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A DATA-DRIVEN CONSTRUCTION OF REDUCED ORDER MODEL FOR SUPERSONIC BOUNDARY LAYER BYPASS TRANSITION

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ABSTRACT

In the present study, a data-driven method for the construction of reduced order model (ROM) for complex flows is proposed. It uses the proper orthogonal decomposition (POD) modes as the orthogonal basis and the dynamic mode decomposition (DMD) method to obtain linear equations for the temporal evolution coefficients of the modes. This method is in no need of the governing equations of the flows involved and saves the effort to derive the projected equations and prove their consistency, convergence and stability as required by the conventional Galerkin projection method, which has been successfully applied to incompressible flows but is hard to be extended to the compressible flows. Using the sparsity promoting algorithm, the dimension of the ROM could be further reduced to the minimum. The ROMs of the bypass transition of supersonic boundary layers at *Ma*=2.25 are constructed with the proposed data-driven method. The temporal evolutions of the POD modes show good agreement with that obtained by direct numerical simulations. The ROM for bypass transition is further investigated as a dynamical system. It is shown that the stability of the ROM dynamical system is determined by its Lyapunov dimension, which could be used as an indicator to determine the ROM dimension.

INTRODUCTION

Modal decomposition methods, such as proper orthogonal decomposition (POD) and dynamic mode decomposition (DMD), are able to capture the important flow structures. Based on these spatial modes, the reduced order model (ROM) can be constructed to approach the temporal evolution of the flow field. ROM is a useful and powerful tool for exploring the physical mechanisms, predicting and designing control schemes of complex flows by constructing a finite-dimensional dynamical system for the flows (Rowley & Dawson 2017). The motivation of the present research is to construct the ROMs for supersonic boundary layer transition which involves very complex physical processes and leaves many open questions concerning transition prediction and control.

The POD-Galerkin method has been successfully applied to the construction of ROMs for various incompressible flows (Holmes, 2012). When it comes to the compressible viscous flows, the ROM is difficult to be obtained with the traditional POD-Galerkin method because of the complexity of the governing equations. The application to compressible viscous flows was firstly proposed and performed by Gloerfelt (2008). However, the ODEs thus obtained are unstable, leading to the exponential growth of temporal coefficients. Although various modifications were proposed, such as taking into account the viscous dissipation of the truncated higher modes by eddy viscosity, the parameters involved were usually set artificially and empirically. Other factors such as inner product and boundary conditions (Kalashnikova et al, 2014) could also influence the stability of ROM. Choosing the factors above properly, the application to laminar subsonic flows is so far successful, while for nonlinear flows, the stability and convergence are still challenging.

DMD is a data-driven method for reduced order representation, which can extract spatial modes as well as their frequencies and growth rates (Schmid 2010). Although it is a powerful method to construct ROM, DMD has the setback that its modes are not orthogonal, which makes it difficult to investigate the interaction between different modes.

In the present study, we proposed a purely data-driven ROM construction method based on the combination of POD, DMD and sparsity-promoting algorithm (Jovanovic et al. 2014): POD is used to obtain the orthogonal basis, DMD to obtain its temporal evolution and sparsity-promoting DMD as the dimension truncation technique. In this method, the governing equations are not involved and the problems related to the projection of governing equations disappear. This method can be easily applied to compressible flows as well as incompressible flows, overcoming the difficulties encountered by the conventional POD-Galerkin method. In the present study, the supersonic boundary layer bypass transition at Mach number *Ma* = 2.25 is considered, and the ROMs are constructed using the proposed construction method to give a longer time prediction of the flow fields.

Principle and Algorithm

The principle of the proposed method will be stated under each step of the algorithm:

(1) Write the data samples with a same time interval obtained by numerical simulations or experimental measurements into the

form of matrices V_{N1}, V_{N2}, each columns of which contains flow quantity of a temporal instantaneous flow field.
(2) Perform SVD via equation to the matrix

$$V_{N1} = U^T \Sigma' V \tag{1}$$

to obtain the orthogonal basis U (POD modes) and the singular values Σ'. The energy of each mode is obtained by $\Sigma = \sqrt{N \Sigma'}$ according to POD.
(3) Perform DMD.

