Stability Analysis of Jets in Crossflow

Marc Regan
Aerospace Engineering & Mechanics
University of Minnesota
107 Akerman Hall
110 Union St SE
Minneapolis, Minnesota 55455-0153
United States of America
rega0113@umn.edu

Krishnan Mahesh
Aerospace Engineering & Mechanics
University of Minnesota
107 Akerman Hall
110 Union St SE
Minneapolis, Minnesota 55455-0153
United States of America
kmahesh@umn.edu

ABSTRACT
Jets in crossflow (JICFs), or transverse jets, are a canonical flow where a jet of fluid is injected normal to a crossflow. The interaction between the incoming flat-plate boundary layer and the jet is dependent on the Reynolds number ($Re = \frac{v_j D}{\nu}$), based on the average velocity ($\bar{v}_j$) at the jet exit and the diameter ($D$), as well as the jet-to-crossflow ratio ($R = \frac{v_j}{u_\infty}$). Megerian et al. (2007) performed experiments at $Re = 2000$ and collected vertical velocity spectra along the upstream shear-layer. They observed that the upstream shear-layer transitions from absolutely unstable between $R = 2$ and $R = 4$. Using an unstructured, incompressible, direct numerical simulation (DNS) solver, Iyer & Mahesh (2016) performed simulations matching the experimental setup of Megerian et al. (2007). Vertical velocity spectra taken along the upstream shear-layer from simulation show good agreement with experiment, marking the first high-fidelity simulation able to fully capture the complex shear-layer instabilities in low speed jets in crossflow. Iyer & Mahesh (2016) proposed an analogy to counter-current monopolar waves formed further downstream. These observations are consistent with the upstream shear-layer wavemaker for the upstream shear-layer is then calculated using the direct and adjoint eigenmodes for case $R = 2$. The results further justify the absolutely unstable nature of the region near the upstream side of the jet nozzle exit.

INTRODUCTION
A jet in crossflow (JICF) is a canonical flow where a wall-normal jet of fluid interacts with an incoming crossflow. The flat-plate boundary layer created by the crossflow interacts with the jet to create a set of complex vortical structures. Shear-layer vortices and the Kelvin-Helmholtz instability often form on the upstream side of the jet path. Further downstream, a counter-rotating vortex pair (CVP) dominates the jet cross section (Kamotani & Greber, 1972; Smith & Mungal, 1998). Horseshoe vortices are also formed near the wall, just upstream of the jet exit (Krothapalli et al., 1990; Kelso & Smits, 1995). These travel downstream as they begin to tilt upward and form wake vortices during ‘separation events’ (Fric & Roshko, 1994) caused by the near-wall adverse pressure gradient. Wake vortices have long been studied in the literature (Kelso et al., 1996; Eiff et al., 1995; McMahon et al., 1971; Moussa et al., 1977). Transverse jets are found in many real-world engineering applications, such as: film cooling, vertical and/or short take-off and landing (V/STOL) aircraft, thrust vectoring, and gas turbine dilution jets. Reviews by Margason (1993), Karagozian (2010) and Mahesh (2013) describe most of the research over the last seven decades.

Low-speed incompressible isodensity JICFs may be characterized by the following: the jet Reynolds number $Re = \frac{v_j D}{\nu}$, based on the average jet exit velocity ($v_j$), the jet diameter ($D$), and the kinematic viscosity of the jet ($\nu$); the jet-to-crossflow velocity ratio $R = \frac{v_j}{u_\infty}$. The jet-to-crossflow velocity ratio may also be defined as $R^* = \frac{v_j \max}{u_\infty}$, based on the maximum velocity at the jet exit.

Megerian et al. (2007) experimentally studied low-speed JICFs at Reynolds numbers of 2000 and 3000 over a range of jet-to-crossflow ratios ($1 \leq R \leq 10$). Vertical velocities were collected at various probe points along the upstream shear-layer to compute velocity spectra. They observed that this region transitions from absolutely to convectively unstable between $R = 2$ and $R = 4$. Megerian et al. (2007) showed that when $R = 2$ the upstream shear-layer has a strong pure-tone mode at a single Strouhal number ($St = f D/v_j$), based on the peak velocity ($v_j$) at the jet exit and the diameter ($D$), from linear stability analysis are compared with experiments (Megerian et al., 2007) and simulations (Iyer & Mahesh, 2016). The eigenmodes are analyzed and show evidence that supports the transition from an absolutely to convectively unstable flow. Additionally, the adjoint sensitivity of the upstream shear-layer is studied for the case when $R = 2$. The location of the most sensitive areas is shown to be localized to the upstream side of the jet nozzle near the jet exit. The wavemaker for the upstream shear-layer is then calculated using the direct and adjoint eigenmodes for case $R = 2$. The results further justify the absolutely unstable nature of the region near the upstream side of the jet nozzle exit.

