HYDRODYNAMIC STABILITY ANALYSIS OF SHEARED CONVECTIVE BOUNDARY LAYER FLOWS IN STRATIFIED ENVIRONMENTS

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ABSTRACT
Hydrodynamic stability analysis is carried out on sheared convective boundary layer (SCBL) flow, in which both sheared stratified flow and thermally convective flow coexist. The linear unstable stratification for the thermal convective flow region is included in the Taylor-Goldstein equations in terms of an unstable factor \( J_b \) in parallel with a stable stratification factor \( J \). New unstable regions corresponding to the thermally convective instability are found above the pure sheared stratified flow regions in the wavenumber \( \tilde{\alpha} \) versus stratification factor \( J \) plane. As the stratification ratio \( J_b/J \) increases, the unstable regions of thermal convection gradually approach and dominate the shear stratified unstable regions. The transition from shear stratified unstable mode to thermal convection unstable mode is also observed in the temporal growth rate \( \tilde{\sigma} \) versus \( J \) plane and in the \( \tilde{\sigma} \) versus \( \tilde{\alpha} \) plane. The eigenfunctions of buoyancy, vertical velocity and buoyancy flux perturbations for sheared stratified dominant mode, the transitional mode and the thermal convection dominant mode in SCBL flow configuration are also discussed.

INTRODUCTION
As a representative geographical flow, sheared stratified (SS) flow in which sheared flow motions occur in a stratified environment has received substantial studies in the past decades. However, in the geographical settings such as the planetary or oceanic boundary layers and engineering settings involving heat transfer from radiation or chemical reaction, the comparable thermally convective flows inevitably coexist with sheared stratified flow, together forming a complex flow configuration, namely the sheared convective boundary layer (SCBL). SCBL flows have significant importance for environmental issues, such as the mixing process of pollutants in atmospheric boundary layers, the heat and mass transfer in the upper ocean, and the mixing in large scale water bodies such as reservoirs, lakes and estuaries. The SCBL flows in fire-induced smoke transportation will potentially increase the fire hazards by changing flow patterns.

As pure thermally convective flows and pure sheared stratified flows have independent unstable modes, i.e., the Rayleigh-Benard instability and shear stratified instability (e.g., the Kelvin-Helmholtz instability and Holmboe instability), the combination of the two basic flow configurations are expected to produce more interactive and complicated unstable modes and corresponding flow patterns. For instance, Raasch & Franke (2011) used high resolution numerical simulation to find the transitional behaviour from a spoke-shape flow pattern in pure thermal convection to a band-like flow pattern in sheared thermal convection. Yang et al. (2010) observed experimentally a unique flow pattern near the interface in fire induced SCBL. Most recently, Stewart et al. (2014) investigated the SCBL flows with two-dimensional direct numerical simulation and found that the Kelvin-Helmholtz instability and the Rayleigh-Benard instability coexist in some stratification conditions, where the presence of the penetrative convection modifies the original Kelvin-Helmholtz eddy structures. Their results also suggest that new unstable modes might occur in the SCBL flow settings, which motivates the current hydrodynamic stability analysis to examine the stability features of the SCBL flow.

This paper will study the hydrodynamics of SCBL flow by introducing an unstable stratification factor \( J_b \) into the Taylor-Goldstein equation that describes the hydrodynamics of shear stratified flow. This idea comes from the hydrodynamic study of two-layer thermal penetrative convective flows where a thermally convective region is capped...
by a stable stratified layer. The difference between the current SCBL flow and such two-layer penetrative convection is whether horizontal shear flow is present. By solving modified Taylor-Goldstein equations for SCBL flows, the instability features of the SCBL, including temporal growth rate and spatial perturbation structures, will be investigated, after a brief introduction of the numerical methodology.

