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ON A NEW SYMMETRY-BASED RANS MODELLING CONCEPT

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ABSTRACT

This work presents an entirely new route to developing turbulence models. The core idea is to assume the most general form of a model equation possible, and to then use the Lie symmetries of the time-averaged Navier-Stokes equations to constrain the form of the equations. In addition to the classical Navier-Stokes symmetries, which are already contained in most existing turbulence models, we integrate into the model so called statistical symmetries, which are specific to the equations describing turbulent statistics. These statistical symmetries are an important recent development, but their high relevance for accurately describing turbulent statistics has already been demonstrated. However, they have not been incorporated into any existing turbulence model yet. It is found that somewhat unconventional model variables are crucial to enable incorporating these statistical symmetries into a turbulence model. A simple prototype of a model obtained in this way is presented.

INTRODUCTION

Even though most natural or technical flows of interest are highly turbulent, there is still no universally accurate and computationally feasible method available for numerically investigating them. The present work is concerned with Reynolds-averaged Navier-Stokes (RANS) modelling, which uses the time-averaged Navier-Stokes equations as its starting point. The RANS equations for incompressible flow read (Reynolds, 1895)

$$\frac{\partial \bar{U}_i}{\partial x_i}=0, \quad \frac{\partial \bar{U}_i}{\partial t}+\frac{\partial \overline{U_i U_j}}{\partial x_j}+\frac{\partial \bar{P}}{\partial x_i}-\nu\frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j}=0, \qquad (1)$$

where U_i is the velocity, P the pressure divided by the density, ν the kinematic viscosity, and t and x_i are the temporal and spatial coordinates, respectively. The bar denotes time-averaging. Usually, the Reynolds decomposition $U_i=\bar{U}_i+u_i$ is employed and with this $\overline{U_iU_j}$ is expanded to $\bar{U}_i\bar{U}_j+\overline{u_iu_j}$, where u_i is the fluctuating part of the velocity. In the present work, this expansion will be omitted when convenient, and therefore we introduce the abbreviations

$$H_{ij}=\overline{U_iU_j} \text{ and } R_{ij}=\overline{u_iu_j}. \qquad (2)$$

The latter is usually referred to as the Reynolds stress tensor, and subsequently, equations written in terms of it are said to be in fluctuation formulation. On the other hand, using H_{ij} directly leads to what is called the instantaneous formulation.

As H_{ij} (or R_{ij}) is unknown, equation (1) cannot be solved without making additional empirical assumptions. This is known as the closure problem of turbulence. Many such assumptions resulting in different turbulence models have been put forward. Most practically used turbulence models fall into two categories, eddy viscosity models and Reynolds stress transport models. Eddy viscosity models usually use the Boussinesq approximation (Boussinesq, 1877)

$$R_{ij}=-\nu_t\left(\frac{\partial \bar{U}_i}{\partial x_j}+\frac{\partial \bar{U}_j}{\partial x_i}\right)+\frac{2}{3}k\delta_{ij}, \qquad (3)$$

or some extension thereof, where δ_{ij} is the Kronecker delta, ν_t is a model variable called the turbulent viscosity, and k is the turbulent kinetic energy, i.e. the trace of the Reynolds stress tensor divided by two, for which a model also has to be formulated. Equation (3) is used to replace the unknown R_{ij} in equation (1) directly. Depending on how ν_t and k are modeled, various models emerge. Reynolds stress transport models take a different approach. No model assumptions are fed into equation (1), but instead, a transport equation for R_{ij}, which can be derived from the original Navier-Stokes equations and equation (1), is considered. This equation contains multiple unknown terms, namely dissipation, pressure-strain-correlation and the triple velocity correlation, for which models have to be formulated. This makes the modelling process somewhat more challenging, but the resulting model can potentially incorporate important turbulent effects more naturally.

Early models were often developed with one specific flow case in mind, and were found to not generalize very well. It was probably first understood by Donaldson & Rosenbaum (1968) that in order for a model to be generally valid, its equations have to be (i) in correct dimensional formulation, (ii) in correct tensorial formulation, and (iii) Galileian invariant, though in the literature it is sometimes understood that (ii) is part of (iii). Galileian invariance implies that the equations must not change if the physical system is translated at a constant velocity, because such a shift does not influence the physical process itself. These conditions are subsumed under the term invariant modelling, and are fulfilled by virtually all modern turbulence models.

