

To address this gap in analyzing nonlinear interactions using existing methods, a new formulation based on Fourier analysis of the momentum fluxes in a flow is proposed– the Fourier-Averaged Navier-Stokes, or FANS equations. Applying a Fourier decomposition in time to the momentum equations themselves, and not just the flow field, unveils properties of the momentum fluxes at different timescales. Thus, critical information about the flow dynamics can be obtained. This leads to valuable insights into the periodic features of the unsteady flow. Unlike other frequency-domain methods, FANS specifically attempts to provide a rigorous link between the flow physics and the governing equations. This methodology has several similarities to the spatially-spectral methods that have enjoyed great success in turbulence and instability analysis (e.g. Barkley & Henderson (1996), chapter 10 of Durbin & Pettersson (2011)), however here the specific focus is on a temporal decomposition. The primary hypothesis for FANS is that many unsteady wakes contain at least one dominant frequency. Examples of such flows are the wakes characterized by Khalid *et al.* (2020), Bai & Alam (2018), and Beneddine (2017). The FANS formulations are likewise capable of handling flows with multiple dominant frequencies, where complicated dynamics may result due to interaction between structures. By virtue of their periodic characteristics, the dynamics of these structures and their resulting effect on the momentum balance of the entire flow would be naturally described by looking at the momentum budget at that frequency.

To introduce FANS and its application, a brief derivation and two case studies are considered. First, FANS formulations are introduced, along with methods to carry out calculations of the key quantities. Then, two case studies on simple periodic flows are considered. By analysing the wake behind a square cylinder and an oscillating foil, connections between FANS-calculated quantities and physical phenomena are established. These highlight how FANS can elucidate mechanisms behind complicated flow physics in certain flow regimes.

FORMULATION

In order to arrive at an expression representing the balance of momentum at each timescale, a periodic flow field (consisting of velocity $\mathbf{u}$, and pressure p) with period $T = 2\pi/\omega$ is assumed. This allows us to write the flow field as a Fourier series in time:

$$\mathbf{u} = \sum_{m=-\infty}^{\infty} \hat{\mathbf{u}}^m e^{j\omega mt}, \quad p = \sum_{m=-\infty}^{\infty} \hat{p}^m e^{j\omega mt}, \tag{1}$$

where $\hat{\mathbf{u}}, \hat{p}$ are the Fourier coefficients corresponding to velocity and pressure, respectively. Assuming incompressible Newtonian flow, the flow field obeys the Navier-Stokes equations,

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u}. \tag{2}$$

In order to isolate the momentum balance at a particular timescale, Eq. 2 is subjected to a Fourier decomposition by applying an inner product with a test function $e^{-j\omega kt}$, exploiting the orthogonality property:

$$\int_0^T e^{-j\omega kt} e^{j\omega mt}\,\mathrm{d}t = \begin{cases} 0 & k \neq m \\ T & k = m \end{cases} \tag{3}$$

The process is applied term by term. For the pressure and diffusion fluxes, this process yields:

$$\begin{aligned}
&\int_0^T e^{-j\omega kt}\left(-\nabla p + \nu\nabla^2\mathbf{u}\right)\mathrm{d}t \\
&= \int_0^T e^{-j\omega kt}\left(-\nabla\sum \hat{p}^m e^{j\omega mt} + \nu\nabla^2\sum\hat{\mathbf{u}}^m e^{j\omega mt}\right)\mathrm{d}t \\
&= \int_0^T \left(-\nabla\sum \hat{p}^m e^{j\omega(m-k)t} + \nu\nabla^2\sum\hat{\mathbf{u}}^m e^{j\omega(m-k)t}\right)\mathrm{d}t \\
&\qquad = \int_0^T \sum(-\nabla\hat{p}^m + \nu\nabla^2\hat{\mathbf{u}}^m)e^{j\omega(m-k)t}\mathrm{d}t
\end{aligned}$$

Note that all summations are indexed by $m \in (-\infty,\infty)$, omitted here for cleanliness. Of the terms in the summation, only the k^{th} term survives the integration due to orthogonality. Thus:

$$\int_0^T e^{-j\omega kt}\left(-\nabla p + \nu\nabla^2\mathbf{u}\right)\mathrm{d}t = -\nabla\hat{p}^k + \nu\nabla^2\hat{\mathbf{u}}^k \tag{4}$$