PERTURBATION EQUATIONS

Based on the Taylor-Goldstein equations for sheared stratified flow and by applying the Squire transformations as detailed in Drazin & Reid (2004), the following perturbation equations, written in matrix form, have been derived by the current study,

\[
\begin{bmatrix}
\nabla_z^2 & I \\
\end{bmatrix}
\begin{bmatrix}
\tilde{\psi} \\
\tilde{b}
\end{bmatrix}
= 
\begin{bmatrix}
-ia(U\nabla_z^2 - U_z) & \frac{\partial^2}{\partial \theta^2} - \frac{i}{\alpha}aU \\
\frac{N^2}{L^2} & -i\alpha U
\end{bmatrix}
\begin{bmatrix}
\tilde{\psi} \\
\tilde{b}
\end{bmatrix}
\tag{1}
\]

where the subscript ‘zz’ denotes the second order differentiation with respect to \( z \), \( \nabla_z^2 \) is the Squire Laplacian operator defined as \( \nabla_z^2 = \partial^2 - \alpha^2 \), \( D = \partial / \partial z \) is the differential operator for the perturbation properties, \( \tilde{\psi} \) and \( \tilde{b} \) (\( \tilde{b} = \gamma \tilde{\theta} \)) are the vertical component of the velocity perturbation and the Squire buoyancy, \( \gamma \) is the thermal expansion coefficient, and \( N^2 \) (\( N^2 = \gamma \theta_b \)) is the local Squire buoyancy Brunt-Väisälä frequency. For sheared stratified flows, the sheared layer thickness, the velocity and temperature changes across the sheared/stratified layer are usually selected as the characteristic length scale \( L \), velocity scale \( \Delta u_{\theta,0} \) and temperature scale \( \Delta \theta_{s,0} \), respectively. \( Fr = \Delta u_{\theta,0} / \sqrt{gL} \) is the Froude number where \( g \) is the gravitational acceleration.

Hazel (1972) suggested that when the basic velocity and background stratification in the sheared/stratified layer are in the form of \( \Delta u_{\theta,0}(z) \) and \( \Delta \theta_{s,0}(z) \), where \( f(z) \) is the hyperbolic function, and if \( (\partial u / \partial z)_{z=0} = 1 \) and \( (\partial \theta / \partial z)_{z=0} = 1 \), where \( z = 0 \) is the central line of the sheared/stratified layer, then the local Richardson number \( Ri_b(z) \) is

\[
Ri_b(z) = \frac{N^2}{(\partial u_{\theta,0} / \partial z)^2} = J (\partial \theta / \partial z) (\partial \theta / \partial z)^{-2},
\tag{2}
\]

in which \( J = g\gamma \Delta \theta_{s,0} L / (\Delta \theta_{s,0})^2 = (g\gamma \Delta \theta_{s,0} L / (L/\Delta \theta_{s,0}))^2 \).

Here the bulk temperature gradient \( \Delta \theta_{s,0} / L \) within the sheared/stratified layer is extracted. In practical stability analysis, \( J \) plays as an effective substitute for \( Ri_b \).

The SCBL flow system in this study is shown in figure 1, where the convective flow region with unstably linear stratification and the length scale \( L_b \) occurs at the bottom of the sheared stratified flow region with smooth tangential stratification and the length scale \( L \). The temperature at the bottom boundary \( \theta_{b,0} \) is larger than the temperature \( \theta_{b,s} \), the flows are unstable at \( z < L_b \). However, when \( z > L_b \), the flows are stably stratified as \( \Delta \theta_{s,b} = \Delta \theta_{s,b} - \Delta \theta_{s,b} > 0 \). In the convective flow region the bulk temperature gradient is \( G_b = -\Delta \theta_{s,b} / L_b \), where the negative sign indicates that the temperature decreases with increasing height. Correspondingly, the bulk temperature gradient for the stably stratified layer is \( G_s = \Delta \theta_{s,b} / L_b \), where the positive sign indicates that the temperature increases with increasing height.