Underlying all three conditions is an important observation: within the scope of classical mechanics, there are transformations that can be applied to a physical system without affecting the physical process inside of the system. For an equation describing a physical process, this imposes the requirement that the equation must also be invariant under the corresponding variable transformation. Dimensional consistency expresses the fact that a physical process is insensitive to the unit system used to describe it, and that the equations therefore must not change if the unit system is changed. Tensorial consistency ensures invariance with respect to rotating the physical system by some fixed angle.

In mathematics, such form invariant transformations are known as symmetries. This term is well-known from geometrical contexts, where it means the same thing: A circle is said to have a rotational symmetry because rotating it does not change its appearance. For symmetries of differential equations, a powerful theory, discussed in such textbooks as Bluman *et al.* (2010), is available. In particular, it is possible to calculate the complete set of symmetries of a given equation algorithmically, i.e. without any ad-hoc assumptions or the need for human intuition. Besides giving a deep insight into the physical problem, these symmetries constitute the ideal basis for generating exact analytical solutions. For example, in turbulence research, this method has been used successfully by Sadeghi *et al.* (2018) to obtain very accurate scaling laws for the temporally evolving jet flow.

In this work, however, the focus will lie on how symmetries can facilitate the development of turbulence models. This approach strongly expands the insight already recognized by the invariant modelling concept, namely the importance of symmetries, by making better use of its mathematical potential. In particular, this allows us to incorporate important features specific to turbulent statistics in addition to general principles of classical physics into the model. Furthermore, we can formalize the modelling process to a great extent, thus creating a very general picture of what a model should look like. Finally, it should be stressed that symmetries affect the physical properties of the entire equation under investigation, not only single terms. In turn, the subsequent modelling methodology generates model equations rather than focusing on each term individually.

MATHEMATICAL BACKGROUND

We presently employ the theory of Lie symmetries, which originates from the analysis of differential equations, in a somewhat unconventional fashion: Rather than extracting symmetries from equations in order to generate solutions, we use symmetries to create new model equations. In other words, we use the symmetries of the unclosed but exact description to generate a closed model containing the same set of symmetries. It is critical for a model to be consistent with the symmetries of the system it intends to describe, because otherwise, under some circumstances, the model will behave differently than the original system.

Generally speaking, given some (differential) equation F in the independent variable $\mathbf{x}$ and the dependent variables $\mathbf{y}$, a transformation of the general form

$$T: \quad \mathbf{x}^* = \mathbf{f}(\mathbf{x},\mathbf{y};a), \quad \mathbf{y}^* = \mathbf{g}(\mathbf{x},\mathbf{y};a), \tag{4}$$

is called a Lie symmetry of this equation if it maps the equation onto itself, i.e.

$$F(\mathbf{x},\mathbf{y}) = 0 \Leftrightarrow F(\mathbf{x}^*,\mathbf{y}^*) = 0, \tag{5}$$

and equation (4) admits transformation group properties, where a is the group parameter. Without loss of generality, Lie symmetries (4) can always be written such that for $a = 0$, equation (4) is the identity transformation, i.e. $\mathbf{x}^* = \mathbf{x}$, $\mathbf{y}^* = \mathbf{y}$. (Bluman *et al.*, 2010)

With these prerequisites, we expand equation (4) into a Taylor series around $a = 0$, yielding

$$x_i{}^* = f_i(\mathbf{x},\mathbf{y};a) = x_i + a\xi_i(\mathbf{x},\mathbf{y}) + O(a^2), \tag{6}$$

$$y_i{}^* = g_i(\mathbf{x},\mathbf{y};a) = y_i + a\eta_i(\mathbf{x},\mathbf{y}) + O(a^2), \tag{7}$$

where

$$\xi_i = \left.\frac{\partial f_i}{\partial a}\right|_{a=0}, \quad \eta_i = \left.\frac{\partial g_i}{\partial a}\right|_{a=0}. \tag{8}$$