The process is similar for the acceleration term:

$$\int_0^T e^{-j\omega kt}\partial_t\sum\hat{\mathbf{u}}^m e^{j\omega mt}\mathrm{d}t = \int_0^T \sum j\omega m\hat{\mathbf{u}}^m e^{j\omega(m-k)t}\mathrm{d}t = j\omega k\hat{\mathbf{u}}^k \tag{5}$$

The convective term is somewhat more complicated due to its nonlinear dependence on $\mathbf{u}$.

$$\begin{aligned}
\int_0^T e^{-j\omega kt}\mathbf{u}\cdot\nabla\mathbf{u}\mathrm{d}t &= \int_0^T e^{-j\omega kt}\sum_m \hat{\mathbf{u}}^m e^{j\omega mt}\cdot\nabla\sum_n\hat{\mathbf{u}}^n e^{j\omega nt}\mathrm{d}t \\
&= \int_0^T \sum_{m,n}\hat{\mathbf{u}}^m\cdot\nabla\hat{\mathbf{u}}^n e^{-j\omega(n+m-k)t}\mathrm{d}t
\end{aligned}$$

Terms in the double summation survive only when $m = k-n$, reducing it to a single summation,

$$\begin{aligned}
\int_0^T e^{-j\omega kt}\mathbf{u}\cdot\nabla\mathbf{u}\mathrm{d}t &= \sum_n \hat{\mathbf{u}}^{k-n}\cdot\nabla\hat{\mathbf{u}}^n \\
&= \mathbf{U}\cdot\nabla\hat{\mathbf{u}}^k + \hat{\mathbf{u}}^k\cdot\nabla\mathbf{U} + \sum_{n\neq 0,k}\hat{\mathbf{u}}^{k-n}\cdot\nabla\hat{\mathbf{u}}^n,
\end{aligned} \tag{6}$$

where terms corresponding to $n = 0$ and k are extracted to highlight the extreme importance of the mean flow $\mathbf{U}$, and to bring the FANS notation in line with other methods, such as Reynolds-Averaged-Navier-Stokes (RANS) and linear stability analysis. Collecting these terms together returns a momentum balance corresponding to a mode $\hat{\mathbf{u}}^m$ in terms of itself and the other Fourier modes:

$$\begin{aligned}
&\underbrace{j2\pi kf\hat{\mathbf{u}}^k}_{\hat{\mathbf{T}}^k} + \underbrace{\mathbf{U}\cdot\nabla\hat{\mathbf{u}}^k + \hat{\mathbf{u}}^k\cdot\nabla\mathbf{U}}_{\hat{\mathbf{C}}^k} = \\
&\qquad \underbrace{-\nabla\hat{p}^k}_{\hat{\mathbf{P}}^k} + \underbrace{\nu\nabla^2\hat{\mathbf{u}}^k}_{\hat{\mathbf{D}}^k} - \underbrace{\sum_{n\neq 0,k}\hat{\mathbf{u}}^n\cdot\nabla\hat{\mathbf{u}}^{k-n}}_{\hat{\chi}^k}
\end{aligned} \tag{7}$$

Here, terms corresponding to convection ($\mathbf{C}^m$, χ^m) and pressure ($\mathbf{P}^m$) are the most important to analyze, as they are representative of the triadic interactions that are highly important in turbulent flows, e.g., Schmidt (2020). However, this derivation is valid for a continuous, periodic flow field. Since almost all fluid dynamics research is built on analyzing snapshots of discrete data, either through simulations or experiment, it is important to translate these results into a discrete domain. Supposing that there is a collection of N evenly-spaced snapshots $\{\mathbf{u}_n\}$ of discrete flow field data, the discrete Fourier transform of this data would be given by

$$\hat{\mathbf{u}}^k = \sum_{m=0}^{N-1}\mathbf{u}^m e^{-j2\pi km/N} \tag{8}$$

Since vectors of dynamics $a_m = e^{-j2\pi km/N}$ are orthogonal under the inner product of $\overline{a}_m\cdot a_n = \delta_{mn}$, this process may be repeated for the discrete analog of the Navier-Stokes equations:

$$\frac{\Delta \mathbf{u}^n}{\Delta t} + C[\mathbf{u}^n, \mathbf{u}^n] = -G[p^n] + \nu L[\mathbf{u}^n] \tag{9}$$