In penetrative convection problems where a thermal convection region is capped by a stably stratified layer, Whitehead & Chen (1972) and Sun (1976) introduced a stability factor \( S \), by replacing the bulk temperature gradient in the Rayleigh number with the bulk temperature gradient in the top stably stratified layer. Inspired by this method, in the current SCBL flow, we replace the bulk temperature gradient \( G_s \), in the convective flow region \( z < L_b \), and define the unstable stratification factor \( J_{b,s} \) for SCBL flow as

\[
J_{b,s} = -\frac{g\gamma \Delta \theta_{s,b} (L/L_b)^2}{L_b} = g\gamma G_b (L/L_b)^2 = G_s J_b \tag{3}
\]

so that when \( z > L_b \), \( N_s^2 = J_b (\partial \theta / \partial z) \) and when \( z < L_b \), \( N_b^2 = J_b (\partial \theta / \partial z) \). It is noted that \( J_b \) and \( J \) have the opposite signs, indicating unstable and stable background stratification, respectively. Therefore, while solving (1), \( N_b^2 \) will become a piecewise function depending on \( z > L_b \) or \( z < L_b \).

METHODOLOGY

The temporal mode of (1), where \( \alpha \) is a real number and \( c \) is a complex number, is solved in this study. It should be noted that the real part of the temporal growth rate, \( \sigma = i \alpha c \), of the perturbation is only determined by the imaginary part of the complex wave speed \( c \), for the wavenumber \( \alpha \) is fixed as a real number.

Matrix methods are used to solve the eigenvalue problems formed by discretising the perturbation equations (1) with uniform grid and using the second-order central difference scheme. The QZ algorithm developed by Moler & Stewart (1973), which is integrated in the LAPACK routine CGGEV, is used as the complex eigenvalue solver. The robustness of the QZ algorithm in hydrodynamic stability analysis has been demonstrated in some recent studies, such as Smyth et al. (2011), Liu et al. (2012), and Thorpe et al. (2013).

The boundary conditions \( \tilde{\psi} = \tilde{b} = 0 \) are applied at both the top and the bottom boundaries. The dimensionless verti-
Figure 2. Contour plots of the real part of temporal growth rate $\tilde{\sigma}$ in the $J$ versus Squire wavenumber $\tilde{\alpha}$ plane with (a) $J_b = 0$, (b) $J_b = (1/6)J$ and (c) $J_b = (1/3)J$. The magnitudes of $\tilde{\sigma}$ are denoted by curves in different colors.

Figure 3. Calculated real part of the temporal growth rate $\tilde{\sigma}$ plotted against $J$ at $\tilde{\alpha} = 0.5$. The results in the SS and RB unstable regions are denoted by the dashed and solid curves, respectively.

RESULTS

Growth Rate $\tilde{\sigma}$

Figure 2 shows the contours of the real part of the temporal growth rate $\tilde{\sigma}$ in the $J - \tilde{\alpha}$ plane for different unstable stratifications in terms of $J_b$. In figure 2(a) where $J_b = 0$ which represents that no convective flux exists, the unstable hemi-ellipse regions over $\tilde{\sigma} = 0 \sim 1$, in which the imaginary part of $\tilde{\sigma}$ is found to be zero, correspond to the stationary Kelvin-Helmholtz instability mode in SS flow as marked by ‘SS’. The shapes of the KH mode regions are similar to the numerical results from Hazel (1972), where the critical stratification factor $J_c$ occurs at a corresponding critical wavenumber $\tilde{\alpha}_c = 0.5$. As $J_b \neq 0$ when a convective flux is introduced, the other large unstable region where $\text{Im}(\tilde{\sigma}) \neq 0$ appears, as marked by ‘RB’ in figure 2(b) and figure 2(c). In figure 2(b), the ‘RB’ region interacts with the ‘SS’ region at $\tilde{\alpha} \approx 0.8$. In figure 2(c) when $J_b$ is one third of $J$, the intersection points move to the small wavenumber range $\tilde{\alpha} \approx 0.25$ and the two contour curves of $\tilde{\alpha} = 0.2$ merge together at $\tilde{\alpha} = 0.45 \sim 0.8$. Inside this merging region, part of the $\tilde{\sigma} = 0.01$ contour, which was in the ‘SS’ region, is overwhelmed by the $\tilde{\sigma} = 0.02$ contour of the ‘RB’ region.

