Self-preservation and the Kolmogorov 1st similarity law

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ABSTRACT

It is demonstrated that the Kolmogorov (1941a) first similarity law, enunciated solely on the basis of phenomenological arguments, can be recovered from the Navier-Stokes equations under the assumption of self-preservation (SP). In particular, it is shown that this law is recovered from the two-point transport equation of the second-order velocity structure function in decaying turbulence which is homogeneous and isotropic. The Kolmogorov length and velocity scales emerge as the "natural" scaling parameters. When SP is complete, *i.e.* SP holds at all scales of motion, the choice of scaling parameters is immaterial because they are all proportional to the Kolmogorov scales. When SP is incomplete, *i.e.* SP is tenable only at small scales and the Kolmogorov scales are the only correct scaling variables for the small scales.

INTRODUCTION

Kármán & Howarth (1938) developed a transport equation for the two-point velocity correlation function ($B_{LL}(r,t) = \overline{u(x,t)u(x+r,t)}$, where L, r and t stand for the longitudinal (mean flow) direction x, the separation between two points and time, respectively; u is the velocity component along x; the overbar represents an ensemble average) for decaying homogeneous and isotropic turbulence (hereafter denoted HIT)

$$\frac{\partial B_{LL}(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\left[B_{LL,L}(r,t) + 2\nu\frac{\partial B_{LL}(r,t)}{\partial r}\right], \quad (1)$$

where $B_{LL,L} = \overline{u(x,t)u(x,t)u(x+r,t)}$ and ν is the kinematic viscosity of the fluid. This equation continues to play a major role in the theory of decaying HIT. An alternative expression for (1), which uses 2nd and 3rd order velocity structure functions $\delta u_2 = \overline{(u(x+r,t)-u(x,t))^2}$ and $\delta u_3 = \overline{(u(x+r,t)-u(x,t))^3}$ was written by Lin (1948). After some manipulations (Danaila *et al.*, 1999), the isotropic transport equation for δu_2 can be obtained

$$-\delta u_3 + 6\nu\frac{\partial \delta u_2}{\partial r} - \frac{3}{r^4}\int_0^r s^4 \frac{\partial \delta u_2}{\partial t} ds = \frac{4}{5}\overline{\varepsilon} r, \quad (2)$$

where $\overline{\varepsilon} = \nu\overline{\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)}$ is the mean turbulent kinetic energy dissipation rate. Since Eqs. (2) and (1) are equivalent, we will use the latter for our analysis. Recently, Djenidi & Antonia (2015) (hereafter denoted DA15) applied a self-preservation (hereafter SP) analysis to (2), using SP expressions for δu_2 and δu_3 *viz.*

$$\delta u_2 = u_2^2(t)f(r^*), \ \ \delta u_3 = u_3^3(t)g(r^*), \quad (3)$$

where $r^* = r/l(t)$; $l(t)$, $u_2(t)$ and $u_3(t)$ are the length scale and velocity scaling functions to be determined while f and g are dimensionless functions. For convenience, we hereafter drop the variable t. Their analysis introduces the skewness of the longitudinal velocity increment, $S(r)(\equiv \delta u_3/\delta u_2^{3/2})$, as a SP controlling parameter. Using (3), one can write $S = c(t)\phi(r/l)$ where $c(t)$ and $\phi(r/l)$ are dimensionless functions of time and (r/l), respectively. Recent measurements on the axis the far-field of a cylinder wake (Tang *et al.*, 2015), which is in SP, show that $S(r)$ collapses onto a single distribution, confirming that $S = c(t)\phi(r/l)$ with $c(t) = constant$ (i.e. $Re_l = constant$), and vindicating the SP analysis of DA15.

The major objective of the present work is to demonstrate that the Kolmogorov first similarity hypothesis (hereafter denoted K41, (Kolmogorov (1941a))) can be subsumed within a general SP framework and can be formulated in a rigorous manner by applying SP to the two-point transport equation for δu_2. The K41 theory not only marked a milestone in the theory of turbulence, it served and continues to serve as a reference to any new theoretical developments. To our knowledge, there was no attempt to show that the first similarity hypothesis of K41 can be deduced the from Navier-Stokes equations. This shortcoming is remedied by the present analysis.