using discrete convection (C), gradient (G) and Laplacian (L) operators, as calculated through whichever method is most appropriate to the data (finite differences, volumes, elements, etc.). Fortuitously, orthogonality of the vectors a_m may be used to find the Fourier modes. This results in a discrete formulation that is similar to the continuous FANS equations:

$$\frac{j2\pi k}{N}\hat{\mathbf{u}}^k + C[\mathbf{U}, \hat{\mathbf{u}}^k] + C[\hat{\mathbf{u}}^k, \mathbf{U}] = -G[\hat{p}^k] + \nu L[\hat{\mathbf{u}}^k] - \sum_{n \neq 0,k} C[\hat{\mathbf{u}}^{k-n}, \hat{\mathbf{u}}^n] \tag{10}$$

FANS formulations are attractive for analyzing unsteady (especially periodic) flows for multiple reasons. For instance, since the regular Navier-Stokes equations may be reconstructed from the decomposed equations (Eq. 7) through a Fourier series, it can be said that all physical interactions represented in the flow are contained in the FANS equations. Another attractive aspect is ease of implementation. FANS-based analysis depends on a discrete Fourier transform of snapshots (usually an FFT) and discrete derivative calculations, both of which are widely available in software packages commonly used for fluid dynamics research. The following sections will provide an example of this analysis on familiar periodic flows to highlight the application of the method.

Existing methods in literature

The above formulation (FANS) bears similarities to existing methods based on a temporal Fourier decomposition that have been discussed in literature. For example, similar formulations have been used as a simulation tool in the past, namely the Spectral Time Discretization (STD) by Carte *et al.* (1995), harmonic balance (HB) technique by Hall *et al.* (2002), and Self-Consistent method (SCM) by Mantič-Lugo *et al.* (2015). The HB and STD are specifically high-accuracy, fast simulation methods for periodic flows, however notably the HB is used for compressible flows with externally forced periodicity, primarily turbomachinery. Meanwhile, SCM can be viewed as an extension of linear stability analysis used to calculate saturation dynamics and sensitivity maps, albeit at significant effort. In this way, the FANS formulations may not constitute a new mathematical approach, however, their application as a tool to obtain physics-based insights using existing data is novel. The previous applications strictly focus on solving for an unknown flow field, whereas the current work contends that the resulting formulations are also useful in post-processing flow field data. A key advantage of using the formulations for postprocessing opens up insights from a larger variety of flows and data acquisition methods. In particular, simulating flows with a large number of harmonics or broadband spectrum may be difficult or infeasible with a time-spectral method. However, investigating physics involving periodically-occuring structures may still be worthwhile. Using the formulations for postprocessing allows these investigations to occur.

CASE STUDIES

The main case studies completed include a two-dimensional direct numerical simulation (DNS) of an oscillating foil with combined pitching and heaving motion at $Re = 1000$ and periodic vortex shedding behind a square cylinder at $Re = 100$. OpenFOAM is used as the main computational platform, although different methods are used for solving the flow. Second order accurate spatial and temporal discretization methods are used for solving the convective, diffusive and advective terms. The Pressure-Implicit with Splitting of Operators (PISO) method is used for coupling velocity and pressure fields. Specific details of each simulation are detailed separately in the coming sections.

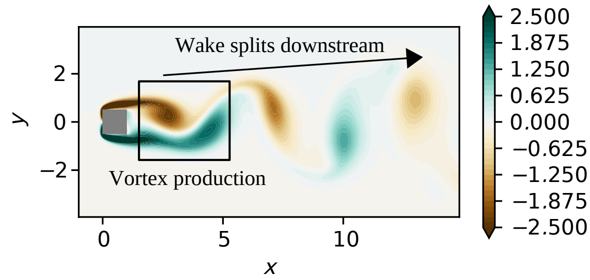


Figure 1. Contours of vorticity trailing a square cylinder. Vortices are produced and shed immediately behind the cylinder, after which time they depart from the centreline at a constant speed.

Square Cylinder: cylinders at low Re are common case studies for investigating periodic flows due to their regularity and simplicity, e.g., Schmidt (2020). For this investigation, a square cylinder oriented normal to the streamwise direction at $Re = 100$ was selected since the resulting flow exhibits purely periodic vortex shedding in a street extending from behind the cylinder, and it is below the threshold for 2D-3D transitions. The incompressible, laminar solver icoFOAM is selected for the simulation. A simple structured mesh with local refinement around the cylinder is used, following the mesh description in Bai & Alam (2018).