Davitian et al. (2010) further characterized the transition from absolute to convectively unstable by examining the spatial development of the fundamental and subharmonic modes that form further downstream along the shear-layer. Additionally, Davitian et al. (2010) examined the response of the upstream shear-layer to strong sinusoidal forcing. They show clear evidence that for flush JICFs, the near-field shear-layer becomes globally unstable when $R^* \leq 3$ at $Re = 2000$. Furthermore, evidence is shown that suggests strong sinusoidal forcing applied to a globally unstable JICF can replace one mode for another with little impact on the overall behavior. These results build on the prior studies of M’Closkey et al. (2002), who...
suggested that strong sinusoidal forcing has little impact on the behavior of JICFs when compared to square-wave forcing. The results by Davitian et al. (2010) and M’Closhkey et al. (2002) highlight the importance of understanding the stability transition of the upstream shear-layer due to the effect it has on the overall controllability of JICFs.

DNSs by Iyer & Mahesh (2016) match the same experimental setup as Megerian et al. (2007) for \( R = 2.4 \) at \( Re = 2000 \). Good agreement was shown between simulation and experiment for both the time-averaged flow, as well as the vertical velocity spectra obtained along the upstream shear-layer. Dynamic mode decomposition (DMD) was shown to capture to complex flow dynamics at the same Strouhal numbers obtained from vertical velocity spectra. Iyer & Mahesh (2016) proposed an analogy to counter-current mixing layers. They assumed the upstream shear-layer acted as a counter-current shear-layer, and therefore characterized the stability using the classic parallel flow analysis by Huerre & Monkewitz (1985). A velocity ratio,

\[
Q = \frac{V_1 - V_2}{V_1 + V_2}
\]  

is defined, where \( V_1 \) and \( V_2 \) are the velocities for the two mixing layers. Huerre & Monkewitz (1985) show that for \( Q > 1.315 \) a mixing layer is absolutely unstable, and for \( Q < 1.315 \) it is convectively unstable. Iyer & Mahesh (2016) show that in their simulations \( Q = 1.44 \) when \( R = 2 \) and \( Q = 1.20 \) when \( R = 4 \). This is consistent with the stability transition for JICFs in the literature, and may suggest that the mechanism that drives stability for free shear-layers may also govern the stability for more complex flows like JICFs.

The stability of JICFs have been studied using linear stability analysis (LSA) by Bagheri et al. (2009). Their analysis marks the first simulation-based Tri-Global LSA of a three-dimensional flowfield assuming no homogeneous directions. Throughout the present work, this type of analysis will be referred to as Global LSA (GLSA). Bagheri et al. (2009) studied the stability of JICFs at \( R^\prime = 3 \) at \( Re_{\delta_i} = u_\delta \delta_i^\prime / \nu = 165 \), which is based on the displacement thickness \( \delta_i^\prime \) at the inlet of the crossflow. GLSA was performed on a steady baseflow obtained using selective frequency damping (SFD) (Åkervik et al., 2006). In their analysis, the jet nozzle was not included, and instead a parabolic velocity profile was prescribed at the jet nozzle exit. The most unstable high-frequency modes were found along the upstream shear-layer, whereas low-frequency wake modes were found downstream. The frequency of the upstream shear-layer was not far from the non-linear shedding frequency; however, the wake mode frequency was far from the non-linear wake frequency. It was suggested by Bagheri et al. (2009) that the difference could be related to the differences between the SFD solution used in GLSA and the time-averaged solution.

Peplinski et al. (2015) studied JICFs at low values of \( R \) in the range between 1.5 and 1.6 using GLSA as well as global adjoint sensitivity analysis (GASA). GASA provides valuable sensitivity information about the flowfield through the use of the adjoint to the linearized Navier-Stokes equations. Peplinski et al. (2015) prescribe a super-exponential Gaussian function at the jet exit. Their results show large streamwise separation between the direct and adjoint eigenmodes induced by the flow advection. Additionally, the adjoint eigenmode is located on the jet exit boundary. This suggests a large sensitivity to the jet exit boundary condition, and may point to a necessary inclusion of the jet nozzle in this simulation.