SELF-PRESERVATION IN HOMOGENEOUS AND ISOTROPIC TURBULENCE

Since in this section we use the SP constraints obtained by DA15, we reproduce briefly their results here for convenience. Substituting (3) and the SP form of $S(r)$ in Eq. (2), we obtain

$$6f'(r^*) - \frac{cu_2 l}{\nu}\phi(r^*)f(r^*)^{3/2} - \frac{3l^2}{\nu u_2^2}\frac{\partial u_2^2}{\partial t}\frac{\Gamma_1}{r^{*4}} + \frac{3}{\nu}l\frac{\partial l}{\partial t}\frac{\Gamma_2}{r^{*4}} = \frac{4}{5}\overline{\varepsilon}\frac{l^2}{\nu u_2^2}r^*, \quad (4)$$

with

$$\Gamma_1 = \int_0^{r^*} s^{*4} f(s^*)ds^*, \ \ \Gamma_2 = \int_0^{r^*} s^{*5} f'(s^*)ds^*, \quad (5)$$

where s^* is a dummy variable of integration. The SP constraints are:

$$cRe_l = C_0,\ \frac{l^2}{\nu u_2^2}\frac{\partial u_2^2}{\partial t} = C_1,\ \frac{l}{\nu}\frac{\partial l}{\partial t} = C_2,\ \frac{\overline{\varepsilon} l^2}{\nu u_2^2} = C_3, \tag{6}$$

where $Re_l = u_2 l/\nu$ is a scaling Reynolds number. The above constraints, *i.e.* C_0, C_1, C_2 and C_3 must be independent of r^*, apply for all separations r^* while the numerical values of the constants depend on the scaling variables. Combining the first and last expressions of (6) and using the definition of Re_l yields

$$l = \left(\frac{\sqrt{C_3}C_0}{c}\right)^{1/2}\left(\frac{\nu^3}{\overline{\varepsilon}}\right)^{1/4}. \tag{7}$$

Now, substituting (7) into the last expression of (6) leads to

$$u_2 = \left(\frac{C_0}{\sqrt{C_3}c}\right)^{1/2}(\nu\overline{\varepsilon})^{1/4}. \tag{8}$$

Recognising that $\eta = \left(\nu^3/\overline{\varepsilon}\right)^{1/4}$ and $v_K = (\nu\overline{\varepsilon})^{1/4}$, the Kolmogorov length and velocity scales, respectively, we can write $l = C_\eta \eta$ and $u_2 = C_{v_K} v_K$; C_η and C_{v_k} are constants whose values vary with the choice of l and u_2. Expressions (7) and (8) reveal that $\overline{\varepsilon}$ and ν are the relevant natural parameters for the velocity increments and showing that η and v_K emerge as natural scaling variables.
We can now discuss the Kolmogorov first similarity hypothesis in relation to the SP analysis.

(i) In considering HIT, Kolmogorov (1941a) enunciated his first similarity law, namely that at a sufficiently large Reynolds number, the probability distribution function of the velocity increment is unambiguously defined by $\overline{\varepsilon}$ and ν. Monin & Yaglom (2007) state that this similarity hypothesis (together with the second hypothesis) *has not (and cannot be) rigorously proved, i.e. derived purely analytically from general laws of mechanics*. However, the above derivation of $\overline{\varepsilon}$ and ν as the controlling parameters for the behaviour of δu is rigorous and based on the Navier-Stokes equations. The parameters $\overline{\varepsilon}$ and ν emerge as ”natural” SP variables, vindicating Kolmorogov's empirical approach when enunciating his first similarity law. Note though that present derivation of this first law does not impose a restriction on the Reynolds number other than that it must be constant under SP. In particular, the derivation does not require the Reynolds number to be very large.

(ii) Kolmogorov obtained his scaling velocity and length scales on dimensional grounds, once he enunciated his first similarity hypothesis. However, as shown above, these scales can be rigorously derived from the SP analysis, which reflects their intrinsic nature as scaling variables, in contrast to, for example, λ, the Taylor microscale, and u. The latter can only be scaling variables once SP is satisfied at all r (DA15); they will be proportional to the Kolmogorov scales. Interestingly, even if complete self-preservation is not achieved at all scales of motion, expressions (7) and (8) reveal that the Kolmogorov scales are the adequate scaling variables, at least in the dissipative range. This explains why, for example, Kolmogorov-normalized velocity spectra in grid turbulence, which does not comply with complete SP, collapse in the dissipative range.