Oscillating Foil: the second case study is a teardrop foil undergoing combined sinusoidal pitching and heaving motion. The reduced frequency of the motion ($f^* = fc/U_\infty$) is 0.4, while the pitching amplitude is 8° and heaving amplitude is $h/c = 0.25$. At $Re = 1000$, this flow is characterized by regular shedding of both leading and trailing edge vortices due to the combined motion. An Overset Grid Assembly in OpenFOAM was used for the simulations. More computational details are described in Verma *et al.* (2022). Five cycles are considered for data analyses after reaching statistical convergence.

ANALYSIS

Square Cylinder: Results of the FANS decomposition of the wake behind a square cylinder are shown in Figures 2 - 4. Fourier decomposition shows the presence of two strong frequencies, corresponding to the primary vortex shedding frequency (Mode 1) and its harmonic (Mode 2). Figure 2a shows the spatial distribution of Mode 1, from which the vortex street can be clearly identified as it splits and moves downstream. Figure 2b shows the corresponding harmonic structure that is strongest in the near-wake, where there are interactions between two vortex streets. Using the momentum budgets in the wake region, it is shown that the main frequency represents a self-sustaining structure, while the harmonic arises naturally due to the convection induced by the velocity fluctuations.

Effect of the nonlinear forcing term (χ_u^1) is not critical in the flow as suggested by the results in Figure 4a, where the wake is largely dominated by the pressure, convective, and temporal terms. Since χ_u^1 is the only connection to other fluctuating modes, and the other terms are linear in $\hat{u}^1$, this indicates that the primary frequency exists as a self-sustaining

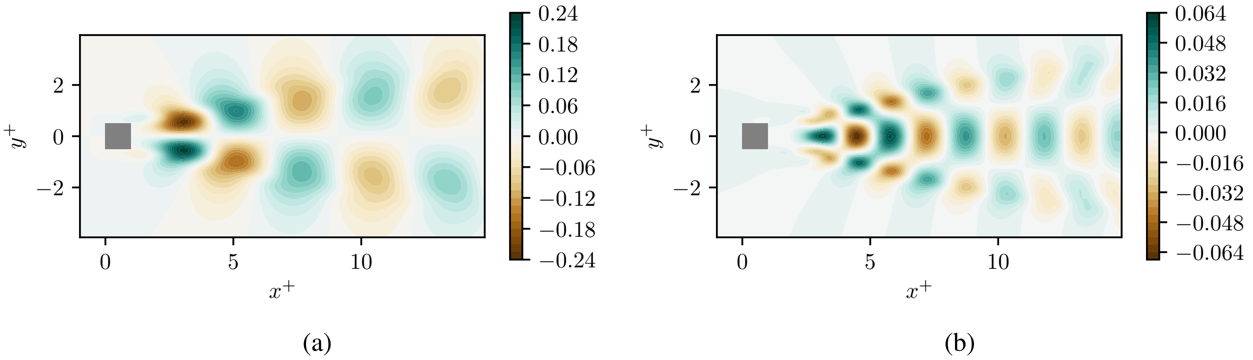


Figure 2. Contours of real parts of $\hat{u}^m$ for streamwise velocity, u, for (a) Mode 1 and (b) Mode 2. Imaginary parts are of similar magnitude, shifted in space by half a vortex length.

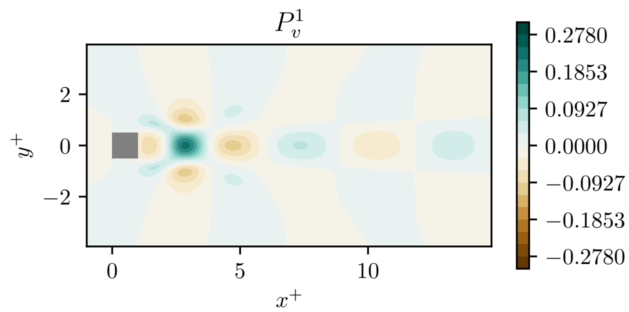


Figure 3. Pressure flux P_v^1 corresponding to mode $\hat{v}^1$, showing strong pressure fluctuations in the region of vortex production.

process. This observation about the vortex street is in agreement with the observations of linear stability analysis reported for example by Noack *et al.* (2003). The pressure flux contour for $\hat{v}^1$, suggests the origin of the split wake. This separation of vortices away from the centreline can be attributed to strong pressure fluctuations in the near-wake, as shown in Figure 3.