The present work studies the stability of JICFs using GLSA for \( R = 2.4 \) and GASA for \( R = 2 \) of the turbulent mean flow. The same nozzle used in experiments by Megerian et al. (2007) is used in our simulations. The nozzle is designed to provide a top-hat profile at the jet exit. Performing GLSA, GASA, or even DNS, of JICF is very computationally expensive. There are 80 million elements in the grid used for the present work, which translates to an eigenvalue problem with a dimension of 240 million. Therefore, a variation of the Arnoldi iteration method (Arnoldi, 1951) is used to efficiently calculate the direct and adjoint eigenvalue spectra and the associated eigenmodes. Once the direct (GLSA) and adjoint (GASA) solutions are known, the wavemaker is computed for the upstream shear-layer to highlight the regions that are most sensitive to localized feedback. A brief discussion section provides conclusion to the presented results.

**NUMERICAL METHODOLOGY**

The incompressible Navier-Stokes (N-S) equations may be written as,

\[
\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = 0,
\]  

where \( \nu \) is the kinematic viscosity of the fluid. Equations 2 are solved using an unstructured, finite-volume algorithm developed by Mahesh et al. (2004). The algorithm has been validated for a number of complex flows, including: a gas turbine combustor (Mahesh et al., 2004), free jet entrainment (Babu & Mahesh, 2004), and transverse jets (Muppidi & Mahesh, 2005, 2007, 2008; Sau & Mahesh, 2007, 2008). The spatial discretization technique focuses on conserving discrete energy, which by design ensures that the flux of kinetic energy only has contributions from the boundary elements. These properties of the numerical algorithm ensure high-fidelity simulation of complex flows at high Reynolds numbers without added numerical dissipation. Adams-Bashforth second-order time integration is used to advance the predictor velocities through the momentum equation. Next, the Poisson equation for pressure is derived by taking the divergence of the momentum equation and satisfying conservation of mass. The pressure field is then used to project the velocity field to be divergence-free.

The N-S equations (eq. 2) can be linearized about a base state,

\[
\tilde{u}_i = \tilde{u}_i(x,y,z), \quad \tilde{\bar{p}} = \bar{p}(x,y,z)
\]  

which varies arbitrarily in space. By decomposing the flowfield into the known base state (\( \tilde{u}_i \)) plus some \( O(\varepsilon) \) perturbation (\( \tilde{u}_i \)), and neglecting the \( \varepsilon^2 \) terms, we arrive at the linearized Navier-Stokes (LNS) equations:

\[
\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial}{\partial x_j} \tilde{u}_i \tilde{u}_j + \frac{\partial}{\partial x_j} \tilde{u}_j \tilde{u}_i = -\frac{\partial \tilde{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} = 0
\]  

If our interest is in the long-time behavior of \( \tilde{u}_i \), then solutions to the LNS (eq. 4) are of the form:

\[
\tilde{u}_i(x,y,z,t) = \hat{u}_i(x,y,z) e^{\omega t} + c.c.
\]  

where \( \omega \) and \( \hat{u}_i \) can be complex. The real part (Re(\( \omega \))) is the growth/damping rate and the imaginary part (Im(\( \omega \))) is the temporal frequency of \( \tilde{u}_i \). By substituting in the ansatz (eq. 5), the
LNS equations (eq. 4) reduce to a linear eigenvalue problem, where \( \omega \) is the eigenvalue and \( \hat{u}_i \) is the eigenmode.

To arrive at the adjoint LNS (ALNS) equations we define the same Lagrangian identity as Hill (1995):

\[
\frac{\partial \hat{u}^*}{\partial t} + \frac{\partial}{\partial x_j} \hat{u}^* \mathbf{n}_j - \hat{\mathbf{u}}^* \frac{\partial \mathbf{n}_j}{\partial x_i} = - \frac{\partial p^*}{\partial x_i} - \nu \frac{\partial^2 \hat{u}^*}{\partial x_i \partial x_j}
\]

The ALNS equations are integrated backwards in time and provide sensitivity information for the corresponding LNS equations. By applying the same ansatz (eq. 5), the ALNS reduce to an eigenvalue problem that can be solved using the same numerical techniques as the direct problem. Additionally, the eigenvalues for the direct and adjoint problems coincide with each other. Therefore each direct eigenmode has a corresponding adjoint eigenmode that provides sensitivity information.