EXPERIMENTAL RESULTS

To test the analytical results of the section above one can for example plot the ratios λ/η and u'/v_K. We note that taking $l = \lambda$ and $u_2 = u'$ in (7) and (8) and combining with (6), we obtain

$$\frac{\lambda}{\eta} = (\sqrt{15}Re_\lambda)^{1/2} \tag{9}$$

$$\frac{u'}{v_K} = (Re_\lambda/\sqrt{15})^{1/2}, \tag{10}$$

since under local isotropy $C_3 = 15$. Accordingly, under SP these ratios must exhibit a plateau, since Re_λ is constant too under SP. Figure 1 reports such plots where the measurements were carried out on the centrelines of the turbulent round jet (Lefeuvre *et al.*, 2015; Djenidi *et al.*, 2016) and a cylinder wake (Tang *et al.*, 2015). While the jet data cover a shorter distance than the wake data, they nevertheless indicate that both λ/η and u'/v_K approach a plateau when $x/D \geq 30$ (D is the nozzle diameter). The wake data show clear evidence of a plateau for both λ/η and u'/v_K when $x/D \geq 200$ (D is the cylinder diameter). Note that according to (9) and (10) the ratio $(\lambda/\eta)_w$ for the wake should be equal to $(Re_{\lambda,j}/Re_{\lambda,w})^{1/2}(\lambda/\eta)_j$, where the subscripts w and j refer to wake and jet (a similar relation holds for u'/v_K). In the present case $(Re_{\lambda,j}/Re_{\lambda,w})^{1/2} = 3$. The presence of a plateau in these distributions indicates that SP is satisfied at all scales but it also vindicates the first similarity hypothesis. This is quite remarkable in particular for the wake flow where Re_λ is about 45. It is even more remarkable that the similarity hypothesis holds on the axis of these turbulent flows which are not isotropic nor homogeneous. This indicates that the first similarity hypothesis not only follows from SP but it does not require neither Re_λ to be large nor the turbulence to be homogeneous and isotropic. This vindicate the results of (Antonia *et al.*, 2014) who showed that, almost irrespectively of the flow or flow region that is considered, Kolmogorov scaling holds at relatively low Reynolds numbers in the dissipative range. However, they also state that the scaling breaks down when Re_λ becomes too small, typically below 20 (see also Djenidi *et al.* (2014)).

The first and second conditions of (6) indicate that for a given Re_λ, $S(r^*)$ should collapse onto a single curve as turbulence decays. These SP conditions also apply for a non HIT such a in cylinder wake (Tang *et al.*, 2015) and a jet flow (Djenidi *et al.*, 2016). To test whether or not the distribution $S(r^*)$ remains unchanged during the decay when SP is achieved, we report in Figure 2 the distributions of $S(r^*)$ (here $r^* = r/\eta$) on a centreline of a turbulent round jet where SP is very well achieved (Djenidi *et al.* (2016)) for two values of the Reynolds number, Re_D, based on the jet nozzle velocity and diameter. We also report in Figure 3 the distributions of $S(r^*)$ measured in grid turbulence (Djenidi *et al.* (2015) for the Reynolds number, Re_M, based of the mesh size of the grid, equal to 13000. As it could have been anticipated from the results of figure 1 there is a relatively good collapse of all the distributions for jet flow (although not shown here, the data on the wake centreline show a very good collapse too; see (Tang *et al.*, 2015)). Notice that the collapse is observed up to the separation r^* where $S(r^*)$ becomes zero, confirming further that SP is satisfied over a very wide range of scales, covering the dissipative, scaling and large scales. For the grid turbulence (Re_λ varies from about 48.5 to about 45 for $Re_M = 13,000$ and from about 19 to about 17 for $Re_M = 4174$), the $S(r^*)$ distributions do not present a collapse, or at least not as good as for the jet, in agreement with the fact that grid turbulence does not decay

