TURBULENT ENERGY DENSITY IN SCALE SPACE BASED ON FILTERED TWO-POINT CORRELATION

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ABSTRACT

The energy spectrum must be useful in describing the energy transfer not only in homogeneous isotropic turbulence but also in inhomogeneous turbulence. Instead of the energy spectrum in the wavenumber space, the energy density in the scale space can be defined using the velocity structure function or the two-point velocity correlation in the physical space. In this work, a new expression for the energy density in the scale space was introduced on the basis of the filtered two-point velocity correlation. In contrast to the previous expressions for the energy density, this expression is expected to be non-negative in homogeneous directions. The transport equation for the energy density was derived and direct numerical simulation data of homogenous isotropic turbulence and turbulent channel flow was used to evaluate the energy transport in the scale space. It was shown that the energy density is positive in all scales and the energy transfer from large to small scales was adequately observed for the turbulent energy.

INTRODUCTION

In order to better understand inhomogeneous turbulence, it must be useful to examine the energy transport not only in the physical space but also in the scale space corresponding to the wavenumber space for homogeneous isotropic turbulence. The second-order structure function $\langle \delta u_i^2 \rangle$ ($\delta u_i = u_i'(\mathbf{x}+\mathbf{r}) - u_i'(\mathbf{x})$) can be considered the scale energy (Hill, 2002; Marati et al., 2004; Davidson, 2004). Its transport equation was investigated in detail in turbulent channel flow (Cimarelli et al., 2012, Cimarelli et al., 2016). However, the structure function is understood as part of the turbulent energy whose scale is equal to or less than $r(=|\mathbf{r}|)$; it is not the energy density in the scale space. As the energy density, or the energy decomposition in the scale space, Davidson (2004) proposed $(3/4)(\partial/\partial r)\langle \delta u_\parallel^2 \rangle$. Paying attention to the two-point correlation $Q_{ii}(\mathbf{r})(=\langle u_i'(\mathbf{x}) u_i'(\mathbf{x}+\mathbf{r})\rangle)$ Hamba (2015) also proposed an expression for the energy density. However, both expressions have a defect that the quantities are not necessarily non-negative. This property is not appropriate as the energy density.

In this work, using the convolution integral we propose another expression for the energy density which is non-negative in homogeneous directions. We expect that this energy density is suitable in discussing the energy transfer in the scale space. We derive the transport equation for the energy density and examine the energy transfer in the scale space using the direct numerical simulation (DNS) data of homogeneous isotropic turbulence and channel flow.

ENERGY DENSITY IN SCALE SPACE

By analogy with the energy spectrum, we require the following properties as the energy density in the scale space (Davidson 2004)

$$E(r) \geq 0 \tag{1}$$

$$\frac{1}{2}\langle u_i'^2 \rangle = \int_0^\infty dr E(r) \tag{2}$$

Paying attention to the two-point correlation we previously proposed an energy density given by

$$E(r) = -\frac{1}{2}\frac{\partial}{\partial r} Q_{ii}(r),\ \ Q_{ii}(r) = \langle u_i'(\mathbf{x}) u_i'(\mathbf{x}+\mathbf{r}) \rangle \tag{3}$$

(Hamba 2015). This expression satisfies the property (2), but not the property (1). $E(r)$ can be negative in the region where $Q_{ii}(r)$ increases with r. This is also shown in the following expression

$$E(r) = \int_{-\infty}^\infty dk \hat{Q}_{ii}(k) \frac{1}{2} k \sin(kr) \tag{4}$$

where $\hat{Q}_{ii}(k)(\geq 0)$ is the Fourier transform of $Q_{ii}(r)$. Since sin(kr) appears in the integral in (4), $E(r)$ can be negative.

In the present work we propose another expression for the energy density as follows

$$E(r) = -\frac{1}{2}\frac{\partial}{\partial r}R^{>}_{ii}(r),\ \ R^{>}_{ii}(r) = \int_{-\infty}^{\infty} d\xi Q_{ii}(\xi)G(\xi,r) \quad (5)$$

where

$$G(\xi,r) = \frac{1}{\sqrt{2\pi}r}\exp\left(-\frac{\xi^2}{2r^2}\right) \quad (6)$$

Here, instead of $Q_{ii}(r)$, we use the filtered two-point correlation $R^{>}_{ii}(r)$. The scale r does not represent the separation distance but the filter width. The new energy density satisfies the properties (1) and (2). Its nonnegativeness is shown by the following expression

$$E(r) = \int_{-\infty}^{\infty} dk\hat{Q}_{ii}(k)\frac{1}{2}k^2 r\exp\left(-\frac{1}{2}k^2r^2\right) \quad (7)$$

Since the integrand in (7) is positive, E(r) is always positive. Note that E(r) can be expressed as (7) when ξ appearing in (5) is taken in a homogeneous direction where the Fourier transform is defined. E(r) may be negative when ξ is in an inhomogeneous direction.