The adjoint velocity field defines the sensitivity of the associated direct mode to an unsteady point force aligned with the adjoint velocity vector. This provides valuable sensitivity information in regards to the underlying flow physics and points to locations in the domain that are sensitive to control.

When performing GLSA and GASA, the rank of the eigenvalue problem can be \( O(10^6 - 10^8) \). Solving eigenvalue problems of this size often prohibits the use of direct methods. In the present work, we use an extension of the Arnoldi iteration method (Arnoldi, 1951) called the Implicitly Restarted Arnoldi Method (IRAM). This method is matrix-free, which, in conjunction with a time-stepper approach, only requires the solution of an LNS time integrator to solve for the leading eigenvalues.

A turbulent mean flow can be used as the base state for GLSA and GASA. This choice of base state is a solution to the Reynolds-averaged N-S equations. Therefore, a non-linear Reynolds stress term is effectively added to the LNS and ALNS equations when the baseflow equations are subtracted. A mode-dependent Reynolds stress term is therefore present in the associated eigenvalue problem that can be solved using the same numerical techniques as the direct problem. Additionally, the eigenvalues for the direct and adjoint problems coincide with each other. Therefore each direct eigenmode has a corresponding adjoint eigenmode that provides sensitivity information.

The adjoint velocity field defines the sensitivity of the associated direct mode to an unsteady point force aligned with the adjoint velocity vector. This provides valuable sensitivity information in regards to the underlying flow physics and points to locations in the domain that are sensitive to control.

When performing GLSA and GASA, the rank of the eigenvalue problem can be \( O(10^6 - 10^8) \). Solving eigenvalue problems of this size often prohibits the use of direct methods. In the present work, we use an extension of the Arnoldi iteration method (Arnoldi, 1951) called the Implicitly Restarted Arnoldi Method (IRAM). This method is matrix-free, which, in conjunction with a time-stepper approach, only requires the solution of an LNS time integrator to solve for the leading eigenvalues.

A turbulent mean flow can be used as the base state for GLSA and GASA. This choice of base state is a solution to the Reynolds-averaged N-S equations. Therefore, a non-linear Reynolds stress term is effectively added to the LNS and ALNS equations when the baseflow equations are subtracted. A mode-dependent Reynolds stress term is therefore present in the associated eigenvalue problem that can be solved using the same numerical techniques as the direct problem. Additionally, the eigenvalues for the direct and adjoint problems coincide with each other. Therefore each direct eigenmode has a corresponding adjoint eigenmode that provides sensitivity information.

The adjoint velocity field defines the sensitivity of the associated direct mode to an unsteady point force aligned with the adjoint velocity vector. This provides valuable sensitivity information in regards to the underlying flow physics and points to locations in the domain that are sensitive to control.

When performing GLSA and GASA, the rank of the eigenvalue problem can be \( O(10^6 - 10^8) \). Solving eigenvalue problems of this size often prohibits the use of direct methods. In the present work, we use an extension of the Arnoldi iteration method (Arnoldi, 1951) called the Implicitly Restarted Arnoldi Method (IRAM). This method is matrix-free, which, in conjunction with a time-stepper approach, only requires the solution of an LNS time integrator to solve for the leading eigenvalues.

A turbulent mean flow can be used as the base state for GLSA and GASA. This choice of base state is a solution to the Reynolds-averaged N-S equations. Therefore, a non-linear Reynolds stress term is effectively added to the LNS and ALNS equations when the baseflow equations are subtracted. A mode-dependent Reynolds stress term is therefore present in the associated eigenvalue problem that can be solved using the same numerical techniques as the direct problem. Additionally, the eigenvalues for the direct and adjoint problems coincide with each other. Therefore each direct eigenmode has a corresponding adjoint eigenmode that provides sensitivity information.

The adjoint velocity field defines the sensitivity of the associated direct mode to an unsteady point force aligned with the adjoint velocity vector. This provides valuable sensitivity information in regards to the underlying flow physics and points to locations in the domain that are sensitive to control.

