International Symposium
On Turbulence and Shear Flow
Phenomena (TSFP-9)
June 30 - July 3, 2015 Melbourne, Australia

LINEAR STABILITY OF THE FLOW IN A TOROIDAL PIPE

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ABSTRACT

While hydrodynamic stability and transition to turbulence in straight pipes — being one of the most fundamental problems in fluid mechanics — has been studied extensively, the stability of curved pipes has received less attention. In the present work, the first (linear) instability of the canonical flow inside a toroidal pipe is investigated as a first step in the study of the related laminar-turbulent transition process. The impact of the curvature of the pipe, in the range $\delta \in [0.002, 1]$, on the stability properties of the flow is studied in the framework of linear stability analysis. Results show that the flow is indeed modally unstable for all curvatures investigated and that the wave number corresponding to the critical mode depends on the curvature, as do several other features of this problem. The critical modes are mainly located in the region of the Dean vortices, and are characterised by oscillations which are symmetric or anti-symmetric as a function of the curvature. The neutral curve associated with the first bifurcation is the result of a complex interaction between isolated modes and branches composed by several modes characterised by a common structure. This behaviour is in obvious contrast to that of straight pipes, which are linearly stable for all Reynolds numbers.

INTRODUCTION

The study of the flow in curved pipes has been the subject of several papers over the last decades: theoretical (Dean, 1927), experimental (Ito, 1987; Kühnen *et al.*, 2014) and numerical (Di Piazza & Ciofalo, 2011; Noorani *et al.*, 2013), results have been presented however, a thorough analysis of the causes and mechanisms behind hydrodynamic instability and transition to turbulence in this flow is still missing. The technical relevance of these flow cases is, in fact, apparent from their prevalence in industrial applications: bent pipes are used in heat exchangers, exhausts, links between straight pipe sections and other devices, while the toroidal pipe is, for example, found in nuclear reactors and tires. For a comprehensive review of the applications, see Vashisth *et al.* (2008).

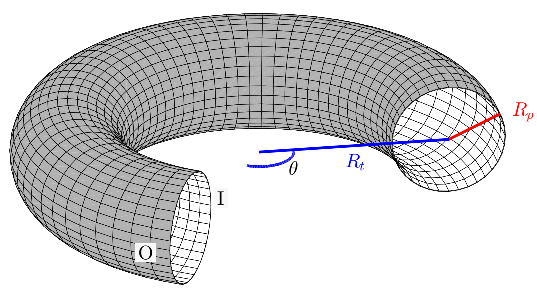

Figure 1. Geometry of the torus, showing the parameters involved in the definition of the curvature $\delta = R_p/R_t$.

As a first step in the investigation of the stability of the flow inside bent pipes, we focus on an idealised toroidal setup. This shape, albeit rarely encountered in industrial applications, is representative of a canonical flow and its study is relevant in the context of the research on the onset of turbulence since it deviates from the thoroughly studied straight pipe by the addition of one parameter only. Moreover, the toroidal pipe constitutes the common asymptotic limit of two important flow cases: the curved pipe and the helical pipe.

The torus, represented schematically in figure 1, is characterised by a single geometrical parameter: the curvature. The curvature is defined as the ratio between the radius of the pipe and that of the torus, *i.e.* $\delta = R_p/R_t$; this and the Reynolds number (*Re*) are the only two parameters defining this flow. The advantage of investigating a flow governed only by these two parameters is that it allows to isolate the effect that the curvature has on the onset of the instability. It will therefore be possible, when studying helical pipes, to discern the flow features produced by the curvature from those produced by the torsion of the helix.