The transport equation for E(r) given by (5) can be written as

$$\begin{aligned}
\frac{D}{Dt}E(r) = &-\frac{1}{2}\int_{-\infty}^{\infty} d\xi_x G_D(\xi,r)\langle u'_k(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})\rangle\frac{\partial}{\partial x_k}U_i(\mathbf{x}) \\
&-\frac{1}{2}\int_{-\infty}^{\infty} d\xi_x G_D(\xi,r)\langle u'_k(\mathbf{x}+\boldsymbol{\xi})u'_i(\mathbf{x})\rangle\frac{\partial}{\partial x_k}U_i(\mathbf{x}+\boldsymbol{\xi}) \\
&-\frac{1}{2}\int_{-\infty}^{\infty} d\xi_x G_D(\xi,r)\nu\langle s'_{ki}(\mathbf{x})s'_{ki}(\mathbf{x}+\boldsymbol{\xi})\rangle \\
&-\frac{\partial}{\partial x_k}\frac{1}{2}\int_{-\infty}^{\infty} d\xi G_D(\xi,r)\langle u'_k(\mathbf{x})u'_i(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})\rangle \\
&-\frac{\partial}{\partial x_i}\frac{1}{2}\int_{-\infty}^{\infty} d\xi G_D(\xi,r)\langle p'(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})+p'(\mathbf{x}+\boldsymbol{\xi})u'_i(\mathbf{x})\rangle \\
&+\frac{\partial}{\partial x_k}\frac{1}{2}\int_{-\infty}^{\infty} d\xi G_D(\xi,r)\nu\langle s'_{ki}(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})+s'_{ki}(\mathbf{x}+\boldsymbol{\xi})u'_i(\mathbf{x})\rangle \\
&+\frac{\partial}{\partial r}\frac{1}{2}\int_{-\infty}^{\infty} d\xi G(\xi,r)\frac{\partial}{\partial \xi_k}\langle (u'_k(\mathbf{x}+\boldsymbol{\xi})-u'_k(\mathbf{x}))u'_i(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})\rangle \\
&+\frac{\partial}{\partial r}\frac{1}{2}\int_{-\infty}^{\infty} d\xi G(\xi,r)\frac{\partial}{\partial \xi_k}[(U_k(\mathbf{x}+\boldsymbol{\xi})-U_k(\mathbf{x}))\langle u'_i(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})\rangle]
\end{aligned} \quad (8)$$

where $G_D(\xi,r) = -(\partial/\partial r)G(\xi,r)$ and $s'_{ij} = \partial u'_i/\partial x_j + \partial u'_j/\partial x_i$. The transport equation for E(r) given by (3) can be obtained by replacing $G(\xi,r)$ by $[\delta(\xi-r)+\delta(\xi+r)]/2$ in (8). By integrating each term from $r=0$ to ∞, (8) is reduced to the transport equation for the turbulent energy $\langle u'^2_i\rangle/2$. On the right-hand side of (8) the first and second terms correspond to the energy production, the third term to the dissipation, the fourth term to the turbulent diffusion, the fifth term to the pressure diffusion, the sixth term to the viscous diffusion in the energy equation. These terms can be considered as the decomposition of the corresponding terms in the scale space. In contrast, the remaining seventh and eighth terms represent the energy transfer in the r space; these terms do not appear in the energy equation. They are useful in assessing the energy cascade among different scales.