DMD performs modal decomposition by assuming that the time interval Δt is small enough so that the two matrices can be related by linear mapping $V_{N2} = A V_{N1}$, which $\boldsymbol{A}$ is the evolving matrix of the dynamical system. Substituting the SVD into the linear mapping, we have the projected version of the evolving matrix

$$A' = U^T A U = U^T V_{N2} V \Sigma^{-1} \tag{2}$$

Eigenvalues and eigenvectors of A' are obtained by solving eigenvalue problem $A' W = W\Lambda$, where Λ is a diagonal matrix with eigenvalues ($\mu_i, i = 1,2,\ldots,N$) on its diagonal line, and **W** is the matrix whose columns are eigenvectors $w_i (i = 1,2,\ldots,N)$.
(4) Truncate A' to lower dimensions according to the SP algorithm results, and obtain the temporal evolution matrix of discrete or continous ROM equations.

The present data-driven ROM construction method employs POD modes as the orthogonal basis and DMD eigenvalues to represent the temporal evolution of the modes. With the POD modes, each temporal flow field can be expressed as a linear combination of the POD modes:

$$v_j = \sum_{i=1}^{N} b_{ij}(\Delta t) u_i, (j = 1,\ldots,N) \tag{3}$$

Substituting the linear expansions into the linear mapping, we get $UB_2 = AUB_1$ (where B_1 and $B_2$are the matrices of temporal evolving coefficients.), which can be further written as

$$B_2 = U^T A U B_1 = A' B \tag{4}$$

indicating that the temporal evolving matrix A' is the same as the matrix defined in DMD. In this sense, temporal evolution of POD modes can be obtained via the DMD eigenvalues and eigenvectors. Therefore, the discrete form of the temporal evolution equation can be expressed as $b(t+\Delta t) = A' b(t)$. It can be further expressed as ODEs, where continuous temporal coefficients can be obtained:

$$\frac{d\, b(t)}{dt} = A_c b(t) \tag{5}$$

The relation between A_c and A could be deduct-ed by simple math, which will be shown in our full paper.
(5) Solve the ROM equations numerically to give temporal evolution of POD orthogonal modes.

With a group of initial values, the temporal coefficients can be obtained via the above linear mapping. The formula can be further expressed as ODEs, where continuous temporal coefficients can be obtained. They can be solved by classical algorithms such as the Runge-Kutta method.

APPLICATION TO SUPERSONIC BOUNDARY LAYER TRANSITION

We construct the ROM for the bypass transition in supersonic boundary layer with the freestream Reynolds number *Re*=635,000/inch, Mach number *Ma*=2.25 and temperature 169.0K. DNS of supersonic boundary layer transition for ideal gas and Newtonian fluid is performed with HOAM-OPENCFD-1.10.4 developed by Li et al (2009). The conservative governing equa-tions are solved numerically with high-order finite difference method. All the flow quantities are non-dimensionalized by the free stream density, velocity and temperature, and the characteri-stic length is 1 inch. The flow parameters are set according to Li et al (2009). The periodic blowing and suction disturbance is introduced at the wall in the region of *x*=4.0~4.5 to induce transition, the frequencies of which are 3.982 and 7.854, each with spanwise wavenumbers of 35.9 and 71.8, which are linearly unstable according to linear stability analysis. The disturbance amplitudes are both set to be 0.004. The vortex structure of the transitionnal region is shown with the second invariant of velocity gradient tensor Q colored by wall-normal (y) coordinate in Fig. 1, and the skin friction coefficient is shown in Fig. 2. In the early transitional region, the streamwise elongated vortex structures are obvious, and further developed into trivial and less organized structures, which is believed to be the end of transition. The skin friction in streamwise direction shows that the transition ends at *x*=7.5. Therefore, subzone in the range of *x*=5.0-6.5 is selected in the analysis since *x*=6.5 is the beginning position of transition.

