NONLINEAR FEATURES IN EXPLICIT ALGEBRAIC MODELS FOR TURBULENT FLOWS WITH ACTIVE SCALARS

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ABSTRACT

A detailed discussion of explicit algebraic turbulence models in the case of active scalars is given. In particular, we discuss the appearance of nonlinearities in the models and the need for explicit solutions of the resulting nonlinear equations. Focussing on a recently published model for two-dimensional stratified flows, we present an intuitive way of approximating the solution of a sixth-order polynomial equation for the production-to-dissipation ratio $(\mathscr{P}+\mathscr{G})/\varepsilon$ of turbulent kinetic energy K. This formulation is shown to be consistent for turbulent channel flow with stable and unstable stratification. The result is important for obtaining a robust model with a correct behaviour of the turbulence production in different limits of shear and buoyancy. The results have recently been published in Lazeroms *et al.* (2015).

INTRODUCTION

Turbulent flows are often modelled using the Reynolds-averaged Navier-Stokes (RANS) approach, where the quantities describing the flow are split into a mean part and a fluctuating part. The RANS equations for the mean quantities containg unknown correlations (the Reynolds stresses and turbulent fluxes), for which appropriate models need to be found. The class of explicit algebraic Reynolds-stress models (EARSM) provides a good compromise between the simplicity of standard eddy-diffusivity models (EDM), and the adequate physics of differential Reynolds-stress models (DRSM). The Reynolds-stress transport equations from a DRSM are approximated by algebraic equations by means of an appropriate equilibrium assumption. In order to avoid numerical problems associated with solving such algebraic equations, one would like to find an explicit solution, hence the term *explicit* algebraic models.

Since Rodi (1972, 1976), many explicit algebraic models have been derived using the weak-equilibrium assumption, under which the advection and diffusion of *dimensionless* fluxes (e.g. the dimensionless Reynolds-stress anisotropy tensor) are neglected. Although this assumption is quite general, it inevitably generates a nonlinear system of equations for the Reynolds stresses. In cases without buoyancy, these nonlinearities can be treated by finding the *exact* solution of a cubic equation for the production-to-dissipation ratio (Girimaji, 1996; Wallin & Johansson, 2000). Retrieving the exact solution is important for obtaining a correct behaviour of the turbulence production for large strain rates, as well as avoiding possible numerical issues associated with solving higher-order equations iteratively. Similar results hold for passive scalars in turbulent flows (Wikström *et al.*, 2000).

In recent years, algebraic models have been derived for turbulent flows with buoyancy, in which temperature acts as an active scalar (So *et al.*, 2002, 2004; Violeau, 2009; Lazeroms *et al.*, 2013; Vanpouille *et al.*, 2013, 2014). In these cases the Reynolds-stresses and turbulent heat flux are mutually coupled, so that the system of equations one needs to solve becomes more complicated. As a result, one retrieves a sixth-order polynomial equation for the production-to-dissipation ratio (in the case of two-dimensional mean flows), for which no exact solution can be found. Many authors avoid this complication by stating that the equation can be solved numerically. By doing so one would lose the practical advantage of a fully explicit algebraic model over DRSMs. Instead, a fully explicit formulation that approximates the solution of the nonlinear equation is preferred.

Here we present a new, intuitive way of formulating the production-to-dissipation ratio in explicit algebraic models, based on previous work for two-dimensional stratified flows (Lazeroms *et al.*, 2013), and show that the approximate solution is close to the exact solution.

