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VORTICITY TRANSPORT: THE TRANSFER OF VISCOUS STRESS TO REYNOLDS STRESS IN TURBULENT CHANNEL FLOW

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1. Introduction

It is 100 years since Taylor (1915) first drew attention to the connection between the transport of vorticity and the Reynolds stress. In a plane, wall-bound shear flow such as Channel flow it is simply[1]

$$\frac{d}{dy}\left(-\overline{u'v'}\right) = \overline{v'\omega'_z} - \overline{w'\omega'_y} \qquad (1)$$

The Reynolds stress is zero at the wall and so it is the gradient in Reynolds stress that transfers the viscous stress to a Reynolds stress. From Eq. 1 it is inextricably linked to the transport of vorticity near the wall. In his extension of the vorticity transport theory to three dimensions, Taylor (1932) developed the correlations through a Lagrangian equation for vortex elements but, after having obtained an expression, commented "In general it is so complicated that it is of little practical use, but in certain special cases considerable simplifications may occur". His comment, in addition to the difficulties with experimental measurement, may explain why so little is known about these two transport terms in plane turbulent shear flows. Experiments identified many characteristic features of near wall flows, however, including for example, the early identification of 'streaks' of low forward momentum near the wall and a 'burst-sweep cycle' found by Kline, Reynolds and Schraub (1967) and various other 'characteristic' structural features as in Wallace, Eckelmann and Brodkey (1972), Blackwelder and Eckelmann (1979), Smith and Metzler (1983) and many others. A broad overview is provided by Wallace (2012). Coles (1978), for example, proposed a model for how the Reynolds stress arises from 'streamwise vortices' but at the time had no way of actually testing the ideas or of completing his 'cartoon', as he described it. The role of streamwise vortices became well-accepted. Nevertheless, experiments did not provide a detailed understanding of the vorticity transport very near the wall and therefore of the transfer mechanism from a viscous stress to a Reynolds stress.

High Reynolds number DNS calculations have no such limitations and transport quantities such as those in Equation (1) can be readily obtained. The early DNS study by Robinson (1991) identified 'coherent motions' in the turbulent boundary layer. Numerical experiments permit 'non-physical' experiments, which allow candidate Reynolds-stress-mechanism hypotheses to be tested, as in the results of Jimenez and Pinelli (1999). Jimenez and Moin (1991) considered the minimal Channel flow dimensions to sustain turbulent motion and, later, Schoppa and Hussein (1997, 2002) and Jimenez and Pinelli (1999) looked closely at the mechanisms which sustain near wall turbulence. Lee and Kim (2002) focused on the viscous sublayer in their study of drag reduction. In the following we revisit the very near wall transfer of viscous stress to Reynolds stress from Taylor's transport of vorticity perspective.

2. Vorticity Transport in Channel flow

For turbulent Channel flow, where the pressure gradient is equal to the gradient in y of the total stress, Eq. (1) leads to

$$\frac{1}{\rho}\frac{d\overline{p}_w}{dx} = \frac{d}{dy}\left(-\overline{u'v'} + \nu\frac{dU}{dy}\right)$$
$$= \overline{v'\omega'}_z - \overline{w'\omega'}_y - \nu\frac{d\overline{\omega}_z}{dy} = -\nu\left(\frac{d\overline{\omega}_z}{dy}\right)_{y=0,2h} \qquad (2)$$

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[1] In summation notation the vector identity

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{\omega}\times\boldsymbol{u}$$

is $u_k\frac{\partial u_i}{\partial x_k} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_k^2\right) + \varepsilon_{ijk}\omega_j u_k$ and since in incompressible fluid the mean and the fluctuation velocity vectors are both solenoidal then $\frac{\partial u'_k u'_i}{\partial x_k} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_k'^2\right) + \varepsilon_{ijk}\omega'_j u'_k$. For streamwise independent turbulent shear flows Eq. 1 then follows for i in the $\boldsymbol{x}$ direction (with the usual notation for Cartesian velocity and vorticity components and averages taken over time or over sufficiently large areas of $\boldsymbol{x}$ - $\boldsymbol{z}$ planes).

since $\left(\overline{v'\omega'_z} - \overline{w'\omega'_y}\right)_{y=0,2h} = 0$ (i.e. at both walls) and $\overline{\omega}_z = -\dfrac{dU}{dy}$. Thus, the total mean vorticity flux, $\overline{v'\omega'}_z - \overline{w'\omega'}_y - \nu \dfrac{d\overline{\omega}_z}{dy}$, is constant across every plane parallel to the wall.