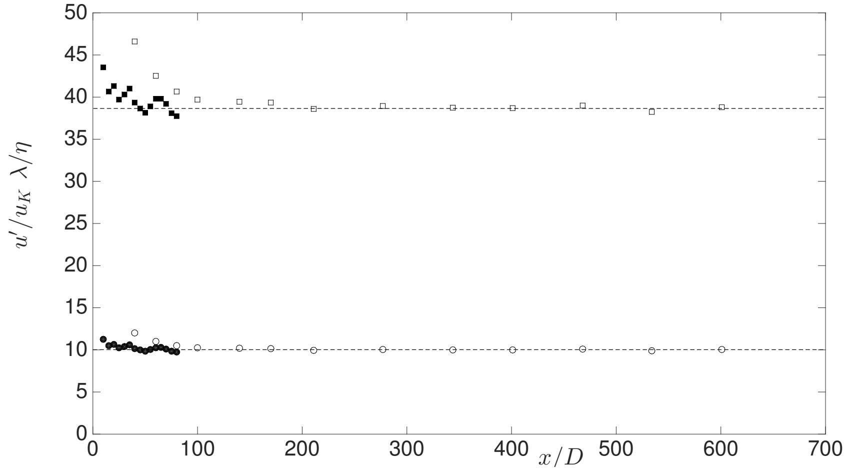

Figure 1. Variation of λ/η (squares) and u'/v_K (circles) on the centreline of an axisymmetric turbulent jet (Closed symbols, $Re_\lambda \simeq 400$) and the centreline of a cylinder wake (Open symbols, $Re_\lambda \simeq 45$). Note that, following (9) and (10), we multiplied both λ/η and u'/v_K for the wake by the factor $(Re_{\lambda,j}/Re_{\lambda,w})^{1/2}$. The dashed straight lines are only used as a visual guide.

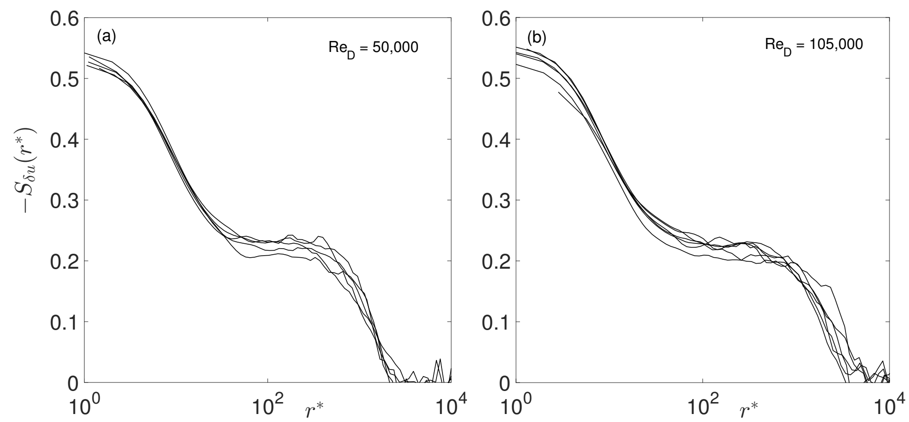

Figure 2. Skewness of the velocity increments, δu, on a centreline of a turbulent round jet over the range $x/D \simeq 30-80$. The separation r is normalized by η.

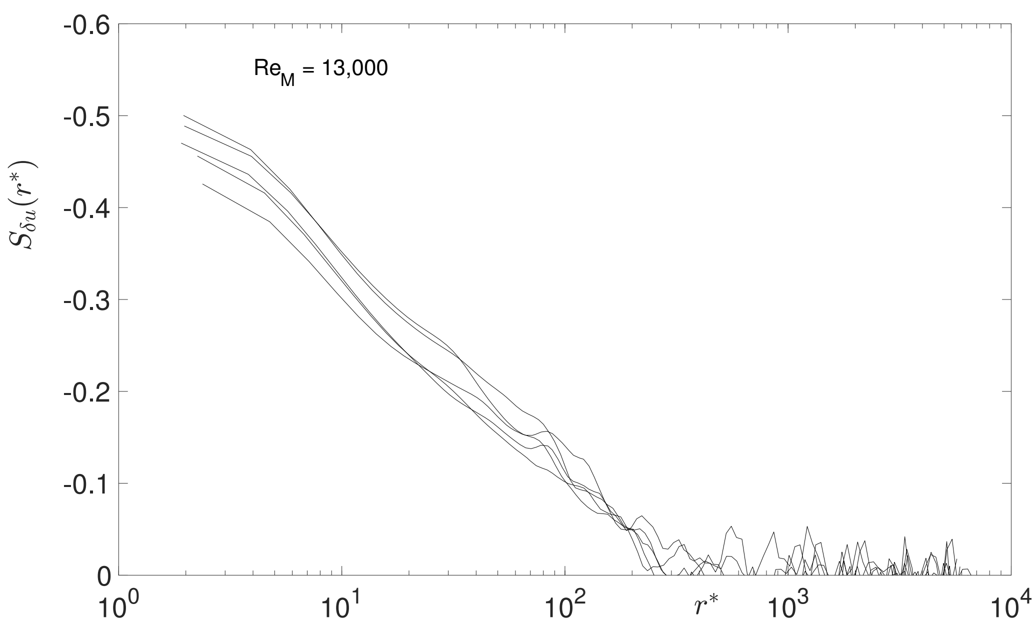

Figure 3. Skewness of the velocity increments, δu, in grid turbulence over a range $x/M \simeq 22-55$. The separation r is normalized by η.