According to Lie's first theorem, the infinitesimals ξ_i and η_i are sufficient to describe the action of the symmetry transformation, or, in other words, the transformation (4) can be uniquely rederived from ξ_i and η_i, and the $O(a^2)$-terms can be omitted without losing information. Inserting (6) and (7) into equation (5) leads to

$$\begin{aligned} F(\mathbf{x}^*,\mathbf{y}^*) &= F\left(\mathbf{x} + a\boldsymbol{\xi} + O(a^2), \mathbf{y} + a\boldsymbol{\eta} + O(a^2)\right) \\ &= F(\mathbf{x},\mathbf{y}) + aXF(\mathbf{x},\mathbf{y}) + O(a^2) = 0, \end{aligned} \tag{9}$$

where X is the so called infinitesimal generator defined by

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i}. \tag{10}$$

This representation is known as the infinitesimal form of a symmetry, while equation (4) is known as its global form. Thanks to Lie's first theorem, the global form and the infinitesimal form can be shown to be completely equivalent, and one can always calculate one from the other. Furthermore, inserting equation (5) into equation (9) cancels the left hand side as well as the leading order term on the right hand side. As the $O(a^2)$-terms can be omitted in equation (9), it follows that F is invariant under a symmetry X if

$$XF\Big|_{F=0} = 0. \qquad (11)$$

Equation (11) conveniently allows examining whether an equation is invariant under a given symmetry, or, if the symmetries are unknown, they can be computed from a given F. On the other hand, if F is unknown, equation (11) can be viewed as a constraint on F, fulfilling which ensures invariance of F under X. This is usually called invariant modelling.

SYMMETRIES OF THE RANS EQUATIONS

The symmetries of the original Navier-Stokes equations were probably first found in Bytev (1972) and read in global form

$$T_t: \ t^* = t + a, \ \ x_i{}^* = x_i, \ \ U_i{}^* = U_i, \ \ P^* = P \qquad (12)$$

$$T_{\mathrm{Gal}_i}: \ t^* = t, \ \ x_i{}^* = x_i + h_i(t), \ \ U_i{}^* = U_i + h_i'(t), \quad P^* = P - x_i h_i''(t) \qquad (13)$$

$$T_{\mathrm{rot}_\alpha}: \ t^* = t, \ \ x_i{}^* = x_j Q_{ij}^{[\alpha]}, \ \ U_i{}^* = U_j Q_{ij}^{[\alpha]}, \quad P^* = P \qquad (14)$$

$$T_P: \ t^* = t, \ \ x_i{}^* = x_i, \ \ U_i{}^* = U_i, \ \ P^* = P + h(t) \qquad (15)$$

$$T_{\mathrm{Sc,ns}}: \ t^* = te^{2a}, \ \ x_i{}^* = x_i e^a \ \ U_i{}^* = U_i e^{-a}, \quad P^* = Pe^{-2a}, \qquad (16)$$

where $\mathbf{Q}^{[\alpha]}$ are constant rotational matrices and $h(t)$ and $h_i(t)$ are arbitrary functions. The global form is given here because it makes the implications of the respective transformations obvious. All of the symmetries (12)–(16) directly translate into a statistical description of turbulence as given by the RANS equations (1). The first symmetry (12) adds a constant to the time coordinate while not transforming the other variables. Evidently, this transformation does not change equation (1), and therefore is a symmetry of this equation. Physically, this corresponds to the principle that there is no absolute origin of time, and physics do not depend on how it is chosen. The generalized Galileian symmetry (13) essentially corresponds to the principle that translation in space does not influence a physical system. In the case of incompressible flow considered here, one can even allow accelerated movement, because its effect can be absorbed into the pressure term. Compressible flows and general classical physical systems are only invariant under a linear movement at constant velocity, i.e. symmetry (13) with $h_i(t) = a_i t$. The rotational symmetry (14) is closely connected to the principle of correct tensorial formulation. It reflects that a fixed rotation of the system has no effect on the physical processes inside of it. Note, however, that a time-dependent rotation does have an effect, and is hence not a symmetry of the Navier-Stokes and the RANS equations. Symmetry (15) is again specific to incompressible flows, in which only pressure differences matter, and thus the absolute value of the backgound pressure can be chosen arbitrarily. Finally, the scaling symmetry given by equation (16) expresses that a rescaling of all variables, if done in a specific way, does not change equation (1). This principle is closely connected to dimensional consistency, because a change of the unit system is nothing but a rescaling of all variables. Evidently, the physical implications of all of the above symmetries are very natural to everyone familiar with fluid mechanics, even without extensive knowledge about the mathematical background. Therefore, these symmetries are generally observed in modern turbulence models.