MODEL DESCRIPTION

In the case of turbulent flows with buoyancy, the goal of any RANS model is to find appropriate expressions for the Reynolds stresses $\overline{u_i u_j}$ and turbulent heat flux $\overline{u_i\theta}$ in order to close the equations for the mean velocity U_i and the mean (potential) temperature Θ. The explicit algebraic model is derived from the transport equations for $\overline{u_i u_j}$ and $\overline{u_i\theta}$, which are strongly coupled through the effect of buoyancy. These equations are reformulated in terms of the Reynolds-stress anisotropy a_{ij} and a normalized heat flux ξ_i:

$$a_{ij} = \frac{\overline{u_i u_j}}{K} - \frac{2}{3}\delta_{ij} \qquad \xi_i = \frac{\overline{u_i\theta}}{\sqrt{KK_\theta}} \tag{1}$$

where $K = \overline{u_k u_k}/2$ is the turbulent kinetic energy and $K_\theta = \overline{\theta^2}/2$ half the temperature variance. As shown in Lazeroms *et al.* (2013, 2015), neglecting advection and diffusion of a_{ij} and ξ_i results in algebraic equations of the following form (using matrix notation):

$$N(\boldsymbol{a},\boldsymbol{\xi})\boldsymbol{a} = \boldsymbol{L}^{(a)}\left(\boldsymbol{a},\boldsymbol{\xi}\ ;\ \boldsymbol{S},\boldsymbol{\Omega},\boldsymbol{\Theta},\boldsymbol{\Gamma}\right) \tag{2a}$$

$$N_\theta(\boldsymbol{a},\boldsymbol{\xi})\boldsymbol{\xi} = \boldsymbol{L}^{(\xi)}\left(\boldsymbol{a},\boldsymbol{\xi}\ ;\ \boldsymbol{S},\boldsymbol{\Omega},\boldsymbol{\Theta},\boldsymbol{\Gamma}\right) \tag{2b}$$

in which the right-hand sides are linear functions of a_{ij} and ξ_i, as well as a set of nondimensionalized tensors and vectors depending on the mean flow: the mean strain-rate tensor S_{ij}, the mean rotation-rate tensor Ω_{ij}, the mean temperature gradient Θ_i, and the buoyancy vector Γ_i. These quantities are defined as:

$$S_{ij} \equiv \frac{\tau}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \qquad \Omega_{ij} \equiv \frac{\tau}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$
$$\Theta_i \equiv \tau\sqrt{\frac{K}{K_\theta}}\frac{\partial \Theta}{\partial x_i} \qquad \Gamma_i \equiv \tau\sqrt{\frac{K_\theta}{K}}\beta_T g_i \tag{3}$$

where τ is the turbulence time scale, β_T the thermal expansion coefficient, and g_j the gravitational acceleration.

Here we focus on the left-hand sides of equations (2), which contain the factors N and N_θ that depend on a_{ij} and ξ_i. Hence, equations (2) are essentially *nonlinear*. However, by assuming that N and N_θ are known coefficients, solving (2) for a_{ij} and ξ_i becomes a linear problem, which can be treated by means of linear expansions. The linear part of the problem has been discussed in previous work (see Lazeroms *et al.* 2013, in which the problem is solved for *two-dimensional flows*). The resulting model for a_{ij} and ξ_i has the following form:

$$\boldsymbol{a} = \sum_{k=1}^{M}\beta_k \boldsymbol{T}^{(k)} \qquad \boldsymbol{\xi} = \sum_{k=1}^{M'}\lambda_k \boldsymbol{V}^{(k)} \tag{4}$$

where $\boldsymbol{T}^{(k)}$ and $\boldsymbol{V}^{(k)}$ are tensorial combinations of S_{ij}, Ω_{ij}, Θ_i and Γ_i.

This method only partly solves the problem, as the factors N and N_θ are still unknown. Hence, the coefficients β_k and λ_k in the aforementioned expansions still depend on these factors. The full expressions for N and N_θ are as follows:

$$N(\boldsymbol{a},\boldsymbol{\xi}) = c_1 - 1 + \frac{\mathscr{P}+\mathscr{G}}{\varepsilon}$$
$$= c_1 - 1 - a_{km}S_{km} - \Gamma_k\xi_k \tag{5a}$$

$$N_\theta(\boldsymbol{a},\boldsymbol{\xi}) = c_{\theta 1} + \frac{1}{2}\left(\frac{\mathscr{P}+\mathscr{G}}{\varepsilon} - 1 - \frac{1}{r}\right)$$
$$= c_{\theta 1} + \frac{1}{2}\left(N - c_1 - \frac{1}{r}\right) \tag{5b}$$

where $\mathscr{P}$ and $\mathscr{G}$ are shear and buoyancy production of turbulent kinetic energy K, respectively, and ε its dissipation rate. Moreover, $\{c_1, c_{\theta 1}, r\}$ are model constants. Hence, N and N_θ are directly related to the total production-to-dissipation ratio $(\mathscr{P}+\mathscr{G})/\varepsilon$. Furthermore, N_θ is a function of N,[1] which means that we only need to consider solving for N. This is the main topic of the current paper.