in conformity with SP (Djenidi *et al.* (2015)). Note the low values of $S(r^*)$ for the grid turbulence at $Re_M = 4127$ which reflect the very weak energy transfer. Djenidi *et al.* (2014) showed that the contribution of the energy transfer in the scale-by-scale energy budget becomes smaller than the contributions from the viscous and (large-scale) non-homogeneous terms at all scales, and that the the Kolmogorov normalised spectra deviate from those at higher Re_λ which is indicative of a breakdown of the Kolmogorov first similarity hypothesis.

It is interesting to note the apparent plateau with a magnitude of about 0.23 in the $S(r^*)$ distributions of the jet flow suggesting a nascent scaling range. Such plateau is absent from grid turbulence because of the low value of Re_λ. Antonia & Burattini (2006) showed that for an inertial range to exist Re_λ should exceed about 10^3 when forcing is applied and 10^6 when the turbulence is decaying. Antonia *et al.* (2015) compiled a series of $S(r/\eta)$ distributions in several flows which include decaying grid turbulence, fully developed channel flows, plane and round jets where Re_λ ranges from 33 to about 1000 (see their figure 9). None of the curves have a clear "inertial range" plateau. They only approach it as Re_λ increases. Even $S(r)$ obtained with the Eddy-Damped Quasi-Normal (EDQNM) simulation of decaying HIT at $Re_\lambda = 25000$ do not exhibit an actual plateau (Bos *et al.*, 2012). These observations vindicate Antonia & Burattini (2006) and show that testing the four-fifths law (or two-thirds law) is particularly difficult, if not impossible.

CONCLUDING DISCUSSION

The relationship between self-preservation and the Kolmogorov first similarity hypothesis is investigated analytically. It is shown for the first time that parameters $\overline{\varepsilon}$ and ν emerge naturally from the Navier-Stokes equations when self-preservation is assumed. This indicates that the first similarity hypothesis is subsumed under the concept of self-preservation. Consequently, the Kolmogorov length and velocity scales formed with $\overline{\varepsilon}$ and ν also emerge as natural scaling parameters. When self-preservation is achieved at all scales of motion, any scaling length and scaling velocity are proportional to the Kolmogorov length and velocity scales. When self-preservation is incomplete, *i.e.* tenable only over length scales ranging from the dissipative scales to a large scale range (the integral length scale for example), the analysis indicates that the Kolmogorov scales would be appropriate scaling variables in that range of scales.

Note that for the sake of simplicity of analysis, we have developed the theoretical arguments in the context of HIT. Nonetheless, Figs. 1 and 2 strongly support the applicability of the present theory for flows which are not necessarily isotropic, such as jet and wake centrelines. The reasons are double.

i) The present analysis can be developed for the transport equation of the second-order moments, in flows slightly inhomogeneous but anisotropic, as already underlined by e.g. Antonia *et al.* (2014) (the transport equation for the third-order moments in anisotropic turbulence is more complex and not considered here). The scaling parameters that emerge from the analysis are then ν and $\overline{\varepsilon}+\overline{\varepsilon}^+$, the mean dissipation rates at the two space points. Along each spatial direction, if turbulence is locally homogeneous, then $\overline{\varepsilon} \sim \overline{\varepsilon}^+$. For strongly inhomogeneous flows, it is expected that when very small scales are approached, the $\overline{\varepsilon}$ and $\overline{\varepsilon}^+$ will become equal. Once again, at least for the second-order statistics, the similarity parameters are those deduced by Kolmogorov on phenomenological basis.

ii) The theory validation we provide here is based on hot-wire measurements, which correspond to selecting one spatial direction, that of the main stream. Taylor's hypothesis was employed to calculate spatial statistics, which are thereof artificially homogeneous. Therefore, our results only concern a single, homogeneous, flow direction, x. These are optimal conditions for the theory to be tested.

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