The Reynolds stress is zero at the wall and on the centerline, so it necessarily has a maximum at some $y = y_m$. That is,

$$\frac{d}{dy}\left(-\overline{u'v'}\right)_{y=y_m} = 0 \qquad (3)$$

This implies from Eqs.(2) and (3) that at this location, $y = y_m$, as at the wall, <u>all the vorticity flux is carried by viscous diffusion</u> because the contribution from the turbulent motion is zero, i.e.

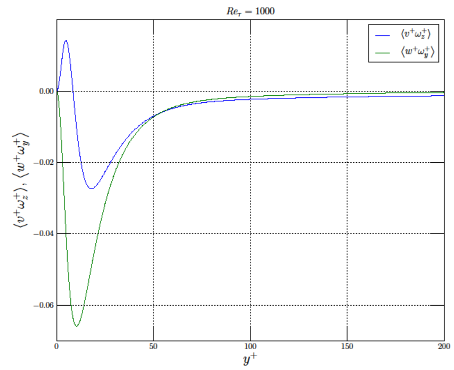
$$\left(\overline{v'\omega'}_z - \overline{w'\omega'}_y\right)_{y=y_m} = 0 \qquad (4)$$

and

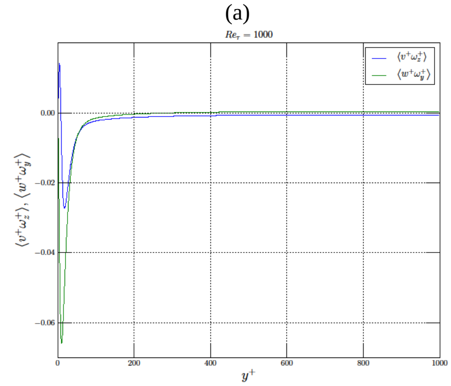
$$\left(-\nu \frac{d\overline{\omega}_z}{dy}\right)_{y=0,2h} = \left(-\nu \frac{d\overline{\omega}_z}{dy}\right)_{y=y_m}$$

Since $\overline{\omega}_z$ is far from zero at $y = y_m$ and $\overline{\omega}_z$ is not at a local maximum but is increasing (becoming less negative) with increasing y, $\overline{v'\omega'}_z$ can be expected to be negative at $y = y_m$ (as expected from Taylor's (1915) analysis, for example). Correspondingly, but somewhat surprisingly, since there is no mean $\overline{\omega}_y$ (or *W*), we anticipate that $\overline{w'\omega'_y}$ is also negative at $y = y_m$ since it is equal to $\overline{v'\omega'}_z$. Interestingly, for $y < y_m$ since $\dfrac{d}{dy}\left(-\overline{u'v'}\right) > 0$ it could then be expected (from Eq. (1)) that in this region, $\overline{w'\omega'_y}$ would be larger in magnitude than $\overline{v'\omega'}_z$, whereas for $y > y_m$, where $\dfrac{d}{dy}\left(-\overline{u'v'}\right) < 0$ then $\overline{v'\omega'}_z$ would be larger in magnitude than $\overline{w'\omega'_y}$!

These simple conclusions are supported by the values of these fluxes determined from DNS calculations for Channel flow (Moser et al (1999), Del Alamo et al (2004), Lee and Moser (2015)). For the particular case of $R_\tau = 1000$ the DNS results are shown in Fig.1, (a) and (b). They show the large negative value of $\overline{w'\omega'_y}$ near the wall, the value of y_m^+ of approximately 50 for this particular R_τ, and the change in relative magnitude of the two fluxes above and below y_m. The largest negative value of $\overline{w^+\omega_y^+}$, is -0.066, occuring at approximately $y^+ = 10$ (half of this peak value occurs at only $y^+ = 5$, approximately).