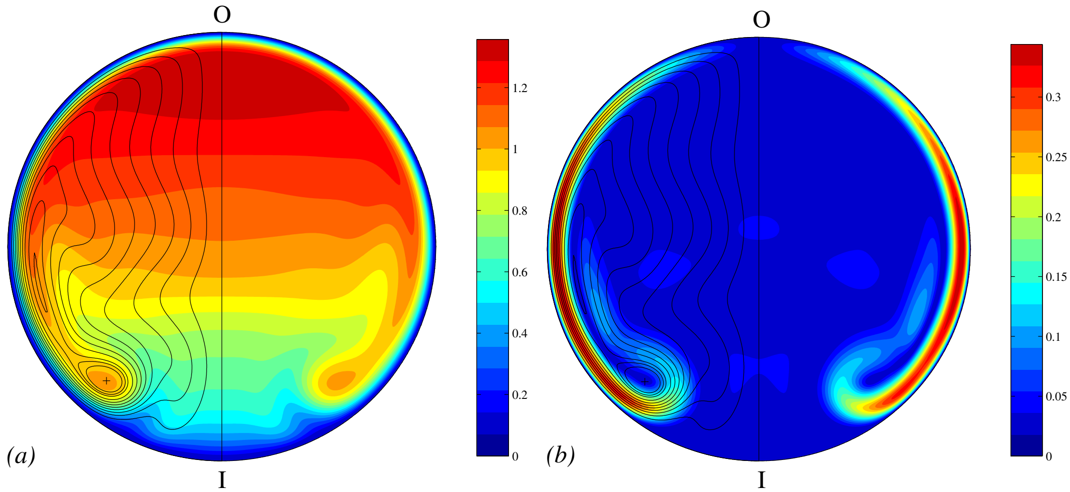

Figure 2. Marginally stable base flow for $\delta = 0.3$, $Re = 3379$. *(a)* Streamwise velocity component and iso-contours of in-plane streamfunction. *(b)* In-plane velocity magnitude and iso-contours of streamfunction. The symbol "$+$" indicates the location of the maximum of the in-plane streamfunction, corresponding to the centre of the Dean vortex.

BASE FLOW

In order to determine the stability properties of the toroidal pipe flow, we investigate the growth of infinitesimal disturbances around a basic state. This base flow, *i.e.* the solution to the steady, incompressible Navier–Stokes equations, has been computed by means of an in-house developed nonlinear solver based on the finite element method (FEM). The solver employs Newton's method to solve the fluid equations and the secant method to impose a fixed bulk velocity in the pipe. A zeroth order continuation method has been employed to explore the parameter space, allowing to use as initial guess for Newton's method a solution computed for a different set of the parameters. This choice, besides reducing the number of iterations needed for convergence, allows the computation of the solutions at high Re, where a time stepping approach would require additional stabilisation to converge to a steady state (Åkervik *et al.*, 2006) and more computational resources.

The considered base flow is invariant with respect to the axial direction, *i.e.* along θ, not because of simplifications but because the steady state solution inherits the rotational symmetry of the torus, a property which has also been confirmed by three-dimensional direct numerical simulation (DNS). For the same reason the base flow is also symmetric with respect to the equatorial plane of the torus, *i.e.* the plane containing the "I/O" labels in the figures, but the computations have been made on a fully circular mesh to allow the existence of non-symmetrical eigenmodes. The flow is maintained in motion by a constant volume force acting in the axial direction and defined by $f = F/R$, where F is a constant determined in order to have unitary bulk velocity, and R is the distance from the centre of the torus. The $1/R$ dependence is needed in order to obtain constant forcing along the axial direction, θ, and thus avoid artificial radial pressure gradients. As mentioned, the secant method has been employed to compute F; this is because the relationship between the force, f, and the average axial velocity inside the torus is in general nonlinear and unknown a-priori.

The base flow is characterised, as first discovered by Dean (1927) using a first-order expansion valid for low δ, by the presence of two counter-rotating vortices, so-called Dean vortices in his honour. These two primary vortices are present at *every* Re and for *any* value of δ (different from zero), and are located symmetrically with respect to the equatorial plane of the torus. The shape of the vortices and the position of their centres depend on both Re and δ. With increasing Re the core region of these vortices becomes increasingly elongated until it reaches a point where it splits into two drop-shaped vortices rotating in the same direction and linked by a critical point in the in-plane streamfunction field. The "splitting" of the Dean vortices takes place before the onset of the linear instability for $0.025 \leq \delta \leq 0.9$ but it is, conversely, delayed after the loss of stability of the steady state solution for very low and very high curvatures. An example base flow with four vortices is depicted in figure 2; note that this base flow is in fact stable, and as such observable in reality. The fact that the splitting of the Dean vortices does not precede the loss of stability of the base flow for all curvatures is a particularly important observation, since it excludes the possibility that the first bifurcation may be due or somehow related to this phenomenon.