Pressure fluctuations are generated by the combined effect of distortions of the mean flow and primary velocity fluctuations in the vortex formation region. Meanwhile, the effect of these pressure fluctuations on the streamwise velocity is highlighted by Figure 4a. Note that the streamwise fluid acceleration (T_u^1) and pressure gradient terms (P_u^1) at the fundamental frequency are in phase as the vortices convect downstream, driven by the mean flow convection (C_u^1). This indicates that the fluctuations of the pressure deficit region serve to dampen the streamwise fluctuations, which slows and distorts vortices. This corresponds to the intuitive idea that pressure deficit behind the square cylinder entrains freestream fluid into the wake region, slowing structures that depart in the streamwise direction, reducing the convection rate of the shed vortices. Here, FANS intuitively represents this physical flow process.

FANS analyses highlight the origins of the harmonic as well. Figure 4b shows the importance of the forcing term χ_u^2 in perpetuating the harmonic. The magnitude of this momentum flux is comparable to the mean-flow convective (C_u^2), pressure (P_u^2), and temporal (T_u^2) terms, indicating that the 'forcing' generated by χ_u^2 continues to create a harmonic even after the wake has split and vortices are no longer interacting. Based on an order-of-magnitude analysis, χ_u^2 is generated entirely by the mechanism associated with the primary frequency, since it is the only other significant mode along with the mean flow:

$$\chi_u^2 = \sum_{n\neq 0,2} \hat{u}_k^n \partial_k \hat{u}^{2-n} \approx \hat{u}_k^1 \partial_k \hat{u}^1 \tag{11}$$

This indicates how FANS analyses highlight the origin of triadic interactions of the fundamental frequency, which leads to the generation of harmonics. This finding corroborates those of other analyses on cylinder wakes, e.g., BMD and self-consistent method (Schmidt, 2020; Mantič-Lugo *et al.*, 2015). These conclusions come directly out of the analysis with little computational effort.

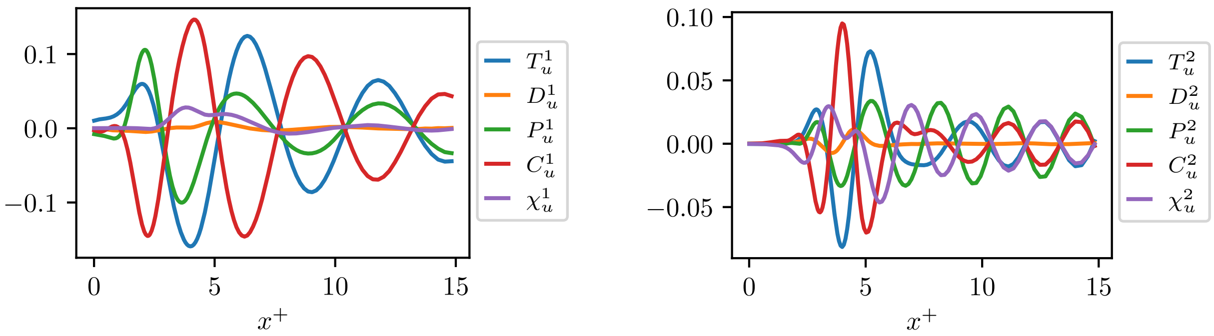
Oscillating Foil: In order to deepen the discussion of FANS and its representation of unsteady wakes, a periodically pitching and heaving foil is analyzed. This oscillatory wake is characterized by pairs of vortices shed during each oscillation cycle from the leading and trailing edges on both top and bottom of the foil, as depicted in Figure 5. The contours of vorticity show strong shearing of the flow between the leading and trailing edge vortices as they develop. These connections link these structures strongly in this region. To isolate the fluid motion around the foil for FANS analyses, a reference frame fixed on the foil is considered. Note that the original formulations (Eqs. 7, 10) will then contain added terms due to the Coriolis and centripetal forces. The contribution of these terms at a harmonic k is denoted as τ^k.

The first mode of the streamwise velocity, corresponding to the foil oscillation frequency, is shown in Figure 6. Effect of the alternated shedding of leading and trailing edge vortices from the foil edges can be seen in their respective locations. Note the meandering of the wake due to the change in reference frame.