(a)



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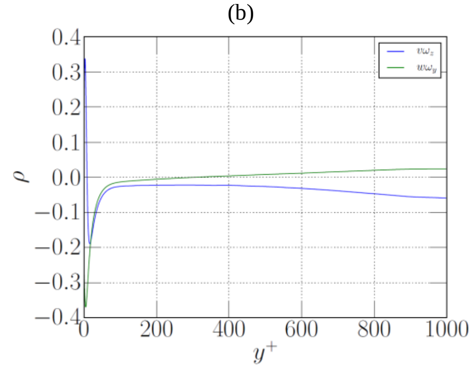

(c)

Fig. 1 The two vorticity flux components of $\dfrac{d}{dy}\left(-\overline{u'v'}\right)$ from Eq.1, $\overline{w'\omega'_y}$ and $\overline{v'\omega'_z}$ vs. y^+ (a) out to $y^+ = 200$ and (b) out to $y^+ = 1000$ (c) The correlation coefficient for the two vorticity flux components out to $y^+ = 1000$.

Fig. 1(c) shows the profile of the corresponding correlation coefficient for the two fluxes (the r.m.s. values of w' and ω'_y at the same y are used to non-dimensionalize the correlation). Note the large coefficient

(-0.37) very near the wall at $y^+ = 5$ for $\overline{w'\omega'_y}$ and the relatively very small coefficient for both fluxes far from the wall, particularly for $\overline{w'\omega'_y}$.[2]

The underlying turbulent structure responsible for the fluxes, is very different for $y << y_m$ from the structure much further from the wall $y >> y_m$. In particular, since both $-\nu \frac{d\overline{\omega}_z}{dy}$ and $\overline{v'\omega'_z}$ are negative then from Eq. (2), $\overline{v'\omega'_z}$ acts as a transport of mean vorticity in the same direction as the viscous diffusion, whereas $\overline{w'\omega'_y}$ acts to transport the mean vorticity in the opposite direction from viscous diffusion! [3]

The effect of the two fluxes and their respective magnitudes near the wall and far from the wall provide insight into a 'counter-gradient' transport of mean vorticity that occurs near the wall. That is, it explains why the mean vorticity near the wall is large but the net flux of mean vorticity perpendicular to the wall remains small and it is the central reason why the mean span-wise vorticity at the wall increases (i.e. the mean vorticity is redistributed back towards the wall in the absence of a mean free-stream pressure gradient) during transition in the boundary layer from laminar to turbulent flow (while farther from the wall the vorticity diffuses more rapidly away from the wall with a corresponding increase in boundary layer thickness).

Note, in passing that, if the Law of the Wall is assumed, $U = u_\tau f\left(y^+\right)$, then since

$$-\frac{1}{\rho}\frac{d\overline{p}_w}{dx} = \frac{u_\tau^2}{h} = \nu\left[\frac{d\overline{\omega}_z}{dy}\right]_{y=y_m} = -\nu\left[\frac{d^2U}{d^2y}\right]_{y=y_m} \qquad (5)$$

$$f''\left(y_m^+\right) = -\frac{1}{R_\tau} \qquad (6)$$

Assuming, tentatively, that the Log Law, $U^+ = f\left(y^+\right) = 1/\kappa \ln\left(y^+\right) + C$, is crudely representative of the mean velocity profile at y_m^+ then

$f''\left(y_m^+\right) = -\frac{1}{\kappa y_m^{+2}}$ so that

$$y_m^+ = \sqrt{\frac{R_\tau}{\kappa}} \quad \text{and} \quad \frac{y_m}{h} = \sqrt{\frac{1}{\kappa R_\tau}} \qquad (7)$$

Thus, assuming the Law of the Wall applies in the region where $y = y_m$, it is interesting that the location of y_m^+ does not become independent of Reynolds number (the ratio of the outer and inner scales).[4] From the further assumption of a Log Law in this region then, for a fixed pressure gradient (constant u_τ for a given h), as $\nu \to 0$, y_m^+ increases as $\nu^{-1/2}$ while $y_m/h \to 0$ as $\nu^{1/2}$.