On the other hand, it is a relatively recent development (Oberlack & Rosteck, 2010; Rosteck & Oberlack, 2011) that the statistical description of turbulence given by the RANS equations and the infinite hierarchy of moment equations contains additional symmetries, which have been shown in Waclawczyk *et al.* (2014) to be connected with the intermittent and non-Gaussian nature of turbulent statistics. These statistical symmetries are given by the transformations

$$T_{\mathrm{Tr,stat},1}: \ t^* = t, \ \ x_i{}^* = x_i, \ \ \bar{U}_i^* = \bar{U}_i + a_i, \ \ \bar{P}^* = \bar{P}, \quad H_{ij}{}^* = H_{ij} \text{ or } R_{ij}{}^* = R_{ij} - a_i\bar{U}_j - a_j\bar{U}_i - a_i a_j \qquad (17)$$

$$T_{\mathrm{Tr,stat},2}: \ t^* = t, \ \ x_i{}^* = x_i, \ \ \bar{U}_i^* = \bar{U}_i, \ \ \bar{P}^* = \bar{P}, \quad H_{ij}{}^* = H_{ij} + a_{ij} \text{ or } R_{ij}{}^* = R_{ij} + a_{ij} \qquad (18)$$

$$T_{\mathrm{Sc,stat}}: \ t^* = t, \ \ x_i{}^* = x_i, \ \ \bar{U}_i^* = e^a\bar{U}_i, \ \ \bar{P}^* = e^a\bar{P}, \quad H_{ij}{}^* = e^a H_{ij} \text{ or } R_{ij}{}^* = e^a(R_{ij} + \bar{U}_i\bar{U}_j) - e^{2a}\bar{U}_i\bar{U}_j. \qquad (19)$$

It is physically irrelevant if one uses H_{ij} or R_{ij} as defined in (2), but the above transformations take a much simpler form when using H_{ij}. This makes H_{ij} the more suitable variable for the subsequent analysis. Symmetries (17)–(19) have no counterpart in the original Navier-Stokes equations, because they only appear if one adopts a statistical view on turbulent flows.

The classical symmetries (12)–(16) have been rarely considered directly when devising turbulence models, but their physical interpretation is intuitively known to most fluid mechanics researchers. In fact, they are mostly equivalent to the conditions imposed by invariant modelling. This has led to each generation of turbulence models fulfilling more and more classical symmetries, and since around the 1970s, virtually every turbulence model correctly contains them. The statistical symmetries (17)–(19), however, cannot be translated into general physical principles but instead are very specific to turbulent flows. Therefore, they have so far not found their way into any existing turbulence model, even though this would be crucially important.

SYMMETRY-BASED MODELLING

When trying to include the statistical symmetries (17)–(19) in a turbulence model following conventional modelling heuristics, it quickly becomes obvious that it is difficult to devise sensible model equations. Instead, we will employ a more formal symmetry-based modelling strategy. The idea is to assume a completely general form of what the final equations could look like, and to successively constrain that form by demanding invariance to a prescribed set of symmetries.

As a simple example, suppose we did not know the precise form of the Euler equations (i.e. the Navier-Stokes equations with $\nu = 0$), but only knew that it was some general function of temporal and spatial coordinates, velocity and pressure, and their first derivatives, i.e.

$$\mathbf{F}\left(x_i, t, U_i, P, U_{i,x_j}, U_{i,t}, P_{,x_i}, P_{,t}\right) = 0, \tag{20}$$

where the comma in the index represents derivation with respect to the following coordinates. We adopt this so called jet notation to denote derivatives because strictly speaking, the following method only works for algebraic equations. The extension to differential equations is accomplished very simply by treating derivatives as new variables, and calculating the action of symmetries on these new variables using the chain rule of differentiation.