The splitting of the vortices is not the only feature of the base flow that depends significantly on the curvature. When Dean first analysed this flow in 1927 he employed an approximation valid for low curvatures in order to analytically solve the Navier–Stokes equations. By using this approximation he obtained a single non-dimensional parameter, defined as $K = 2\delta Re^2$, which was used to fully characterise the problem, and was later called Dean number. By extending the study to non-negligible curvatures it was observed that, even when comparing base flows at constant Dean number, the curvature plays a primary role. In fact, starting at least from $\delta = 0.25$, the peak of streamwise velocity sensibly starts moving from the outer region of the torus towards the core of Dean's vortices, highlighted by pluses in figure 2, and it then jumps at the cores for $\delta \approx 0.5$. This is not the only feature of the base flow that is affected by the curvature: as a further example the magnitude of the in-plane velocity components, the so-called secondary flow (see *e.g.* figure 2*b*), is also greatly affected, increasing from less than 10% of the bulk velocity for $\delta = 0.01$ to nearly 80% for $\delta = 1$. Other characteristics of the base flow which are not possible to describe by a single scaling parameter include, but are not limited to, the position of the core of the vortices, their aforementioned splitting, and the structure of the secondary flow. A thorough description of these features, though, goes beyond the scope of this paper. The examples given will suffice to show the need to treat the curvature as a parameter with the same relevance as the Reynolds number, an importance which is even more evident in the stability analysis.

The study of the possible presence of multiple steady state solutions presenting a different number of vortices, following Yang & Keller (1986) and Daskopoulos & Lenhoff (1989), is underway, but it appears not to be linked to the stability properties of the currently observed steady state solution.

MODAL STABILITY ANALYSIS

To carry out the modal stability analysis, the Navier–Stokes equations have been linearised in the neighbourhood of the base flow, and the perturbation fields have been expanded in a Fourier series along the homogeneous θ direction, introducing the wave number α. Wave numbers between zero and two hundred have been investigated so far, with no necessity to analyse negative wave numbers since their spectra are the complex conjugate of the respective positive wave numbers. Note that only integer wave numbers have been studied due to the fact that, differently from an infinitely long helix, the ideal toroidal geometry does not allow for solutions in the form of waves with non-integer wave number.

A first calculation of the eigenvalues associated with the linearised Navier–Stokes equations for $\delta = 0.3$ and $Re = 3379$ revealed the presence of a pair of complex-conjugate, marginally stable eigenvalues. This pair of eigenvalues is characterised by a wave number $\alpha = 7$, temporal frequency $|\omega_r| = 2.82$, and indicates that the flow is undergoing a Hopf bifurcation. Figure 3 shows the eigenvalue spectrum computed for this curvature and for a slightly higher value of the Reynolds number; the two critical eigenvalues, which have now become unstable, are highlighted by arrows. From figure 3 it is possible to observe that other eigenmodes, corresponding to different wave numbers, are close to the unstable semi-plane. In particular, another pair of modes associated with $\alpha = 8$ and $|\omega_r| = 3.36$ has nearly crossed the real axis and a third pair, with $\alpha = 6$, $|\omega_r| = 2.29$, is not far from becoming unstable as well. This observation will be further explained in the following.

Figure 4*a* depicts the axial velocity component of the real part of the direct eigenmode corresponding to the marginally stable eigenvalue highlighted in figure 3. It can be observed that this eigenfunction reaches its extremum in the proximity of the two principal Dean vortices and its intensity decays in the rest of the domain. This is in accordance with the fact that, as observed in experiments and DNSs (see *e.g.* Kühnen *et al.*, 2014; Di Piazza & Ciofalo, 2011), the first instability in bent pipes manifests itself by oscillations of the two vortices. Figure 4*a* also shows that the unstable mode is anti-symmetric with respect to the equatorial plane of the torus. This feature implies that this is a symmetry-breaking mode and that the flow for $\delta = 0.3$ and $Re > 3379$ is characterised by anti-symmetric oscillations. The anti-symmetry of this eigenfunction is in general accordance with the observations by Di Piazza & Ciofalo (2011), although their simulations presented an unstable, periodic flow only for a higher Re and for a different wave number.