The simple prediction from Eq. (7) can be compared with the numerical results at $R_\tau = 180$, 550, 1000 and 5186 for which the calculation gives $y_m^+ = 31.7$, 44.2, 53.4 and 104.6, respectively, and for which Eq. (7) gives 21.6, 37.8, 51.0 and 116.2 assuming $\kappa = .384$ (Lee and Moser (2015)). Only at $R_\tau = 5186$ do the numerical results show a significant region of the velocity profile ($y^+ \geq 300$) which is accurately logarithmic. The accuracy of these simple estimates of y_m^+ is therefore limited but the trend with R_τ is well captured.

3. The probability density function for $w^+\omega_y^+$.

Fig. 2(a) shows the probability density function for $w^+\omega_y^+$, i.e. $P(w^+\omega_y^+)$, at $y^+ = 5$ for three different Reynolds numbers. Fig. 2(b) shows a 'Cumulative Integral', defined as $-\int_\infty^{w^+\omega_y^+} \tau P(\tau) d\tau$), which illustrates the contribution of the large negative values of $w^+\omega_y^+$ to the mean value. Several important points can be made. The similarity of the pdf for $w^+\omega_y^+$ at three different values of R_τ (including the very low value of 180) is remarkable. This supports an underlying basis for a common mechanics and, correspondingly, the Law of the Wall very close to the wall. The pdf has a peak value at zero and a pronounced asymmetry. Since the mean value at $R_\tau = 1000$ is -0.041 (the value of the Cumulative Integral as $w^+w_y^+$ ® - ¥) it is remarkable that the

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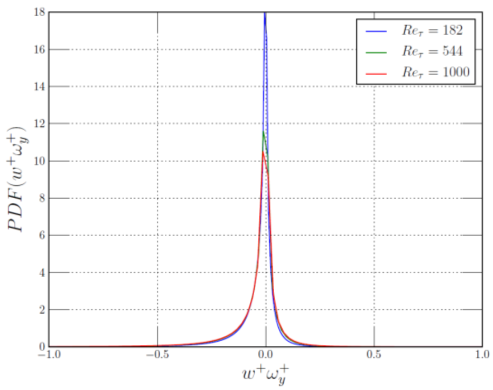
[2] Note also the large <u>positive</u> coefficient ($+0.33$) at only $y^+ = 2.5$ for $\overline{v'\omega'_z}$ (i.e. very close to the wall the flux of spanwise vorticity is positive (back towards the wall). As discussed below this too has the same origin as the high correlation for $\overline{w'\omega'_y}$.)

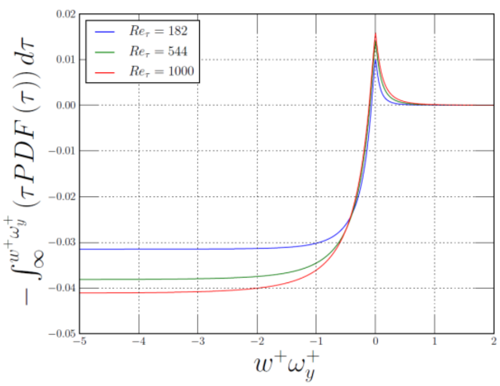
[3] An alternative description in terms of the 'vortex force', $\boldsymbol{u} \times \boldsymbol{\omega}$, is that the vortex force $\boldsymbol{v} \times \boldsymbol{\omega}_z$ points downstream (so that a negative value points upstream and acts to slow the fluid down (oppose the pressure gradient force)) while $\boldsymbol{w} \times \boldsymbol{\omega}_y$ points upstream and a negative value points downstream and it requires a large gradient in viscous stress to balance the sum of this component of the vortex force and the pressure force which both act in the same direction.

[4] This dependence of y_m on R_t for channel flow also leads, as expected, to - $r\overline{u¢v¢}(y_m)$ ® τ_w as $R_\tau \to \infty$.

Cumulative Integral does not reach 90% of this asymptotic value (i.e -.037) until $w^+ w_y^+ < -1.0$, approximately, which is nearly 25 times larger than the mean value!