We now impose the constraint that $\mathbf{F} = 0$ in (20) be invariant under all classical symmetries (12)–(16). (As the Euler Equations are not time-averaged, we do not expect invariance under the statistical symmetries.) Using the invariant surface condition (11), this leads to a system of linear partial differential equations. To understand the implications of each symmetry, we consider them one at a time. We start with translation in time given by symmetry (12), the infinitesimal generator of which as defined in (6)–(8) and (10) can be calculated to be

$$X_t = \frac{\partial}{\partial t}. \tag{21}$$

Demanding invariance of $\mathbf{F} = 0$ under X_t, according to equation (11), implies

$$X_t \mathbf{F} = \frac{\partial}{\partial t}\mathbf{F}\left(x_i, t, U_i, P, U_{i,x_j}, U_{i,t}, P_{,x_i}, P_{,t}\right) = 0, \tag{22}$$

which means that t has to be removed from the list of variables. The infinitesimal generator of symmetry (13) with $h_i(t) = a_i$ reduces to $X_{\text{Gal}} = \frac{\partial}{\partial x_i}$, which analogously to X_t eliminates x_i from the set of variables in $\mathbf{F}$. Similarly, symmetry (15) with $h(t) = a$ eliminates P from the list of variables, and, using the chain rule of differentiation, symmetry (15) with $h(t) = at$ eliminates $P_{,t}$ from (20). Next, we demand invariance with respect to the general Galileian symmetry, i.e. equation (13) for arbitrary $h_i(t)$ again using (6)–(8) and (10). The infinitesimal generator then becomes

$$X_{\text{Gal}} = h_i(t)\frac{\partial}{\partial x_i} + h_i'(t)\frac{\partial}{\partial U_i} - x_i h_i''(t)\frac{\partial}{\partial P} \tag{23}$$

Applying X_{Gal} to $\mathbf{F}$ in (20) again requires the chain rule of differentiation, yielding

$$X_{\text{Gal}}\mathbf{F} = h_i'(t)\frac{\partial \mathbf{F}}{\partial U_i} + (h_i''(t) - U_{i,x_j} h_j'(t))\frac{\partial \mathbf{F}}{\partial U_{i,t}} + h_i''(t)\frac{\partial \mathbf{F}}{\partial P_{,x_i}} = 0. \tag{24}$$

Equation (24) can be solved using the method of characteristics, leading to

$$\mathbf{F}\left(U_{i,t} + U_j U_{i,x_j} + P_{,x_i}, U_{i,x_j}\right) = 0. \tag{25}$$

The functional form of $\mathbf{F}$ is thus reduced to only two variables, which already closely resemble the final form of the Euler equations. A further constraint is imposed by the rotational symmetry (14), the infinitesimal generator of which is

$$X_{\text{rot}_\alpha} = \varepsilon_{ij\alpha} x_i \frac{\partial}{\partial x_j} + \varepsilon_{ij\alpha} U_i \frac{\partial}{\partial U_j}, \tag{26}$$

where ε_{ijk} is the permutation symbol. The resulting system of partial differential equations $X_{\text{rot}_\alpha}\mathbf{F} = 0$ is fulfilled if

$$U_{i,t} + U_j U_{i,x_j} + P_{,x_i} = 0, \tag{27}$$

$$\mathbf{F}\left(U_{i,x_i}\right) = 0. \tag{28}$$

Finally, symmetry (16) constrains this free function $\mathbf{F}$, leading to

$$U_{i,x_i} = 0. \tag{29}$$

The Euler equations have thus be constructed from their symmetries only. This observation by itself already illustrates the relevance of symmetries, but unfolds great potential in turbulence modelling and other modelling challenges. If one accepts the proposition that the symmetries of a physical system are not just abstract mathematical properties of the equations describing it, but in fact contain the underlying physical axioms in a distilled form, a powerful modelling strategy can be put forward. (i) Given an equation system for which simplified model equations are to be devised, one may first calculate the symmetries of the non-simplified equations. (ii) A set of reduced variables on which the model should depend is defined. (iii) The most general form of model equations that are invariant under the symmetries of the exact system (or a selected subset thereof) is obtained as shown above.

This procedure is now employed in order to develop the skeleton of a new turbulence model. A simplistic Reynolds stress transport model without terms accounting for viscous effects would take the general form

$$\mathbf{F}\left(x_i, t, \bar{U}_i, \bar{P}, H_{ij}, \overline{U_i P_{,x_j}}, \bar{U}_{i,x_j}, \bar{U}_{i,t}, \bar{P}_{,x_i}, \bar{P}_{,t}, H_{ij_{,x_k}}, H_{ij_{,t}}, \overline{U_i P_{,x_j}}_{,x_k}, \overline{U_i P_{,x_j}}_{,t}\right) = 0. \tag{30}$$