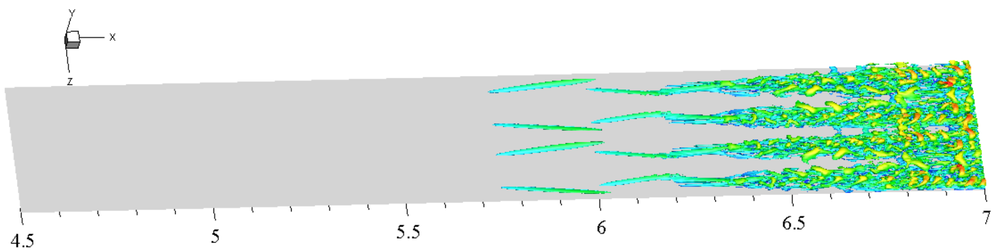


Fig. 1 Vortex structure (Q=2.0) in transition area

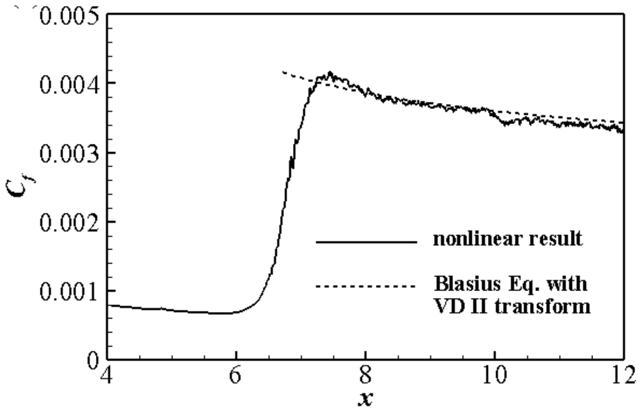


Fig. 2 Skin friction coefficient

The fluctuation of velocity components, density and temperature are used to constructing the ROM. The time interval Δt ¿0.02 and 201 temporal samples in total are included. The en-ergy of POD modes are shown in Fig. 3(a), and the total energy loss of the first i modes are shown in Fig. 3(b) defined as

$$El_i = \left(1 - \sum_{j=1}^{i} \sigma_j / \sum_{j=1}^{N} \sigma_j\right) \times 100\% \quad (6)$$

The mode energy decreases slowly with the increase of the mode number, indicating the existence of complicated flow structures in the field caused by nonlinear effects. If the ROM could capture 90% of the total energy, 13 modes are needed (as shown in Fig. 3 (b)). DMD frequency spectrum is shown in Fig. 3 (c), the modes selected with SP algorithm are shown with solid circles. The energy of the low-frequency modes is higher than that of the high-frequency modes (Fig. 3 (c)). The results of SP algorithm indicate that 19 modes are need-ed to capture the most important structures in the flow field. As shown in Fig. 3 (d), 19 modes are able to capture 88% of the total energy.