(a)



(b)

Fig. 2 (a) The Probability Density Function, i.e. $P(w^+\omega_y^+)$ at $y^+ = 5$ and (b) A 'Cumulative Integral' from the pdf of $w^+\omega_y^+$ at $y^+ = 5$

The distribution is very far from Gaussian! It seems that the mean is determined by negative (and weaker positive) 'events' which not infrequently are relatively very large compared with the mean and that in each event the negative value substantially exceeds the positive.

This statistical result demands a physical explanation.

The mechanics can be readily inferred from detailed views of the velocity field and vorticity field on $y-z$ planes through regions of large negative values of $w^+\omega_y^+$ ($w^+\omega_y^+ \leq -1.0$) on this plane. Both from individual realizations and from statistical correlations the mechanics results from the central role played by streamwise vorticity. Local concentrations of streamwise vorticity give rise to a compatible velocity field (qualitatively arising from a local application of the Biot Savart relation as if the flow were 2D) which lifts up and pushes down spanwise vorticity, which by the tilting mechanism gives rise to wall normal vorticity. This then correlates with the spanwise component of velocity (w) which also arises from the streamwise vorticity (hence the high correlation and correlation coefficient). The asymmetry in the correlation (larger negative values than positive) arises because the lift up away from the wall is larger than the push down towards the wall (due to the image vorticity for zero velocity at the wall) and for the same sign of w the value of $w\omega_y$ is therefore larger on the side of ω_x which lifts spanwise vorticity away from the wall. An example of a characteristic 'event' is shown in Fig 3. It shows the streamwise vorticity, the velocity vectors in the plane, the spanwise velocity, the wall normal vorticity and contours of $w\omega_y$. The role of the streamwise vorticity in producing the large correlation at $y^+ = 5$ is plain.

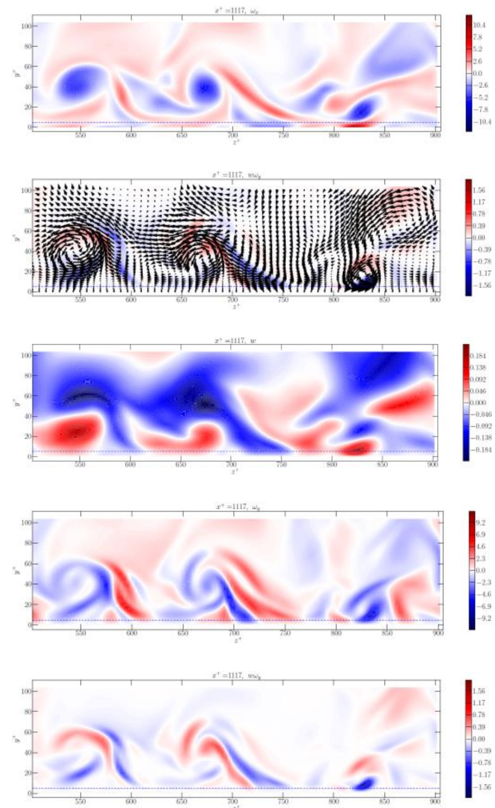


Fig. 3. From top to bottom are contours at a given time on a y-z plane, 500 wall units wide by 100 high of (a) streamwise vorticity (b) the velocity vectors on the plane (c) spanwise velocity (d) wall normal vorticity (e) $w^+\omega_y^+$. The line $y^+ = 5$ on the plane is marked.

The plot of the wall normal vorticity shows how vortex lines are wrapped around the structure and contribute wall normal vorticity as well as streamwise vorticity and, of course, spanwise vorticity. The resulting structure is therefore oblique to both the wall and to the streamwise direction. Fig 4 shows the structure made evident by a contour plot of λ_2 (Hussein and Jeong 1997).

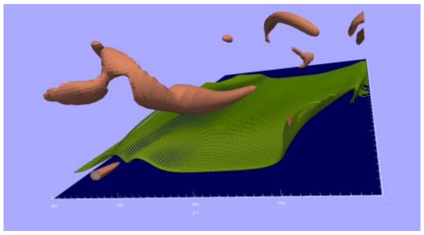


Fig 4 An isometric view of the vortical structure with vortex lines passing through $y^+ = 10$, $z^+ = 900$.