When performing GLSA and GASA, the rank of the eigenvalue problem can be \( O(10^6 - 10^8) \). Solving eigenvalue problems of this size often prohibits the use of direct methods. In the present work, we use an extension of the Arnoldi iteration method (Arnoldi, 1951) called the Implicitly Restarted Arnoldi Method (IRAM). This method is matrix-free, which, in conjunction with a time-stepper approach, only requires the solution of an LNS time integrator to solve for the leading eigenvalues.
Table 1. Simulation parameters $R$ and $R^∗$ are jet-to-crossflow ratios based on the average jet exit velocity and the peak jet exit velocity, respectively. Jet to crossflow ratios ($R$) of 2 and 4 are studied at a Reynolds number of 2000, based on the average velocity ($v_{jet}$) at the jet exit and the jet exit diameter ($D$). Also shown is an alternative jet to crossflow ratio ($R^∗$), based on the jet exit peak velocity ($v_{jet,max}$). The momentum thickness of the laminar crossflow boundary layer is described at the jet exit when the jet is turned off.

<table>
<thead>
<tr>
<th>Case</th>
<th>$R$</th>
<th>$R^*$</th>
<th>$Re$</th>
<th>$θ_{bl}/D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R2$</td>
<td>2</td>
<td>2.44</td>
<td>2000</td>
<td>0.1215</td>
</tr>
<tr>
<td>$R4$</td>
<td>4</td>
<td>4.72</td>
<td>2000</td>
<td>0.1718</td>
</tr>
</tbody>
</table>

Figure 2. GLSA eigenvalue spectrum for JICF at a Reynolds number of 2000 for case $R2$ (a) and $R4$ (b). The growth rates and Strouhal numbers are normalized appropriately. Vertical velocity spectra from Iyer & Mahesh (2016) are shown (dash-dotted lines) for comparison. Note that $St_2 = 1.3$ is not shown in (a) as it would obscure the low frequency results. The eigenvalues with positive growth rates are unstable.

CONCLUSIONS

Performing GLSA of low-speed JICFs, using turbulent mean flows as the base states, has been shown to produce upstream shear-layer eigenmodes that oscillate at frequencies close to those observed in experiments (Megerian *et al.*, 2007) and simulations (Iyer & Mahesh, 2016) for $R$ values of 2 and 4. Additional unstable eigenmodes have been shown that occupy the wake for $R = 2$ that dominate far downstream. Also, for case $R4$, the downstream shear-layer modes have been shown to be more unstable than upstream shear-layer modes and span a range of frequencies. The transition from sensitive to localized feedback. It is clear that two lobes make up the wavemaker; one on the upstream side of the shear-layer and one on the downstream side that extends into the jet nozzle. Perturbations to the base state that travel through the wavemaker are subject to localized feedback. Therefore, perturbing just inside jet nozzle on the upstream side, or the boundary layer just upstream of the jet exit, could cause localized feedback to amplify the direct upstream shear-layer eigenmode (Figure 3a). Additionally, the location of the wavemaker confirms the absolutely unstable nature of the flow in the vicinity of the jet exit because the wavemaker is located at the origin of the direct upstream shear-layer eigenmode.
Figure 3. Real part of the eigenmodes for case $R_2$ are shown with positive and negative isocontours of $\tilde{u}$ and $\tilde{v}$ contours of the base state in the background. Mode (a) corresponds to the most unstable and highest frequency upstream shear-layer mode. Modes (b-e) are lower frequency and originate near the downstream shear-layer and travel far downstream. Modes (d) and (e) also show a connection between near-wall motions and motions in the jet wake. Note that the zero-frequency modes are not shown.

absolute to convective instability observed in simulations and experiments have been further justified through the use of GLSA.

GASA analysis for case $R_2$ has been shown to coincide with the GLSA spectrum. This allows the conclusion that the upstream shear-layer eigenmode is sensitive to forcing along the upstream side of the jet nozzle close to the jet exit. Furthermore, the computation of the wavemaker shows that the upstream shear-layer direct mode is subject to localized feedback near its point of origin. This further justifies the conclusion that in the vicinity of the jet exit, case $R_2$ behaves as an absolutely unstable flow.

Figure 4. Real part of the eigenmodes for case $R_4$ are shown with positive and negative isocontours of $\tilde{u}$ and $\tilde{v}$ contours of the base state in the background. Modes (a-g) correspond to the higher frequency downstream shear-layer modes. Mode (h) is associated with the upstream shear-layer.

Figure 5. Real part of the adjoint eigenmode for case $R_2$ is shown with positive and negative isocontours of $\tilde{u}$ and $\tilde{v}$ contours of the base state in the background. The eigenmode has an associated non-dimensional eigenvalue $\omega = 0.050 \pm 0.61$, which coincides with the direct eigenmode and provides sensitivity information.
REFERENCES