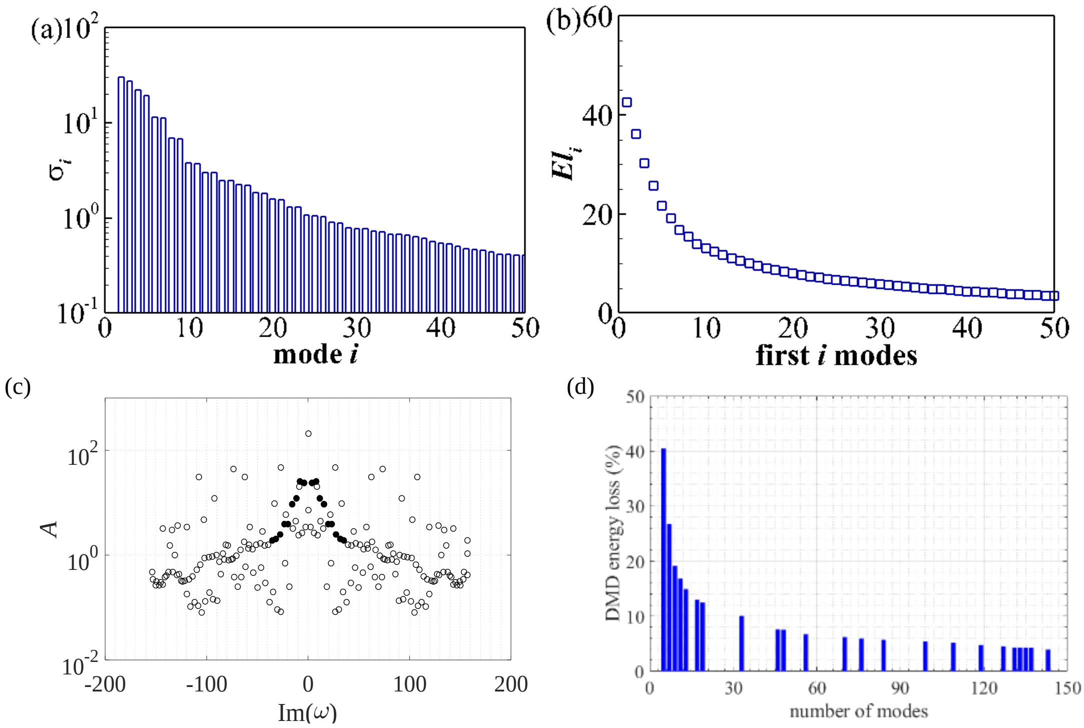


Fig. 3 Energy of POD modes (a), total energy loss of the first i modes (b), DMD frequency spectrum (c) and energy loss of a certain number of modes (d), solid circles in (c) are modes selected by SP algorithm

The vortical structures of the first four POD modes with the highest energy are shown in Fig. 4 by the iso-surfaces of Q, the second invariant of the velocity gradient tensor. The structures are mainly streamwise elongated vortices. They become stronger downstream, and correspondingly, the velocity disturbance around the vortices are more significant as indicated by the contours of streamwise disturbance velocity u. With the increase of the mode number (decrease of mode energy), the vortex structures become smaller and appear further downstream.

We select 11, 15 and 21 modes respectively to construct the ROMs, in order to examine the influence of the number of modes. (The mode number provided by SP algorithm is used as reference, not a criterion, so it does not need to be checked whether 19 is the precise number of modes to establish a suitable ROM.) The temporal coefficients for each mode are obtained by solving equation. The results are compared with the projected DNS data, as shown in Fig. 5. The red symbols represent the projected DNS data in the time range where POD and DMD is performed (hereinafter referred to as TRI), and the green symbols represent those beyond the time range (hereinafter referred to as TRO). For the temporal evolution of the lower modes 2~8, all the ROMs can yield an accurate prediction both in TRI and TRO. But for mode 10, obvious discrepancy can be observed between the result predicted by the ROM with 11 modes and the DNS data. Increasing the number of modes included in the ROM, the accuracy for mode 10 can be enhanced as shown by Fig.5 (II-e) and (III-e).