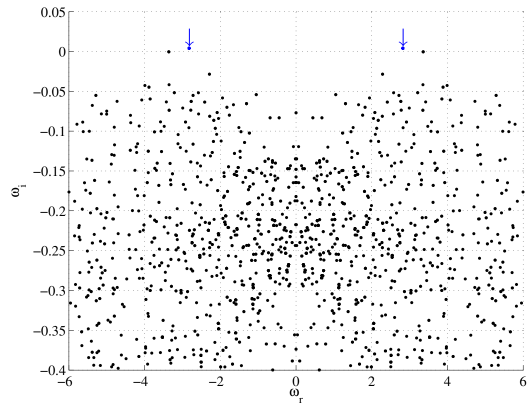

Figure 3. Spectrum for $\delta = 0.3$, $Re = 3400$. The pair of critical modes is highlighted in blue and by arrows is characterised wave number $\alpha = 7$ and belongs to F3A (cf. figure 7).

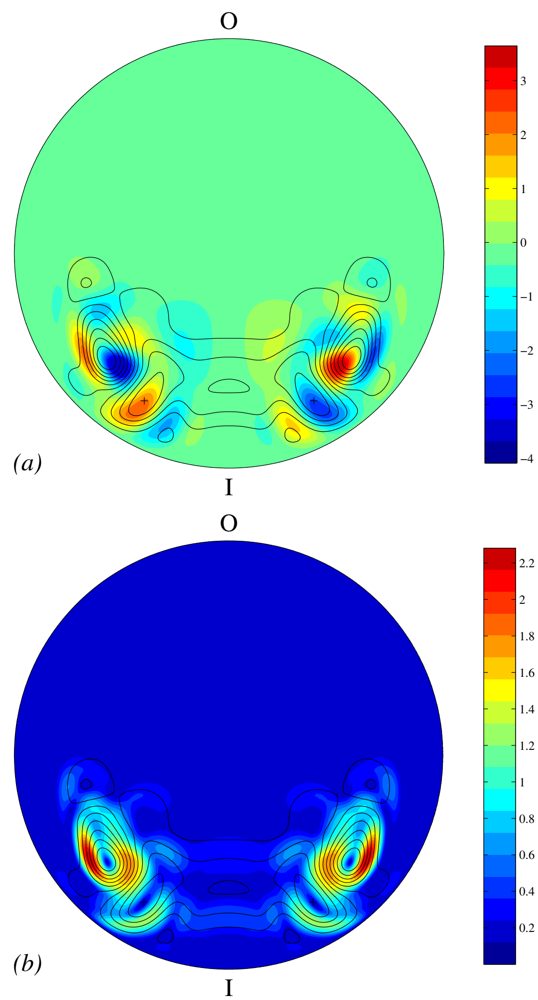

Figure 4. Axial velocity component *(a)* and in-plane velocity magnitude *(b)*, with superposed iso-contours of in-plane streamfunction, of the real part of the marginally stable eigenvector for $\delta = 0.3$, $Re = 3379$. A different representation of this eigenfunction is presented in figure 8.

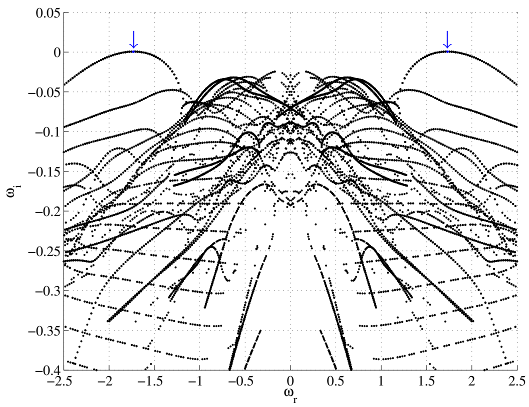

Figure 5. Spectrum for $\delta = 0.01$, $Re = 4300$. The pair of critical modes is highlighted in blue and by arrows and has wave number $\alpha = 74$. It is part of the branch that constitutes F4S (cf. figure 7), a family of symmetric modes. The wave number monotonically grows along the branch with increasing $|\omega_r|$.