To further study the interactions between the two instability regions, figure 3 shows $\text{Re}(\tilde{\sigma})$ plotted against $J$ at the critical wavenumber $\tilde{\alpha}_c = 0.5$ for a series of $J_b/J$ ratios. As $\text{Im}(\tilde{\sigma}) = 0$ solutions for the ‘SS’ mode and the $\text{Im}(\tilde{\sigma}) \neq 0$ solutions for the ‘RB’ mode are denoted by dashed and solid curves, respectively. It is found that as different $J_b/J$ are applied, the ‘SS’ curve retains its original shape at $J_b = 0$ although different ‘RB’ curves are obtained. Therefore, the ‘SS’ mode is independent of the ‘RB’ mode. As the ‘SS’ mode branch decreases quickly with increasing $J$ and the ‘RB’ mode branch increases gradually with increasing $J$, the two branches intersect near the critical stratification factor $J = 0.25$ for SS flow. Near the intersection point, as $\text{Re}(\tilde{\sigma})$ in the ‘RB’ mode is comparable to that in the ‘SS’ mode, a narrow transition region where both ‘SS’ and ‘RB’ have equal influence is created. As $J_b/J$ increases, the ‘RB’ mode curves gradually rise upwards and cap the ‘SS’ mode curves, thereby moving the intersection point upwards as well. As a result, some unstable regions which were in the ‘SS’ mode are replaced by the ‘RB’ mode, e.g., $J_0 = (1/3)J$ but belong to the ‘RB’ for $J_0 > (10/3)J$.

Figure 4 shows the dispersion relations between $\text{Re}(\tilde{\sigma})$ and wavenumber $\tilde{\alpha}$ at $J = 0.2$. It can be considered as a horizontal slice plot of figure 2 at $J = 0.2$. Similar to the $J \sim \tilde{\sigma}$ plot in figure 3, the dispersion curves for the ‘SS’ and the ‘RB’ modes are represented by dashed and solid curves, respectively. Different from figure 3, two intersection points between the ‘SS’ mode dispersion curve and each ‘RB’ mode dispersion curve are found at small wavenumber range and large wavenumber range respectively, forming two transitional regions in the $\tilde{\alpha} \sim \tilde{\sigma}$ plane. As $J_b$ increases, the ‘RB’ dispersion curves gradually rise based on their common origin point and move the two intersection points to larger $\text{Re}(\tilde{\sigma})$. Consequently, more and more the ‘SS’ mode dispersion curve is dominated by the ‘RB’ mode dispersion curve, e.g., when $J_b = (8/3)J$ the most parts of the ‘SS’ dispersion curve are covered by the ‘RB’ curves.

To quantify how the ‘RB’ mode gradually overwhelms
At $J = 0.2$, the strong variation section of $w'$ starts from the bottom of the domain for both the 'SS' and 'RB' modes. The 'SS' mode curve and the 'RB' mode curve intersect at $J_\alpha = 0.5$ and $J = 0.2$ where the transition from the 'SS' mode to the 'RB' mode occurs, and the $\alpha$ curve denoted by the red solid curve.

The 'SS' mode as $J_\alpha = 0.5$ and $J_\alpha = 0.2$ for both the 'SS' and 'RB' modes at $\tilde{z}$ varies drastically near $z = -2$, which is the vertical height of the interface between the initial stable and unstable stratified layer. As $J$ increases to 0.2, for the 'RB' mode as shown in figure 6(b), $b'$ varies dramatically near $z = -2$, which is the vertical height of the interface between the initial stable and unstable stratified layer. For the 'SS' mode as shown figure 6(c), the variation of $b'$ remains at the central shear layer but its amplitude becomes positive compared to the negative one as shown in figure 6(a). As $J$ increases to 1.0 where the 'RB' mode dominates, the absolute magnitude of the phase change (dashed curve) evolves to be larger than the amplitude change (solid curve) of $b'$ at $z = -2$. Meanwhile, the amplitude variation along $z$ in figure 6(d) keeps a similar profile to that shown in figure 6(b), and also shows a similar $b'$ profile to that Sun (1976) where only penetrative convection occurs.