The momentum flux terms ($\hat{P}^1, \hat{C}^1$) dominate the momentum transport phenomenon at the foil oscillatory frequency in Figures 7 and 8. This is due to flow stagnation in the narrow nose region at the leading edge, the location of which moves during the cycle by the foil rotation. This oscillation contributes to shedding of the leading edge vortices.

The motion of leading edge vortices along the foil body is an important factor leading to changes in the thrust generated by an oscillating foil (Verma *et al.*, 2022). Thus, it would be interesting to see if such a motion can be isolated by flux terms in FANS formulations.

As seen in Figure 9, the effect of this motion is more apparent at higher harmonics due to the interaction between the foil and leading edge vortices. This vortex motion drives the inter-harmonic triadic coupling along the foil boundary. In



(a) (b)

Figure 4. Real parts of momentum budgets of $\hat{u}^m$ behind the cylinder along $y^+ = 1$ for (a) Mode 1 and (b) Mode 2.

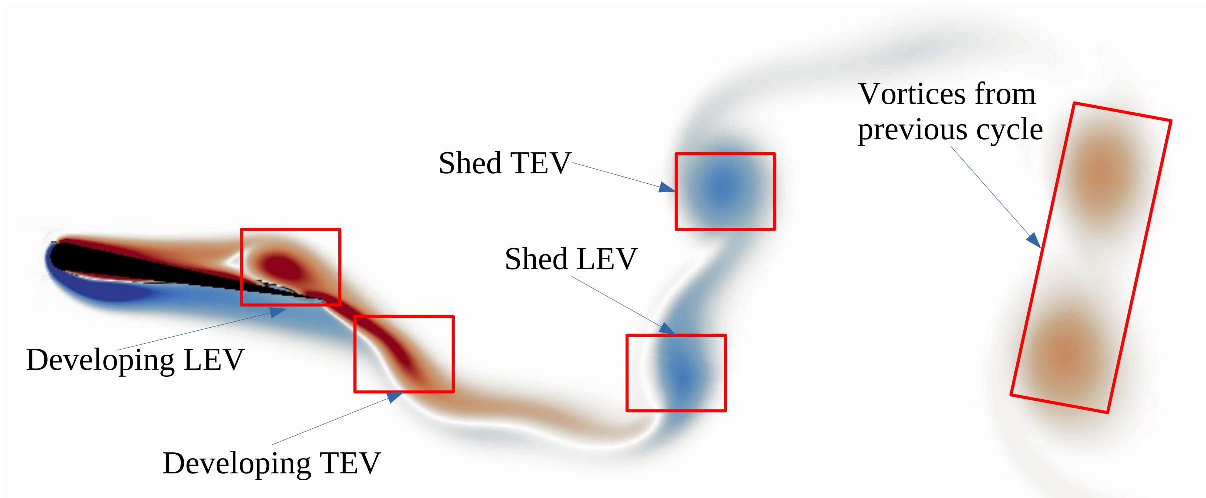


Figure 5. Vortex shedding pattern of foil undergoing simultaneous heaving and pitching