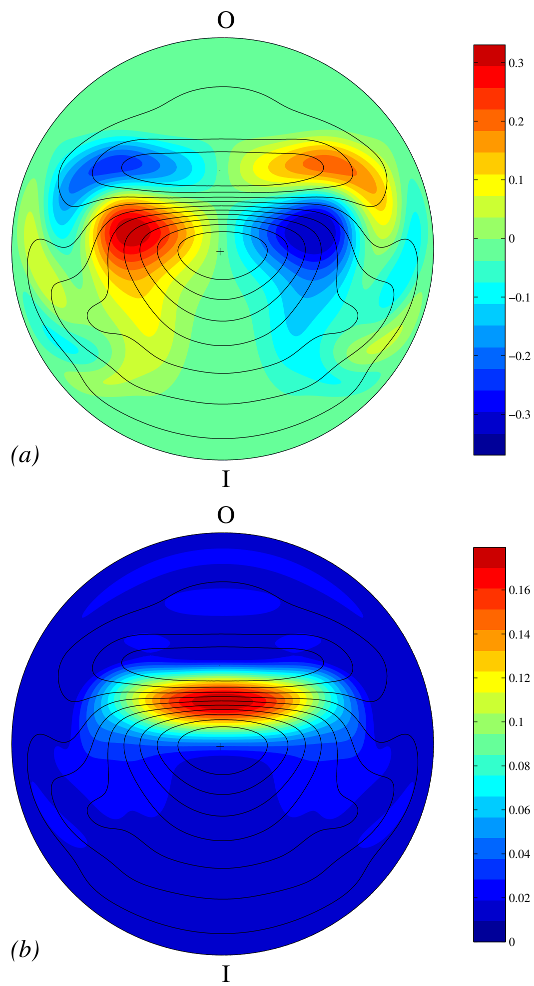

Figure 6. Streamwise velocity component *(a)* and in-plane velocity magnitude *(b)*, with superposed iso-contours of in-plane streamfunction, of the real part of the marginally stable eigenvector for $\delta = 0.01$, $Re = 4257$.

More specifically, Di Piazza & Ciofalo (2011) presented DNSs for two values of curvature: 0.1 and 0.3. Their results for $\delta = 0.3$ indicate a change from steady to periodic flow for $Re \approx 4575$, a value 35% higher than our results show; moreover, in their simulation the flow exibits a travelling wave characterised by wave number 8, in contrast to the present $\alpha = 7$. For $\delta = 0.1$ they observe a periodic flow for $Re > 3800$ with wave number 13 but only when starting from a quasi-periodic flow for higher Reynolds number and slowly decreasing Re (they observe a hysteresis which makes it impossible to reach the periodic regime when starting from low Re). This is, again, in contrast with our results that present a periodic flow, with no hysteresis, for $Re > 3331$ and $\alpha = 17$. The only feature in common between the two sets of results is the symmetry characteristic of the oscillating flow in the periodic regime.

The analysis carried out at this first point in the parameter space shows that the flow is in fact *modally unstable* and suggested the study of other values of δ to completely characterise the instability. Figure 7 summarises the results by presenting the neutral curve in the parameter space. In the figure, each line corresponds to the neutral curve associated to a single pair of complex conjugate eigenmodes, tracked in parameter space with a zeroth-order continuation algorithm. The complete neutral curve for the flow is thus formed by the lower envelope of the lines. The flow turns out to be modally unstable for all curvatures investigated, and several eigenmodes contribute to the instability. The analysis of the critical modes revealed that three isolated modes, with corresponding neutral curves depicted with green lines in figure 7*a*, and five families of modes, depicted with black and blue lines in figure 7*a*, constitute the complete neutral curve for this flow (in the range of curvatures investigated). The rightmost isolated mode, the critical one for $\delta \geq 0.98$, is the only mode characterised by $\alpha = 0$, *i.e.* it consists of a stationary, pulsating mode, with a non-zero temporal frequency. The unique characteristics of this mode can be (easily) attributed to the singular geometrical condition encountered by the toroidal pipe when approaching unitary curvature. When lowering δ the wave number of the critical mode increases non-monotonically, as it is possible to observe from figure 7*b*, exceeding $\alpha = 118$ for $\delta \leq 0.005$, and, correspondingly, the eigenvector represents a travelling wave. The frequency of the critical mode also changes with δ and so does the topology of the spectrum as it is possible to observe by comparing figures 3 and 5.