We now demand invariance with respect to all classical symmetries like before, and additionally, we want the model to be invariant under the statistical symmetries (17)–(19). Such a model would not only have built into it the axioms of classical physics represented by symmetries (12)–(16), but also essential properties of turbulent statistics such as intermittency and non-Gaussianity. In a similar fashion as above, the classical translational symmetries of time, space and pressure as shown in (12), (13) and (15) quickly reduce the set of possible model variables to

$$\mathbf{F}\left(\bar{U}_i, \overline{U_i P_{,x_j}}, \bar{U}_{i,x_j}, \bar{U}_{i,t}, \bar{P}_{,x_i}, H_{ij}, H_{ij_{,x_k}}, H_{ij_{,t}}, \overline{U_i P_{,x_j}}_{,x_k}\right) = 0. \tag{31}$$

Unlike before, however, we now have additional translation symmetries, namely the statistical symmetries (17) and (18). Its application written in generator form as

$$X_{\mathrm{Tr,stat},1} = \frac{\partial}{\partial \bar{U}_i} \text{ and } X_{\mathrm{Tr,stat},2} = \frac{\partial}{\partial H_{ij}} \tag{32}$$

eliminates $\bar{U}_i$ and H_{ij} from (31). Next, we apply the Galileian group, which, if $\bar{U}_i$ was still allowed to be present, would yield terms of the type

$$\mathbf{F}\left(\cdots, H_{ij_{,t}} + \bar{U}_k H_{ij_{,x_k}} + \cdots\right) = 0. \tag{33}$$

Equation (33) is essentially the form of the left-hand side terms found in classical Reynolds stress transport models, which break the statistical symmetries, written in terms of H_{ij} as defined in (2). With $\bar{U}_i$ eliminated by the statistical symmetry (32), however, we obtain

$$\mathbf{F}(\bar{U}_{i,x_j}, \bar{U}_{i,t} + H_{ij_{,x_j}} + \bar{P}_{,x_i}, H_{ij_{,t}} + \lambda_k H_{ij_{,x_k}} + \overline{U_i P_{,x_j}} + \overline{U_j P_{,x_i}} + \lambda_i H_{jk_{,x_k}} + \lambda_j H_{ik_{,x_k}} - \bar{U}_{i,x_k}\lambda_j\lambda_k - \bar{U}_{j,x_k}\lambda_i\lambda_k) = 0, \tag{34}$$

in which we have introduced the abbreviation

$$\lambda_i = \bar{U}^{-1}_{i,x_k}(-\bar{U}_{k,t} - \bar{P}_{,x_k}). \tag{35}$$

The first two terms in (34) naturally have to appear, because they will simply yield equation (1) with (2). This is expected, because no modelling of the RANS equations is needed in a Reynolds stress transport model. Empirical model assumptions are only made in the equation for the second moments. The crucial part of (34) is therefore the third term, in which the expression abbreviated with λ_i appears. Notice how this expression appears in places where one would naturally expect $\bar{U}_i$. Of course, $\bar{U}_i$ cannot appear directly, because in the instantaneous formulation, that would violate the statistical translation symmetry (17). In that sense, λ_i can be seen as its replacement. Looking more closely at expression (35), it turns out that it transforms like $\bar{U}_i$ under all classical symmetries (12)–(16), but is invariant under the statistical symmetries (17)–(19). One could say that the algorithm found a way of replacing $\bar{U}_i$ by a combination of variables that has similar but more suitable symmetry properties.

However, it is also the case that λ_i can only be uniquely determined if the velocity gradient is invertible, but this is generally not the case. As a consequence, any equation containing λ_i may yield non-unique and unphysical results. Before going any further, we have to conclude that no meaningful equations can be constructed from (34). Obviously, the assumed form (30) was too narrow, and we need to introduce more model variables.

At the same time, we can extract an important insight from this failure: In order to devise a model that is invariant under all classical and statistical symmetries, we have to introduce a new model variable that behaves like $\bar{U}_i$ under all classical symmetries, but is invariant under the statistical symmetries. We are going to call this variable $\mathfrak{U}_i$, but we postpone giving a precise definition. Instead, we repeat the modelling procedure with the set of variables extended by $\mathfrak{U}_i$ and, as will prove useful, a pressure-like variable $\mathfrak{P}$ that transforms like $\bar{P}$ under all classical symmetries while being invariant under the statistical ones. In concrete terms, we now ask which equations of the general form