For homogeneous isotropic turbulence the above transport equation is rewritten as

$$\frac{\partial}{\partial t}E(r) = -\varepsilon_E(r) + T_E(r) \quad (10)$$

$$\varepsilon_E(r) = \frac{1}{2}\int_{-\infty}^{\infty} d\xi_x G_D(\xi,r)\nu\langle s'_{ki}(\mathbf{x})s'_{ki}(\mathbf{x}+\boldsymbol{\xi})\rangle \quad (11)$$

$$T_E(r) = \frac{\partial}{\partial r}\frac{1}{2}\int_{-\infty}^{\infty} d\xi G(\xi,r)\frac{\partial}{\partial \xi_k}\langle (u'_k(\mathbf{x}+\boldsymbol{\xi})-u'_k(\mathbf{x}))u'_i(\mathbf{x})u'_i(\mathbf{x}+\boldsymbol{\xi})\rangle \quad (12)$$

Only the dissipation term $\varepsilon_E(r)$ and the transfer term $T_E(r)$ remain. If the external force exists, the forcing term $F_E(r)$ is added to the right-hand side of (10).

HOMOGENEOUS ISOTROPIC TURBULENCE

The transport equation given by (8) represents the energy transfer in the physical and scale spaces. We first examine the energy transfer in the scale space for homogeneous isotropic turbulence. We examine DNS data of homogeneous isotropic turbulence using 512^3 grid points. We will show results of steady turbulence with external forcing at $2.5 < k < 4.5$. The Reynolds number R_λ is 121.

Figure 1 shows the energy density E(r) given by (5) multiplied by r for log scale. The profile of rE(r) shows its maximum at $r \simeq 0.6$. The inertial range of the energy density should be proportional to $r^{-1/3}$; the corresponding profile of $rE(r) \propto r^{2/3}$ is also plotted in Fig. 1. In the small scale at $r < 0.07$, the energy density decays faster than that for the inertial range, indicating the dissipation range.

For the homogeneous isotropic turbulence, the transport equation for E(r) is given by (10). Since external forcing is applied, the forcing term $F_E(r)$ is added to the dissipation and transfer terms in (10). Figure 2 shows the three terms multiplied by r. The forcing term is positive at the large scale around $r = 0.6$ which corresponds to the peak of rE(r) in Fig. 1. The transfer term is negative at the large scale and positive at the small scale, representing the energy transfer from large to small scales. In the dissipation range the positive transfer term and the negative dissipation term balance to each other. This situation is the same as the energy cascade from low to high wavenumbers.

CHANNEL FLOW

Next, we examine DNS data of channel flow to assess the energy density in inhomogeneous turbulence. The size of the computational domain is $L_x \times L_y \times L_z = 2\pi \times 2 \times \pi$. The number of grid points is $N_x \times N_y \times N_z = 512 \times 192 \times 512$. The Reynolds number based on the friction velocity u_τ and the channel half width $L_y/2$ is set to $Re_\tau = 395$. Physical quantities are nondimensionalized using u_τ and $L_y/2$. The periodic boundary conditions are used in the streamwise and spanwise directions and no-slip conditions are imposed at the wall ($y = \pm 1$). Statistics are obtained by averaging over x-z plane and over a time period of 20. Although the Reynolds number is not very high, the data can be used to assess the expressions for the energy density. It remains as future work to analyze a flow at higher Reynolds number.

In channel flow the energy density E(r) depends on the coordinate y in the wall-normal direction. Here we will show results at $y^+ = 144$. The energy density also depends on the direction of the vector $\mathbf{r}$. In this work we focus on the spanwise direction $\mathbf{r} = r_z \mathbf{e}_z$ and the wall normal direction $\mathbf{r} = -r_y \mathbf{e}_y$. The spanwise direction is examined because negative velocity correlation is clearly seen and we can investigate the difference between the previous and new expressions for the energy density. The wall normal direction is also examined because the energy density was introduced to apply to inhomogeneous directions.

First we show the result of the energy density for $\mathbf{r} = r_z \mathbf{e}_z$. Figure 3 shows the two-point correlation $Q_{ii}(y,r_z)/2$ as a function of r_z at $y^+ = 144$. Negative values are clearly seen around $r_z = 0.6$. This negative correlation is mainly caused by the streak structure of the streamwise velocity fluctuation u'_x. Figure 4 shows the energy density $E(y,r_z)$ given by (3) as a function of r_z. At the small scale $r_z < 0.6$, the energy density is positive. However, the value is negative at $0.6 < r_z < 1.2$ because the correlation $Q_{ii}(y,r_z)/2$ increases with r_z in this region as shown in Fig. 3. The previous energy density given by (3) is not always positive and needs to be improved.