The origin of this streamwise vorticity is not yet certain (one can, however, see a nascent 'offspring' in Fig 4) but once initiated with sufficient amplitude there is a process of 'self-amplification'. The circulation in a y-z plane tends to increase with downstream distance through the helical wrapping of vortex lines. Additional streamwise vorticity increases this circulation, which will tend to wrap more lines etc. The structure lifts away from the wall because of the addition of wall normal vorticity. The 'Streak Transient Growth' mechanism described by Schoppa and Hussein (2002) appears to be this process.

The footprint of the structures that give rise to large values of $w^+\omega_y^+$ near the wall can be seen from the diffusion of the square of the streamwise vorticty at the wall, as shown in Fig 5. One can see the oblique angle with respect to the streamwise direction, the relatively sparsely spaced large events, and also the occurence of events that appear to be spatially related in some way to neighboring events.

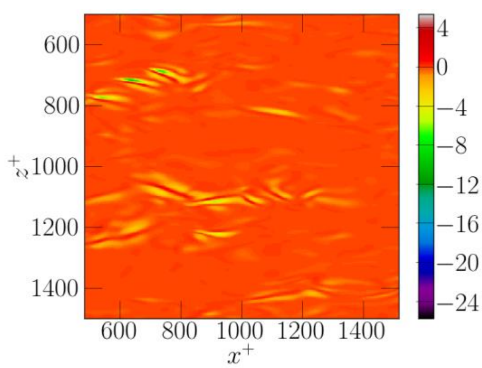


Fig 5 Contour plot at the wall of the 'viscous transport' $\nu\left[\partial\left(\frac{1}{2}\omega_x'^2\right)\Big/\partial y\right]_{y=0}$ at a given time, $500 \le x^+ \le 1500$,. $500 \le z^+ \le 1500$, $R_\tau = 180$

4. Summary and some Concluding Remarks

The DNS results over a range of Reynolds numbers for turbulent channel flow have provided a unique opportunity to consider the vorticity flux terms that account for the gradient in Reynolds stress. Taylor (1915 and 1932) drew attention to these vorticity flux terms. In 1915 he provided a 2-D model for the correlation $\overline{v'\omega_z'}$ and his attempt in 1932 to include the other term led him to comment subsequently on the 'intractability of the equations' and the need for further assumptions with a view to simplification. Of course these fluxes near the wall could not be measured. Simple arguments in this paper show that it is $\overline{w'\omega_y'}$ that is the dominant term near the wall. The DNS results provide, for the first time, details of these vorticity fluxes over a range of Reynolds numbers. Close to the wall the transfer from a viscous stress at the wall to a Reynolds stress results from the gradient in Reynolds stress and therefore the behavior near the wall of $\overline{w^+\omega_y^+}$. It has a maximum value at $y^+ = 10$, and even at $y^+ = 5$, the correlation is substantial; it is found to have statistical characteristics which reflect a contribution from 'large events'. For example, at $R_\tau = 1000$ and $y^+ = 5$ the mean value is .041, but approximately 10% of the contribution to the mean value comes from negative values of the correlation that are more than 25 times larger than the mean value. The focus has been on the vorticity field and the mechanism that accounts for the vorticity transport (flux) close to the wall. The essential role of streamwise vorticity, its origin and development have been discussed.

As has been recognized, once the transfer of viscous stress to Reynolds stress is understood, ways to affect it, as in the case of the Thom's effect (the reduction in shear stress at moderate Reynolds number with the addition of a relatively small concentration of polymer) or re-laminarization through a pressure gradient or other future strategies, can be anticipated.

The same mechanisms are expected near the wall for other wall-bound turbulent shear flows.
At high Reynolds number, for example, is there some 'cancellation', far from the wall, of the effects of positive and negative streamwise vortical structures but a remaining ω_z component from the 'heads' of the near wall structures? This component is only cancelled from sources on the opposite wall for channel flow (and not cancelled in the boundary layer). Is it connected with the dominance of the $\overline{v'\omega_z'}$ flux far from the wall and the emergence in a boundary layer of a large scale structure?

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