$$\mathbf{F}\,(x_i, t, \bar{U}_i, \bar{P}, H_{ij}, \mathfrak{U}_i, \mathfrak{P}, \bar{U}_{i,x_j}, \bar{U}_{i,t}, \bar{P}_{,x_i}, \bar{P}_{,t}, \mathfrak{U}_{i,x_j}, \mathfrak{U}_{i,t}, \mathfrak{P}_{,x_j}, \mathfrak{P}_{,t}, H_{ij_{,x_k}}, H_{ij_{,t}}, \overline{U_i P_{,x_j}}, \overline{U_i P_{,x_{j,x_k}}}, \overline{U_i P_{,x_{j,t}}} = 0 \tag{36}$$

are invariant under all classical and statistical symmetries, i.e. (12)–(16) and (17)–(19). The additional model variables vastly increase the solution space of the ensuing PDE system, which now also contains practically useful model equations. As a possible solution, we obtain the closed set of model equations

$$H_{ij_{,t}} + \mathfrak{U}_k H_{ij_{,x_k}} + \overline{U_i P_{,x_j}} + \overline{U_j P_{,x_i}} + \mathfrak{U}_i H_{jk_{,x_k}} + \mathfrak{U}_j H_{ik_{,x_k}} - \bar{U}_{i,x_k}\mathfrak{U}_j\mathfrak{U}_k - \bar{U}_{j,x_k}\mathfrak{U}_i\mathfrak{U}_k = 0, \tag{37}$$

$$\overline{U_i P_{,x_{j,x_k}}} - \mathfrak{U}_i \bar{P}_{,x_j x_k} - \mathfrak{P}_{,x_j}\bar{U}_{i,x_k} = 0, \tag{38}$$

$$\mathfrak{U}_{i,x_i} = 0,\ \mathfrak{U}_{i,t} + \mathfrak{U}_j\mathfrak{U}_{i,x_j} + \mathfrak{P}_{,x_i} = 0. \tag{39}$$

completed by equation (1). For better comparison with existing Reynolds stress transport models, we rewrite equation (37) in terms of the Reynolds stress tensor R_{ij}, also returning to the normal notation for derivatives and obtain

$$\begin{aligned}\frac{\partial R_{ij}}{\partial t} &+ \bar{U}_k\frac{\partial R_{ij}}{\partial x_k} \\ &= \frac{\partial \bar{U}_j}{\partial x_k}(\bar{U}_i - \mathfrak{U}_i)(\bar{U}_k - \mathfrak{U}_k) + \frac{\partial \bar{U}_i}{\partial x_k}(\bar{U}_j - \mathfrak{U}_j)(\bar{U}_k - \mathfrak{U}_k) \\ &+ \frac{\partial R_{ik}}{\partial x_k}(\bar{U}_j - \mathfrak{U}_j) + \frac{\partial R_{jk}}{\partial x_k}(\bar{U}_i - \mathfrak{U}_i) - \frac{\partial R_{ij}}{\partial x_k}(\bar{U}_k - \mathfrak{U}_k) \\ &- \overline{u_i\frac{\partial p}{\partial x_j}} - \overline{u_j\frac{\partial p}{\partial x_i}},\end{aligned} \tag{40}$$

The equation for the velocity-pressure-correlation (38), written here in a slightly unusual form as a single term, can also be written in fluctuation form,

$$\frac{\partial\left(\overline{u_i\frac{\partial p}{\partial x_j}}\right)}{\partial x_k} + \frac{\partial^2 \bar{P}}{\partial x_j \partial x_k}(\bar{U}_i - \mathfrak{U}_i) + \frac{\partial \bar{U}_j}{\partial x_k}\left(\frac{\partial \bar{P}}{\partial x_j} - \frac{\partial \mathfrak{P}}{\partial x_j}\right) = 0. \tag{41}$$

In both cases, it is interesting to note that the fluctuation formulation contains prominently the velocity difference $\bar{U}_i - \mathfrak{U}_i$. This expression is of course itself a velocity, and, as is readily verified, is Galileian invariant.