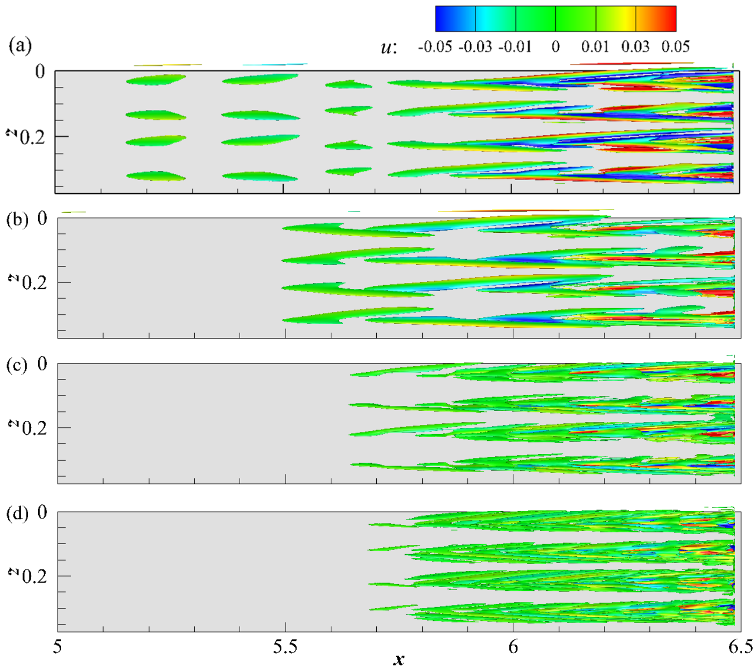


Fig.4 Iso-surfaces of Q=0.0001 colored by u. (a-d) mode 2, 4, 6, and 8 respectively.

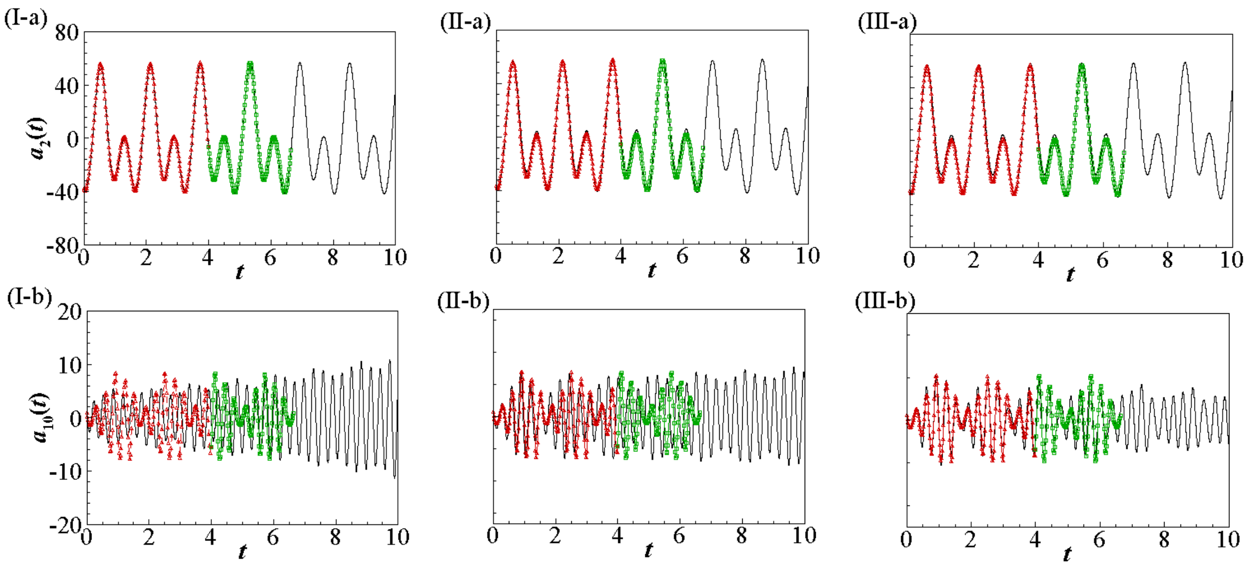


Fig. 5 Temporal coefficients of mode 2 (a) and mode 10 (b), with 11 (I), 15 (II) and 21 (III) modes. —△— projected DNS data in the time range that POD and DMD is performed, —□— beyond the time range that POD and DMD is performed, ——ROM