Out of the five families of modes, three are formed by anti-symmetric eigenfunctions, and are indicated with FxA in figure 7*a*, and two families contain symmetric modes, indicated with FxS. The three isolated modes are all anti-symmetric. The symmetry characteristic of the critical mode, and its relationship with the curvature, could be of importance when considering the mixing of fluid inside the pipe: an anti-symmetric mode would be more efficient than a symmetric mode. It remains to be seen how this property of the eigenfunctions is affected by torsion, when considering helicoidal pipes. Modes belonging to the same family are characterised by monotonically increasing wave number with decreasing curvature, but step jumps are present at the boundaries between neighbouring families and between families and isolated modes (highlighted by red dashed lines in figure 7*b*). Continuous, if not for the fact that only discrete wave numbers have been employed in the present analysis, branches of eigenvalues are present for low values of δ, as observable in figure 5 (here "continuous branches" should

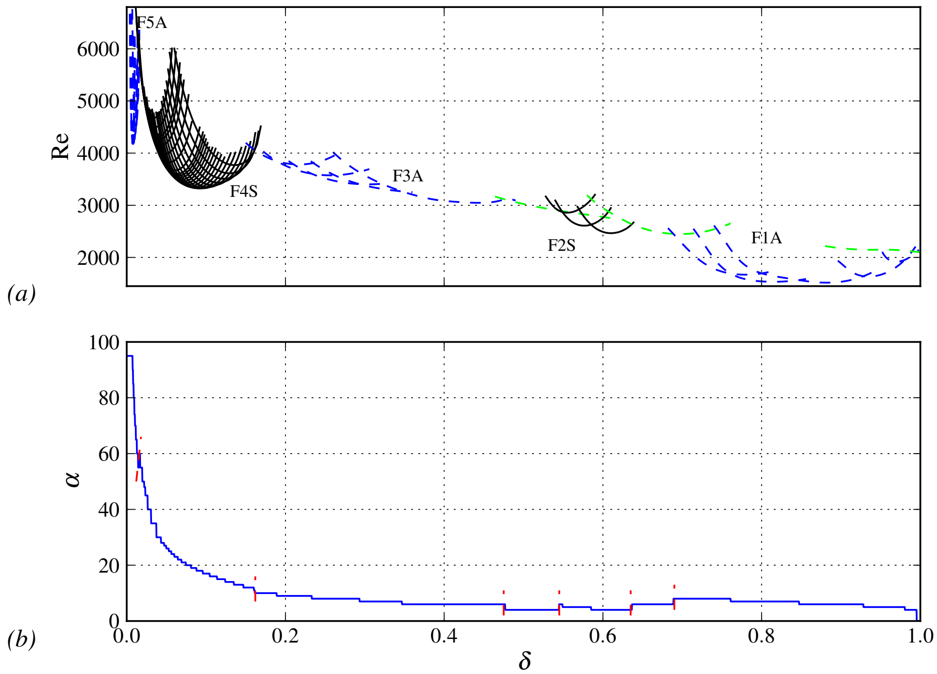

Figure 7. *(a)* Neutral curve of linear stability for the bent pipe in the $\delta - Re$ plane. Each line represents the neutral curve associated with a different eigenmode, the complete neutral curve for this flow is thus formed by the envelope of the lines. Continuous lines correspond to symmetric modes while dashed lines indicate anti-symmetric modes. The curves are not interpolated, *i.e.* they are drawn employing straight segments connecting computed solutions. *(b)* Envelope of the wave numbers associated with the modes forming the neutral curve. The dashed (red) lines mark the boundaries between the different families of modes.

not be intended as branches which correspond to spatially unbounded eigenfunctions, as encountered in boundary layers). When increasing δ these branches are "lost" amongst the other eigenvalues constituting the spectrum, and isolated pairs of complex conjugate eigenvalues substitute them as critical modes.

In order to verify the correctness and relevance of the results, a series of nonlinear DNSs have been performed for values of the parameters just above critical. The spectral element code nek5000 (Fischer *et al.*, 2008) has been chosen for this purpose; employing a fully three-dimensional mesh of the whole torus, as to allow for travelling waves of any wave number, and the flow has been driven at constant mass flux, as in the setup for the linear stability analysis. At least one DNS per isolated mode and per family has been run with the exception, at the moment of writing, of family F5A. First of all, the DNSs have, with excellent agreement, confirmed the values of curvature and Reynolds number at which the instability takes place. Moreover, the DNSs have also shown that the nonlinear, unstable flow is affected by periodic oscillations caused by a travelling wave with wave number, temporal frequency and symmetry properties corresponding to those of the critical modes resulting from the linear stability analysis.