Figure 5 shows terms in the transport equation (8) for $E(y,r_z)$ given by (3) as functions of r_z at $y^+ = 144$. In Fig. 5 each term is multiplied by r_z as is the case of Fig. 2. At the scale $r_z < 0.4$ the role of each term can be explained naturally as follows. The production term shows positive values at the large scale $r_z \simeq 0.2$. The transfer term is negative at the large scale and positive at the small scale, representing the energy transfer from large to small scales. The dissipation term shows negative values at the small scale $r_z \simeq 0.05$. This situation is similar to the case of homogeneous isotropic turbulence with external forcing shown in Fig. 2. In contrast, it is difficult to explain the energy transport at the scale $r_z > 0.4$. Each term fairly oscillates at this scale because long time average is required to obtain good statistics at the very large scale. Moreover, the production term shows negative values and the transfer term shows positive values. These profiles may indicate that a negative energy density is produced here and it is transferred to smaller scales, but its physical meaning is not clear.

Figure 6 shows the energy density $E(y,r_z)$ given by (5) as a function of r_z. In contrast to Fig. 4, the profile shows positive values in the whole region because of the new definition of the energy density. Figure 7 shows terms in the transport equation (8) for $E(y,r_z)$ given by (5) as functions of r_z. Smooth profiles are obtained even at the very large scale. This is because the values of $Q_{ii}(\xi)$ in various scales affects the value of $E(y,r_z)$ in (5) in contrast to (3). The production term correctly shows positive values in the whole region. The transfer term shows the energy cascade from large to small scales. This situation is similar to the case of homogeneous isotropic turbulence shown in Fig. 2. In addition, we can see small positive values of the transfer term at the very large scale $r_z > 0.7$. In contrast to Fig. 5, this positive value is expected to be physically meaningful; it suggests a slight backward energy cascade to the very large scale.

Next we show the result of the energy density for $\mathbf{r} = -r_y \mathbf{e}_y$. We consider the direction approaching the wall to examine the effect of the wall in detail. Figure 8 shows the two-point correlation $Q_{ii}(y,r_y)/2$ as a function of r_y at $y^+ = 144$. The value $r_y = 0.36$ where the correlation vanishes corresponds to the lower wall. The correlation $Q_{ii}(y,r_y)/2$ monotonically decreases as r_y increases. It decreases rapidly at $r_y = 0.33$ near the wall. Figure 9 shows the energy density $E(y,r_y)$ given by (3) as a function of r_y. At large scales it decreases as r_y increases. However, it rapidly increases at $r_y = 0.33$ due to the rapid decrease of $Q_{ii}(y,r_y)/2$ shown in Fig. 8. This rapid change of profile is because the energy density depends on the statistics at $y' = y - r_y$ directly.

This tendency is also seen in the transport equation. Figure 10 shows terms in the transport equation (8) for $E(y,r_y)$ given by (3) as functions of r_y at $y^+ = 144$. At $r_y < 0.3$ the behavior of the production, dissipation, and transfer terms is similar to those in Fig. 5; the energy produced at the large scale is transferred to the small scale. However, at $r_y > 0.3$ each term rapidly changes. Therefore, the energy density given by (3) is too sensitive to the field near the wall because E(r) is directly affected by the correlation $Q_{ii}(r)(= \langle u'_i(\mathbf{x})u'_i(\mathbf{x}+\mathbf{r})\rangle)$.

Figure 11 shows the energy density $E(y,r_y)$ given by (5) as a function of r_y. Since r_y is not the separation distance but the filter width appearing in the convolution integral in (5), it can be greater than 0.36. In contrast to Fig. 9, the profile smoothly decreases at the large scales because of the new definition of the energy density. It is similar to the profile of $E(y,r_z)$ shown in Fig. 6. Figure 12 shows terms in the transport equation (8) for $E(y,r_y)$ given by (5) as functions of r_y. Smooth profiles are obtained in the whole region. The profiles are similar to the case of homogeneous isotropic turbulence shown in Fig. 2. In contrast to Fig. 7, the transfer term is negative at the very large scale; only the forward energy cascade is seen. These results show that the new definition of the energy density given by (5) is expected to be useful in investigating the energy transfer even in inhomogeneous directions.

CONCLUSIONS

A new expression for the energy density in the scale space was introduced on the basis of the filtered two-point velocity correlation. The transport equation for the energy density was evaluated using DNS data of homogeneous isotropic turbulence and channel flow. It was shown that the energy density takes positive values in the whole region and the energy transfer in the scale space can be examined appropriately even in inhomogeneous direction.