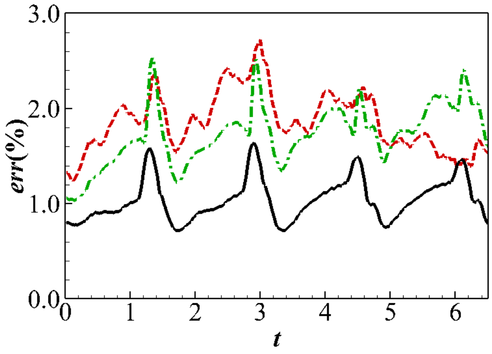
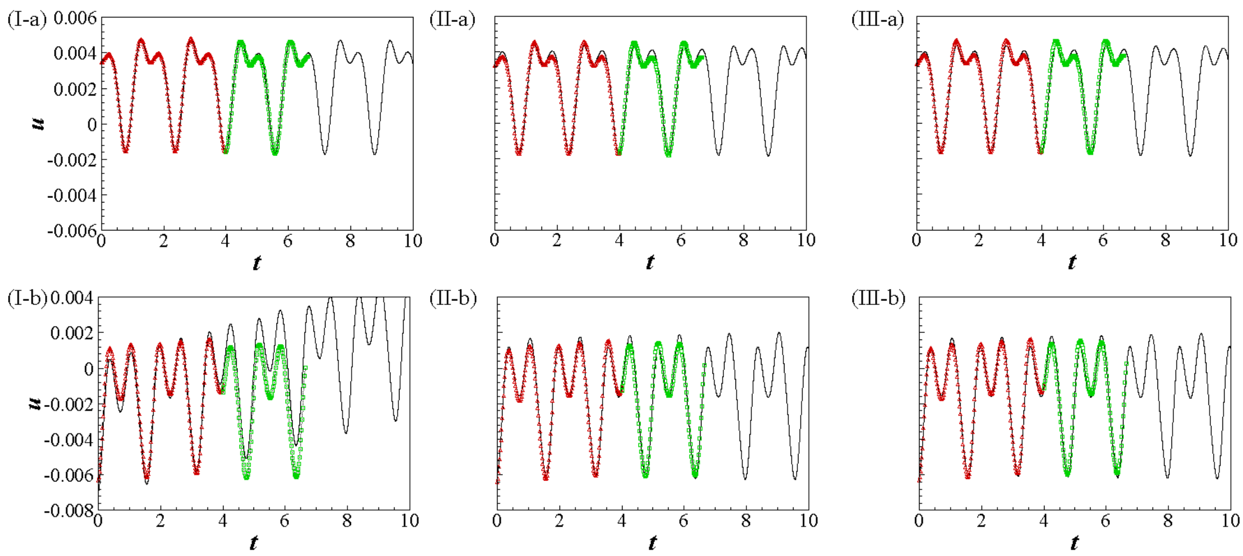


Fig. 6 Error of ROMs. Red dashed line: 11 modes; green dashed-dot line: 15 modes; black solid line: 21 modes



In order to quantify the accuracy of the ROMs, we define the averaged relative error as

$$err(t)=\left(\sum_{i=1}^{N}\left|a_i(t)-a_{pi}(t)\right|^2/N\right)^{\frac{1}{2}}/\left(\sum_{i=1}^{N}a_{pi}(t)^2\right)^{1/2}$$

where N is the number of modes, and a_{pi} represents the projected DNS data on POD mode i. As shown in Fig. 6, the relative error is less than 3% in all cases, and generally decreases when more modes are used in the ROMs. Notably, the relative error with 15 modes is larger than that with 11 modes at some time instants, especially near the peaks and in TRO region. This is caused by the inaccurate prediction of the higher modes as discussed above.