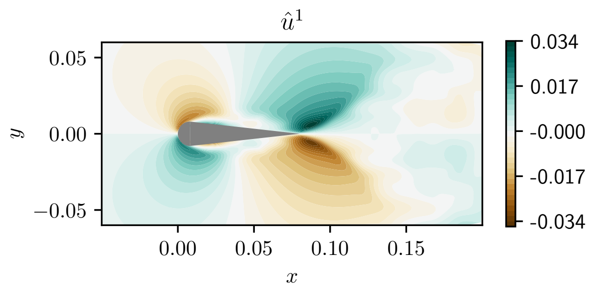


Figure 6. Real part of the first mode of the streamwise velocity fluctuations

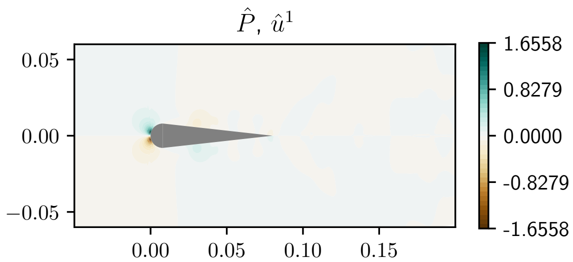


Figure 7. Streamwise pressure gradient corresponding to mode 1

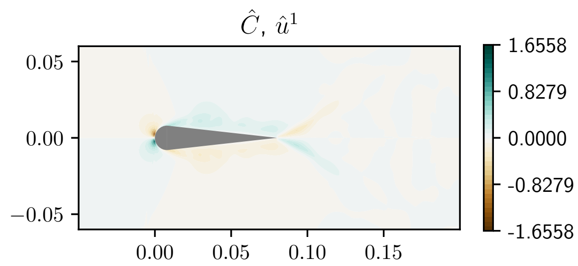


Figure 8. Streamwise convective flux corresponding to mode 1

turn, this convective coupling generates strong pressure gradients on the foil (Figure 10), which have a pronounced effect on thrust generations. The appearance of these structures in FANS analyses shows how it may be connected to important performance characteristics, such as thrust generation of an oscillating foil and the utilized power. Finally, Figure 9 shows a significant degree of triadic interaction just past the trailing edge through $\hat{\chi}^3 \approx \nabla \cdot (\hat{u}^1 \hat{u}^2)$. This can be inferred due to the high saturation of contours in the wake behind the foil. As the color scheme is referenced to the maximum momentum flux in this region, dark contours are indicative of locations where the inter-harmonic coupling is dominant. These convective couplings suggestive of the shearing between the shed trailing and leading edge vortices, earlier seen in the developing vortex pat-

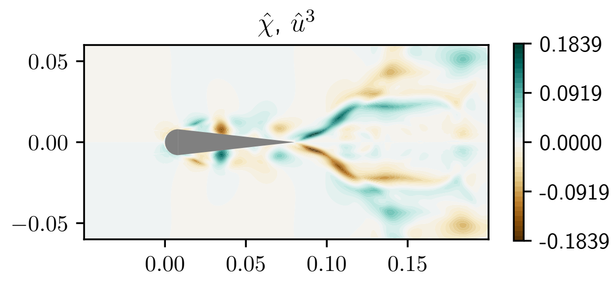


Figure 9. Inter-harmonic streamwise momentum flux corresponding to the 3rd harmonic of an oscillating foil

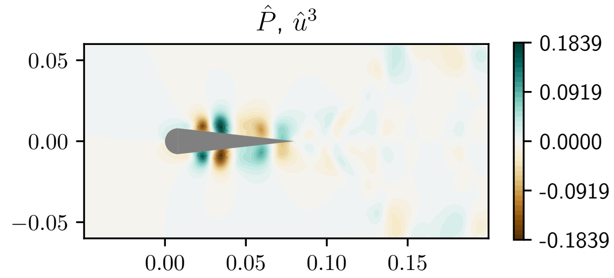


Figure 10. Streamwise pressure gradient corresponding to the 3rd harmonic of an oscillating foil

tern in Figure 5. As this connection between vortices occurs only during select portions of the cycle, it is natural that the resulting coupling would be found at a timescale corresponding to a higher harmonic. This suggests that FANS can highlight complicated interactions between vortex structures, as well as complementing other forms of wake analyses.

Conclusion

The Fourier-Averaged Navier-Stokes equations represent a novel method based on Fourier decomposition of a flow to gain insights into the dynamics of periodic phenomena. In FANS operations, the momentum budgets corresponding to individual frequencies are extracted. Here, the FANS formulations are presented and a basic analysis is performed by applying them to the periodic wake of a square cylinder and oscillating foil. These results show that the primary frequency, corresponding to the vortices traveling downstream, are largely unaffected by higher frequency content in the wake. This agrees with linear stability analysis. Meanwhile, FANS builds upon these findings by showing that the flow harmonic is generated naturally by a convective process of the large-scale structures, which is an important source of secondary velocity fluctuations in the wake. Likewise, the role of convective- and pressure-based interactions around and behind an oscillating foil were shown in connection to the motion of the leading and trailing edge vortices. In this way, FANS-based analysis is a convenient method to analyze nonlinear interactions by leveraging information from the governing equations. Each of the above findings indicates that FANS is capable of recreating the results of other methods, using techniques that are widely utilized in a fluid dynamics context. The simplicity and connection to physical phenomena presented by FANS formulations make it an attractive technique for analyzing periodic and quasi-periodic flows.

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