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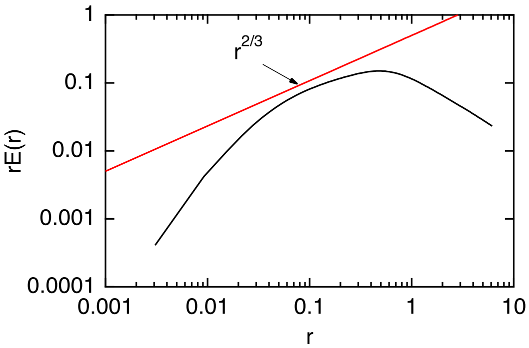

Figure 1. Profile of the energy density $E(r)$ given by (5) for homogeneous isotropic turbulence.

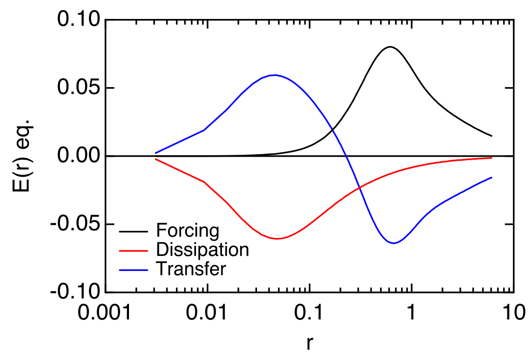

Figure 2. Profiles of terms in the transport equation (8) for homogeneous isotropic turbulence

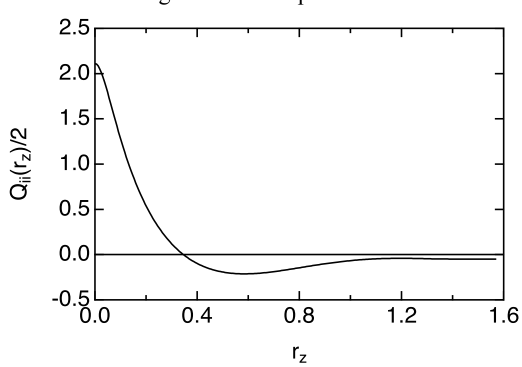

Figure 3. Profile of the velocity correlation $Q_{ii}(y,r_z)/2$ at $y^+ = 144$ in channel flow.

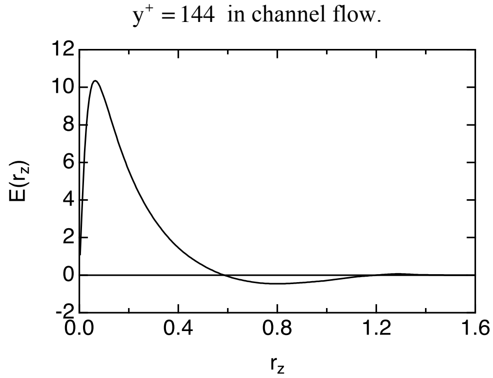

Figure 4. Profile of the energy density $E(y,r_z)$ given by (3) at $y^+ = 144$ in channel flow.

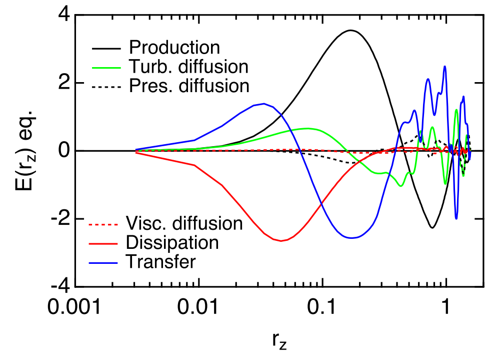

Figure 5. Profiles of terms in the transport equation (8) for $E(y,r_z)$ given by (3).

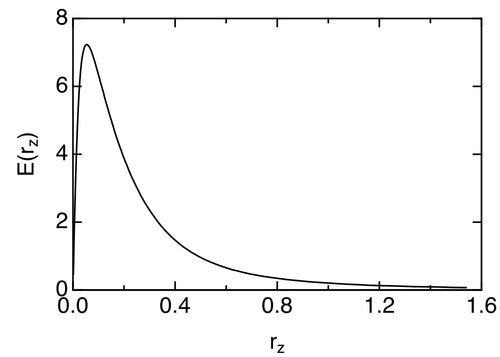

Figure 6. Profile of the energy density $E(y,r_z)$ given by (5) at $y^+ = 144$ in channel flow.