THE NONLINEAR EQUATION

Now we discuss the nonlinear part of the problem, focussing on (two-dimensional) horizontal parallel flows with buoyancy, in which the velocity gradient and temperature gradient are aligned with gravity, while the flow is in the perpendicular direction (e.g. horizontal channel flow). By inserting the linear expansions (4) for a_{ij} and ξ_i into equation (5a) (and using (5b) to eliminate N_θ), one retrieves a polynomial equation for N. In the current case with buoyancy, it turns out that the polynomial equation has *degree 6*, which means that no exact solution can be found for its roots. It is at this point where many authors (e.g. So *et al.* 2002, 2004; Vanpouille *et al.* 2013, 2014) decide to solve the remaining equations for $(\mathscr{P}+\mathscr{G})/\varepsilon$ numerically, which in our opinion does not yield an *explicit* algebraic model and might yield numerical difficulties. The great advantage of EARSMs lies in the fact that no additional iterations should be necessary to evaluate the closure, other than the inevitable addition of differential equations for the turbulent scales, such as the $K-\varepsilon$-model. The other extreme is to use a (constant) *ad hoc* value for $(\mathscr{P}+\mathscr{G})/\varepsilon$ (suggested by e.g. Violeau 2009), which makes the model internally inconsistent and gives a wrong behaviour of the turbulence production at large strain rates (as explained by Girimaji 1996; Wallin & Johansson 2000).

Since the exact solution for N cannot be found, it is our aim to devise an *approximate* expression satisfying two requirements: (1) it should be fully explicit in the mean quantities (for computational reasons), and (2) it should be as close to the exact solution as possible (to obtain a consistent model with a correct asymptotic behaviour).

We begin the search for an approximation for N by considering the following two important limits:

1. Zero buoyancy ($\Gamma_i \to 0$ and $\mathscr{G}/\varepsilon \to 0$), or *shear-dominated* regime, occuring e.g. in near-wall regions of wall-bounded flows.
2. Zero shear ($S_{ij} \to 0$ and $\mathscr{P}/\varepsilon \to 0$), or *buoyancy-dominated* regime; this limit can occur locally (e.g. in the centre of channel flow) or in larger regions when

[1] This requires an additional assumption for one of the model constants, otherwise (5b) also involves the production-to-dissipation ratio of K_θ. See Lazeroms *et al.* (2013, 2015).

convection effects are strong and the mean gradients are small due to mixing.

In the first limit, one obtains the case studied independently by Girimaji (1996) and Wallin & Johansson (2000), in which the equation for N is a third-order polynomial. This equation can be solved exactly, and the solution, here called $N^{(S)}$, is given in the aforementioned papers. It turns out that the second limit of zero shear also gives a (different) third-order polynomial equation, which can be solved in a similar way (Lazeroms *et al.*, 2015). We shall call the exact solution of this equation $N^{(B)}$. Since this method essentially models $\mathscr{P}/\varepsilon$ and $\mathscr{G}/\varepsilon$ separately, we can write (cf. the definition of N in equation (5a)):

$$N^{(0)} = N^{(S)} + N^{(B)} - c_1 + 1 \tag{6}$$

as our first approximation. The advantage of this method is that one automatically obtains the correct behaviour in the two aforementioned limits. Furthermore, it turns out that $N^{(0)}$ can also be used in regions where both shear and buoyancy are important as long as the stratification is unstable (i.e. $\Gamma_k\Theta_k \geqslant 0$).