Of course, this model prototype does not contain viscous terms, and subsequently also dissipation effects are not accounted for in equations (37) and (40). As these terms have very simple transformations under the considered symmetries, it would have not posed a problem to incorporate them in the analysis, but it would have made the derivation much longer without yielding more insight. One should not have the impression that it would be difficult to include these terms into the current model. On a similar note, the given equation system is somewhat minimal in that it does not contain additional model terms: all terms present play together in fulfilling the considered symmetries, and omitting any one of them would cause the breaking of one ore more symmetries. Useful turbulence models generally contain additional terms to model specific turbulent effects, even if removing these terms would not break any symmetries. Such terms are certainly necessary in the present model as well in order to predict flows accurately.

Nevertheless, a comparison with the exact equations and how existing turbulence models treat them is useful. The exact equation for R_{ij} reads

$$\frac{\partial R_{ij}}{\partial t} + \bar{U}_k \frac{\partial R_{ij}}{\partial x_k} = -R_{ik}\bar{U}_{j,x_k} - R_{jk}\bar{U}_{i,x_k} - \varepsilon_{ij}$$
$$-\overline{u_i \frac{\partial p}{\partial x_j}} - \overline{u_j \frac{\partial p}{\partial x_i}}, + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial R_{ij}}{\partial x_k} + R_{ijk}\right), \qquad (42)$$

in which ε_{ij} denotes the dissipation. Setting aside dissipation and viscous diffusion for now, we can attempt to establish connections between the terms in equation (40) and equation (42). First, the closure of the pressure-strain correlation is not performed in equation (40), but in equation (41), so there is a direct correspondence between these terms. The first two right-hand side terms of equation (42) are commonly referred to as the production term. As this term is closed, i.e. it only contains known properties, it is normally left as it is. However, as soon as modelling assumptions for the other terms in equation (42) are introduced, the production term in its original form breaks the statistical symmetries (18) and (19). Therefore, the modelling formalism yields the similar but unproblematic term

$$\bar{U}_{j,x_k}(\bar{U}_i - \mathfrak{U}_i)(\bar{U}_k - \mathfrak{U}_k) + \bar{U}_{i,x_k}(\bar{U}_j - \mathfrak{U}_j)(\bar{U}_k - \mathfrak{U}_k), \quad (43)$$

in which instead of R_{ij}, the product $\mathfrak{U}_i\mathfrak{U}_j$ appears. This is necessary for preserving the statistical symmetries (18) and (19).

The only term left to discuss in equation (42) is then the derivative of the triple correlation, which has to correspond to the remaining terms in (40):

$$\frac{\partial R_{ijk}}{\partial x_k} \approx \frac{\partial R_{ik}}{\partial x_k}(\bar{U}_j - \mathfrak{U}_j) + \frac{\partial R_{jk}}{\partial x_k}(\bar{U}_i - \mathfrak{U}_i) - \frac{\partial R_{ij}}{\partial x_k}(\bar{U}_k - \mathfrak{U}_k). \qquad (44)$$

In the model by Launder, Reece and Rodi Launder *et al.* (1975), the closure assumption

$$R_{ijk} \approx \frac{2}{3} C_s \frac{k^2}{\varepsilon}\left(\frac{\partial R_{jk}}{\partial x_i} + \frac{\partial R_{ik}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_k}\right) \qquad (45)$$

in which C_s is a model parameter, is employed. Compared to other unclosed terms, relatively little attention seems to be focused on this term in the context of classical modelling (Wilcox, 1994), so that we must take care not to draw too strong conclusions from this comparison. Obviously, the statistically invariant model does not require the dissipation as a scale-providing variable to formulate a possible closure for the triple correlation, but instead uses the velocity difference $\bar{U}_i - \mathfrak{U}_j$. Apart from that, it is notable that both expressions prominently contain spatial derivatives of the Reynolds stress tensor, though the form is rather different.

Even though equation (40) arises only from symmetry analysis, a term-by-term comparison with the exact equation (42) does allow an identification of similarities.

CONCLUSION

A statistically invariant turbulence model framework was successfully developed using a symmetry-based modelling strategy. This prototype of a new class of models can potentially incorporate important features of turbulence in an intrinsic manner. It is important to note that this is accomplished by encoding these principles on the level of symmetries, rather than attempting a term-by-term modelling on the level of equations. Operating on this higher level of abstraction enables extracting the essential features of the exact equations and preserving them in the model equations in an algorithmic way. This type of modelling approach is not restricted to turbulence modelling, but could be used in other modelling challenges as well.

The model skeleton presented is only a basis that will be extended and tested in the future. Implementing a further developed model into a numerical solver framework is also planned.

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