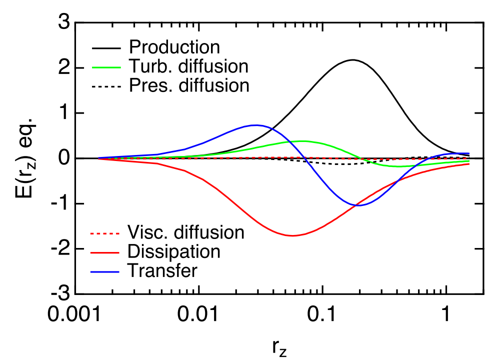

Figure 7. Profiles of terms in the transport equation (8) for $E(y,r_z)$ given by (5).

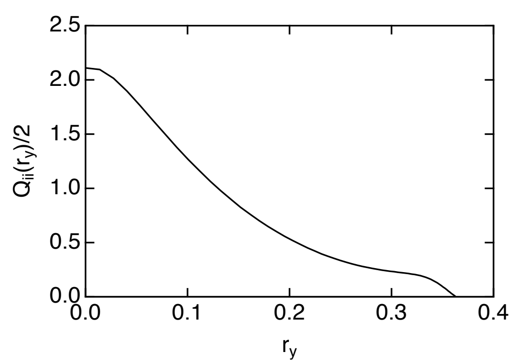

Figure 8. Profile of the velocity correlation $Q_{ii}(y,r_y)/2$ at $y^+ = 144$ in channel flow.

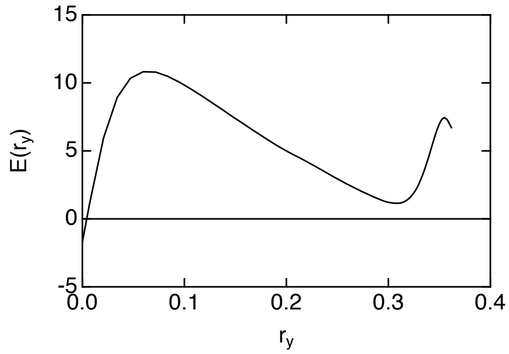

Figure 9. Profile of the energy density $E(y,r_y)$ given by (3) at $y^+ = 144$ in channel flow.

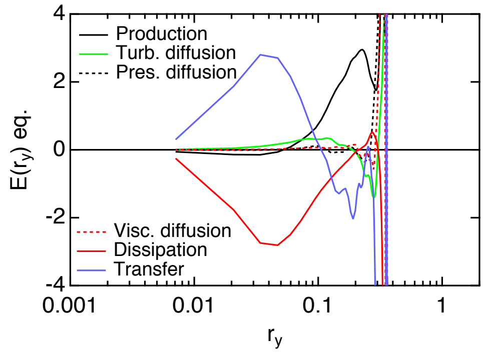

Figure 10. Profiles of terms in the transport equation (8) for $E(y,r_y)$ given by (3).

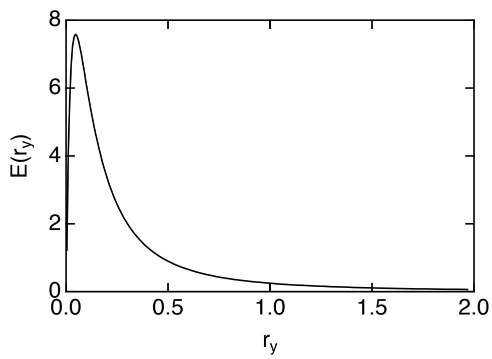

Figure 11. Profile of the energy density $E(y,r_y)$ given by (5) at $y^+ = 144$ in channel flow.

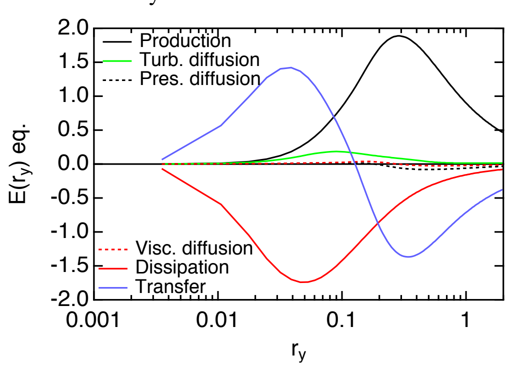

Figure 12. Profiles of terms in the transport equation (8) for $E(y,r_y)$ given by (5).