However, when the stratification is stable (i.e. $\Gamma_k\Theta_k < 0$), additional corrections are needed. As shown in the next section, it turns out that the exact solution $N^{(S)}$ significantly overpredicts the shear production the case of stable stratification. Therefore, a correction to this term is derived by incorporating some of the neglected buoyancy terms in the exact equation, which in the case of stable stratification should counteract the shear production. The details of this correction are given in Lazeroms *et al.* (2015); the corrected expression will be called $\widetilde{N}^{(S)}$ and subsequently, we have:

$$\widetilde{N}^{(0)} = \widetilde{N}^{(S)} + N^{(B)} - c_1 + 1 \tag{7}$$

The approximation can be improved by using a *one-step iteration*, i.e. by inserting (7) into the right-hand side of (5a) in order to get the next approximation. Therefore, our final expression for N, combining the shear-dominated and buoyancy-dominated regimes both for stably and unstably stratified flows, is expressed as follows:

$$N = \begin{cases} N^{(0)}, & \mathbf{\Gamma}\cdot\mathbf{\Theta} \geqslant 0 \qquad \text{(unstable/neutral)} \\ f(\widetilde{N}^{(0)}), & \mathbf{\Gamma}\cdot\mathbf{\Theta} < 0 \qquad \text{(stable)} \end{cases} \tag{8}$$

where f represents the right-hand side of (5a).

NUMERICAL EVALUATION: CHANNEL FLOW

The method for approximating the solution for N given by (8) has been investigated numerically in a number of test cases. These tests focus on the comparison of the obtained values for N with the production-to-dissipation ratio following from the full model, i.e.:

$$\frac{\mathscr{P}+\mathscr{G}}{\varepsilon} = -a_{km}S_{km} - \Gamma_k\xi_k \tag{9}$$

Note that the value of $N - c_1 + 1$ will only be identical to $(\mathscr{P}+\mathscr{G})/\varepsilon$ if we have the exact solution for N. Since we only have an approximation for N, the two values will in principle be different. In order to have an approximately self-consistent model, we require that the values for $N - c_1 + 1$ and $(\mathscr{P}+\mathscr{G})/\varepsilon$ be close to each other.

The model has been evaluated in the case of turbulent channel flow, both with stable and unstable stratification, i.e. an infinite horizontal channel containing a temperature difference $\Delta T = T_{\text{up}} - T_{\text{down}}$ between the two walls, in which the flow is forced by a constant streamwise pressure gradient. The flow is determined by the friction Reynolds number, $Re_\tau \equiv u_\tau h/\nu$, the friction Richardson number $Ri_\tau \equiv \beta_T g \Delta T h/u_\tau^2$, and the Prandtl number $Pr \equiv \nu/\kappa$, where u_τ is the friction velocity, h the channel half-height, ν the kinematic viscosity and κ the molecular heat diffusivity. The calculations were performed using a $K-\omega-K_\theta$-model (including near-wall corrections) on a collocated grid of 201 grid point along one half of the channel. The stratification depends on the sign of Ri_τ (i.e. the sign of ΔT). Nine cases were studied: the neutral case ($Ri_\tau = 0$), three stably stratified cases ($Ri_\tau = 120,\ 480,\ 960$), and four unstably stratified or convective cases ($Ri_\tau = -25,\ -50,\ -100,\ -200$). Figure 1 shows the results of the calculations for the mean velocity profile, in which the effects of the stratification can clearly be seen. The stably stratified cases show an increase of velocity in the centre of the channel due to the damping effect of turbulent motions, while the unstable cases show a decrease of the velocity due to increasing turbulent mixing. Note that the model yields a particularly good agreement with DNS data in the stably stratified cases. More details can be found in Lazeroms *et al.* (2013, 2015).