Flow reconstruction is performed to further verify the ROMs, as is shown in Fig. 7. The temporally evolving POD modes are added together to obtain the predicted flow fields. The streamwise velocity u given by the ROMs with 11, 15 and 21 modes is compared with the original DNS data at 2 selected points at x=5.275, 6.058, y=0.01, z=0.175 inside the boundary layer and at the center in spanwise direction. The ROM with 11 modes loses its accuracy and tends to increase sharply at x=6.058, while for the ROMs with 15 and 21 modes the error is exterminated. The reason could be found when it is studied as a dynamical system. For the ROM with 11 modes, the summation of the Lyapunov exponent is still positive (3.645×10^{-3}), indicating the system is unstable, while for the ROM with 15 modes, the summation of the Lyapunov exponent is negative (-9.412×10^{-2}), like the ROM with 21 modes shown above. This suggests that the summation of the Lyapunov exponents could be used as an indicator for the truncation of the ROM dimension. In order to obtain a temporally stable ROM, the summation of the Lyapunov exponents should be negative. In other words, the dimension of the ROM should not be less than its Lyapunov dimension, otherwise the system is unstable.

CONCLUSIONS

In the present study, we proposed a data-driven method to construct ROMs for complex flows. It uses the POD modes as the orthogonal basis and obtains the time evolution ODEs by DMD. This ROM construction method is purely data driven and therefore can be easily applied to flows governed by complex equations such as compressible flows.

The ROMs of the bypass transition induced by nonlinear disturbances in supersonic boundary layer at Ma=2.25 were built other words, the dimension of the ROM should not be less than its Lyapunov dimension, otherwise the system is unstable.

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The ROMs of the bypass transition induced by nonlinear disturbances in supersonic boundary layer at *Ma*=2.25 were built using the proposed method. By comparing the temporal coefficients of POD modes and the reconstructed flow fields given by the ROMs with those given by DNS data confirms the ability and accuracy of the present method. The influence of ROM dimension on the predicted flow was further examined by checking the time history of the streamwise velocity *u* at different streamwise locations inside the boundary layer in the bypass transition case. It was found that downstream near the transition region, more modes are needed to give a satisfying prediction because of the intensified complexity of the flow. The Lyapunov exponents and Lyapunov dimension were calculated to quantitatively measure the complexity of the ROM dynamical system. It was observed that the ROM of the bypass transitional boundary layer is a dissipative dynamical system with high Lyapunov dimension, and its stability can be determined by the Lyapunov dimension, which can be used to determine the ROM dimension.

The present method was also applied to the bypass transition of hypersonic boundary layer at *Ma*=6 (not shown here). Because the disturbance frequencies differ in several orders in hypersonic flow, the data samples with longer time span and finer time interval are needed. Due to the limitation of the computational resources, the ROM we obtained currently can only provide an accurate prediction for the short-term behavior. A probable remedy for the present ROM construction method is to use the multi-resolution dynamic mode decomposition method proposed by Kutz et al. [20], which is now under our research.

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REFERENCES

Rowley, C.W. and Dawson, S.T., 2017. Model reduction for flow analysis and control. Annual Review of Fluid Mechanics, 49, pp.387-417.

Holmes, P., 2012. Turbulence, coherent structures, dynamical systems and symmetry. Cambridge university press.

Gloerfelt, X., 2008. Compressible proper orthogonal decomposition/Galerkin reduced-order model of self-sustained oscillations in a cavity. Physics of Fluids, 20(11), p.115105.

Kalashnikova, I., Barone, M.F., Arunajatesan, S. and van Bloemen Waanders, B.G., 2014. Construction of energy-stable projection-based reduced order models. Applied Mathematics and Computation, 249, pp.569-596.

Schmid, P. J. (2010). Dynamic mode decomposition of numerical and experimental data. Journal of fluid mechanics, 656, 5-28.

Jovanović, M.R., Schmid, P.J. and Nichols, J.W., 2014. Sparsity-promoting dynamic mode decomposition. Physics of Fluids, 26(2), p.024103.

Xin-Liang, Li, et al. "Acoustic calculation for supersonic turbulent boundary layer flow." Chinese Physics Letters 26.9 (2009): 094701.