CONCLUSIONS

The present results rigorously show, for the first time, that the flow in a bent pipe is indeed modally unstable, as opposed to straight pipe flow. In fact, the linear stability analysis reveals that the flow undergoes a Hopf bifurcation in the whole range of investigated curvatures, and settles onto a periodic regime. The comparison between the linear stability analysis and the set of nonlinear DNSs aimed at validating it, shows the reliability of the obtained results and the relevance of the linearly-critical modes for the nonlinear flow. The computed neutral curve highlights the complexity of the first instability of this flow, not only showing the non-triviality of the relationship between the curvature and the Reynolds number, but also revealing a complex picture between the identity of the critical mode, its wave number and its symmetry properties. The computed results reveal a new picture of the dynamics of the incompressible flow inside bent pipes, but do not conclude the investigation. As a first consideration, the link between a straight pipe and a toroidal one with vanishing curvature has to be better understood; moreover, now that the influence of the curvature on the flow is clear, it is possible to investigate helical pipes, adding the pitch of the helix as a third parameter.

ACKNOWLEDGMENTS

This work is funded by The Swedish Research Council Grant No. 621-2013-5788. Jing Gong at PDC-HPC is acknowledged for assistance concerning technical aspects in making the in-house developed code run on the PDC-HPC resources.

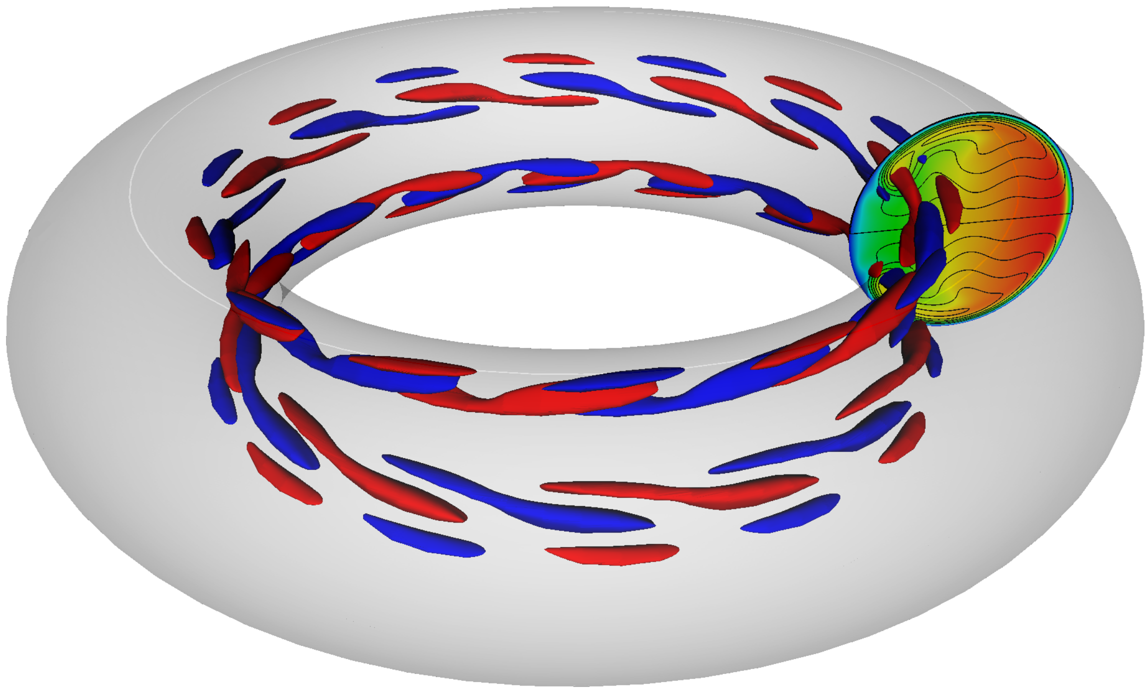

Figure 8. Eigenfunction corresponding to the marginally stable mode for $\delta = 0.3$, $Re = 3379$, $\alpha = 7$, belonging to F3A (cf. figures 3, 4 and 7). On the pipe section are plotted the streamwise velocity component and contours of the in-plane streamfunction of the corresponding base flow (cf. figure 2*a*). In red and blue isocontours of the radial velocity component of the eigenfunction (for two opposite values of magnitude). It is possible to observe that the oscillations induced by this mode are mainly located in the vicinity of the cores of the Dean vortices and that this mode is anti-symmetric with respect to the equatorial plane of the torus.

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