As mentioned, we are mainly interested in the outcome of the formulation for N given by (8). Figure 2 shows a comparison of N, $\widetilde{N}^{(0)}$, $N^{(S)}$, and $N^{(B)}$ with the value of $(\mathscr{P}+\mathscr{G})/\varepsilon$ following from the explicit algebraic model for a selection of values for Ri_τ. In the neutral case (Figure 2(a)), the formulation for N is identical to the exact solution $N^{(S)}$ for the shear-dominated regime, meaning that the curves for $N - c_1 + 1$ and $(\mathscr{P}+\mathscr{G})/\varepsilon$ are exactly the same. In both the stably and the unstably stratified cases (Figures 2(b)-(d)), we see that the approximation given by (8) gives values that are very close to the curve of $(\mathscr{P}+\mathscr{G})/\varepsilon$. We conclude that the new formulation yields a model that is very close to being self-consistent, except very close to the wall ($y = 0$) where the near-wall corrections are active.

It is interesting to consider the behaviour of the two solutions $N^{(S)}$ and $N^{(B)}$, since they are exact in their respective limits. In the stably stratified case (Figure 2(b)), $N^{(B)}$ gives a relatively small, negative contribution to the production-to-dissipation ratio, corresponding to a negative buoyancy term in the K-equation that damps turbulence. However, we see that $N^{(S)}$ significantly overpredicts the shear production, as mentioned in the previous section. This is the reason why we used the corrected version $\widetilde{N}^{(0)}$ from equation (7) (the blue dashed line in Figure 2(b)). To further improve the approximation, the one-step iteration explained in the previous section is needed.

On the other hand, Figures 2(c)-(d) show that $N^{(B)}$ has a large positive contribution to the production of turbulence in the unstably stratified cases. The buoyancy-dominated regime is reached in a significant portion of the channel around $y = h$. In this region where $N^{(B)}$ is significant, the contribution of $N^{(S)}$ is nearly zero. However, the roles of $N^{(S)}$ and $N^{(B)}$ are reversed towards the wall ($y = 0$), where the shear becomes much more important. The fact that the shear- and buoyancy-dominated regimes switch places in a

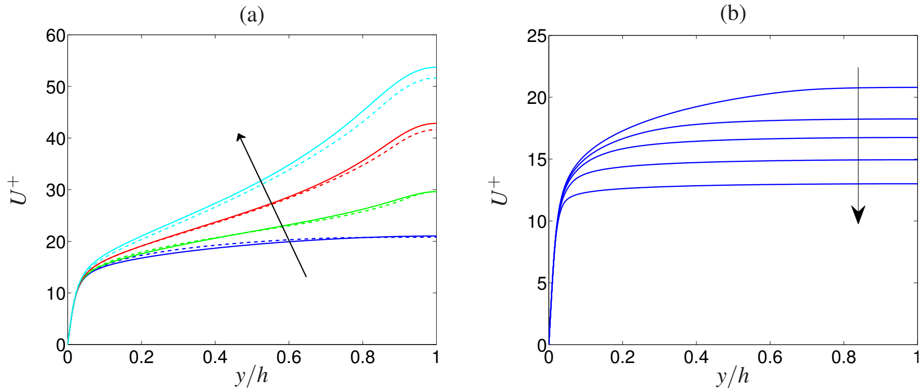

Figure 1. Profiles of mean velocity $U^+ = U/u_\tau$ in turbulent channel flow for $Re_\tau = 550$, $Pr = 0.7$ and different values of Ri_τ: (a) stable stratification, the model (dashed) is compared with DNS (solid) by García-Villalba & del Álamo (2011); (b) unstable stratification. The arrows point in the direction of increasing $|Ri_\tau|$.

relatively simple way is the main reason for using the simple addition given by (6) for convective flows.

In Lazeroms *et al.* (2015), we also discuss the interesting test case of an idealized diurnal cycle in the atmospheric boundary layer. The flow is forced by a constant outer velocity and the buoyancy effects are determined by a sinusoidal surface temperature, which creates a periodic alternation between stable and unstable stratification. Even in this non-stationary case containing rather complex balances between shear and buoyancy, the formulation for N is shown to give nearly self-consistent results.

CONCLUSION

The method described here provides an approximately self-consistent formulation of the factor N corresponding to the total production-to-dissipation ratio of turbulent kinetic energy. This result is important for having a correct asymptotic behaviour for large strain rates, and it contributes greatly to the robustness and practicality of the model. The method yields a fully explicit algebraic turbulence model for which no additional numerical iterations are required. Furthermore, the formulation is directly derived from the equations and does not contain any additional empirical assumptions, such as the ones considered in earlier work (Lazeroms *et al.*, 2013). Even though the current investigations only addressed parallel flows, we presented a systematic approach that might inspire similar derivations for more complex flow cases. Nevertheless, the model gives very satisfying results for parallel flows with buoyancy, which already makes it useful for a wide range of applications, including parametrizations of turbulence in the atmospheric boundary layer.

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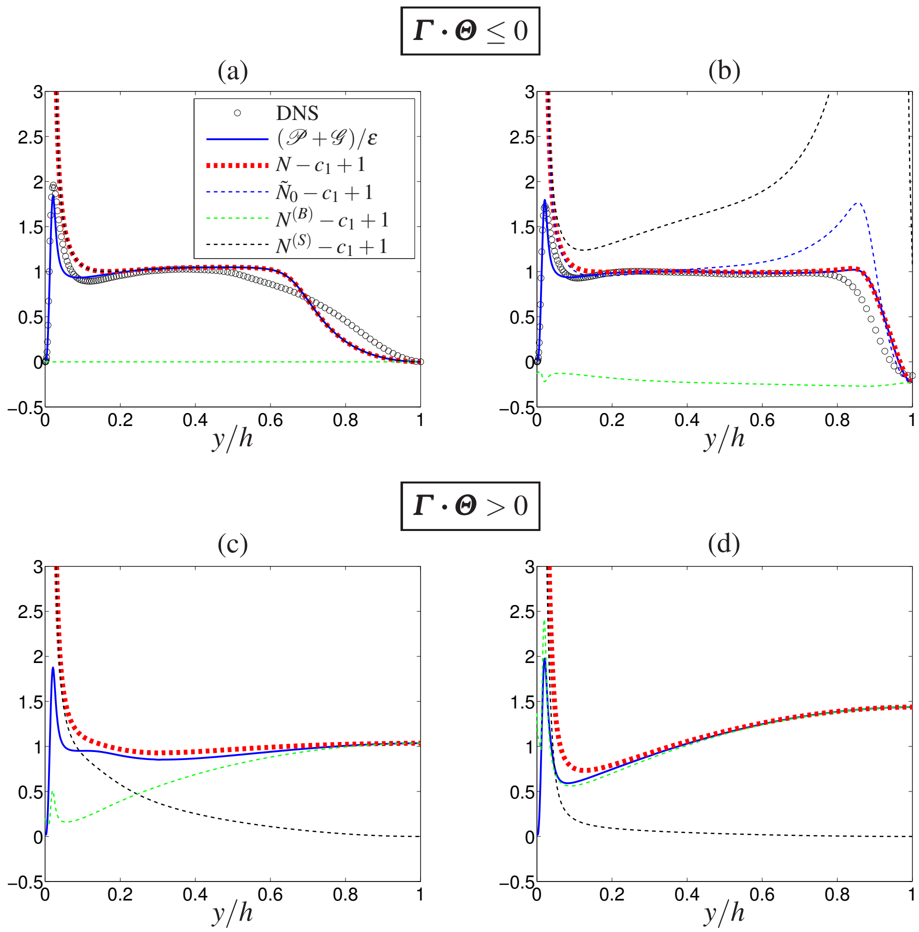

Figure 2. Comparison of $(\mathscr{P}+\mathscr{G})/\varepsilon$ resulting from the explicit algebraic model with $N - c_1 + 1$ from equation (8) and other approximation levels, in turbulent channel flow with $Re_\tau = 550$, $Pr = 0.7$ and (a) $Ri_\tau = 0$, (b) $Ri_\tau = 480$, (c) $Ri_\tau = -50$, (d) $Ri_\tau = -200$. Also shown are (circles) DNS by García-Villalba & del Álamo (2011) for the neutral and stable case.