Figure 6 shows the calculated eigenfunctions for $b'$ at $\alpha_0 = 0.5$ and $J_\alpha = 3$ but with three typical $J$ values as discussed above. The solid and the dashed curves represent the real and the imaginary parts of $b'$, which correspond to the amplitude and the phase of $b'$, respectively. At $J = 0.1$ where the 'SS' mode dominates, $b'$ concentrated on the initial central shear/stratified layer, despite slight deviations of phase (dashed curve) due to weakly convective flow induced by the bottom unstable stratified layer. As $J$ increases to 0.2, for the 'RB' mode as shown in figure 6(b), $b'$ varies drastically near $z = -2$, which is the vertical height of the interface between the initial stable and unstable stratified layer. For the 'SS' mode as shown figure 6(c), the variation of $b'$ remains at the central shear layer but its amplitude becomes positive compared to the negative one as shown in figure 6(a). As $J$ increases to 1.0 where the 'RB' mode dominates, the absolute magnitude of the phase change (dashed curve) evolves to be larger than the amplitude change (solid curve) of $b'$ at $z = -2$. Meanwhile, the amplitude variation along $z$ in figure 6(d) keeps a similar profile to that shown in figure 6(b), and also shows a similar $b'$ profile to that Sun (1976) where only penetrative convection occurs.

Based on figures 3 and 5, the eigenfunctions for buoyancy perturbation $b'$, vertical velocity perturbation $w'$ in (1) and their product, and buoyancy flux perturbation $B' = \theta'w'$ are studied at the critical wavenumber $\alpha_0 = 0.5$ and stratification ratio $J_\alpha/J = 3$, which is the intersection point found in figure 5. Three typical $J$ values are selected at $J = 0.1$ where the ‘SS’ mode dominates, at $J = 0.2$ where the transition from the ‘SS’ mode to the ‘RB’ mode occurs, and at $J = 1.0$ where the ‘RB’ mode dominates. For $J = 0.2$, the Re[$\theta'$] for the ‘SS’ and ‘RB’ modes are very close to each other, therefore the eigenfunctions for both modes are shown.

Figure 6 shows the calculated eigenfunctions for $b'$ at $\alpha_0 = 0.5$ and $J_\alpha = 3$ but with three typical $J$ values as discussed above. The solid and the dashed curves represent the real and the imaginary parts of $b'$, which correspond to the amplitude and the phase of $b'$, respectively. At $J = 0.1$ where the ‘SS’ mode dominates, $b'$ concentrated on the initial central shear/stratified layer, despite slight deviations of phase (dashed curve) due to weakly convective flow induced by the bottom unstable stratified layer. As $J$ increases to 0.2, for the ‘RB’ mode as shown in figure 6(b), $b'$ varies drastically near $z = -2$, which is the vertical height of the interface between the initial stable and unstable stratified layer. For the ‘SS’ mode as shown figure 6(c), the variation of $b'$ remains at the central shear layer but its amplitude becomes positive compared to the negative one as shown in figure 6(a). As $J$ increases to 1.0 where the ‘RB’ mode dominates, the absolute magnitude of the phase change (dashed curve) evolves to be larger than the amplitude change (solid curve) of $b'$ at $z = -2$. Meanwhile, the amplitude variation along $z$ in figure 6(d) keeps a similar profile to that shown in figure 6(b), and also shows a similar $b'$ profile to that Sun (1976) where only penetrative convection occurs.

Figure 7 shows the calculated eigenfunctions for $w'$ at $\alpha_0 = 0.5$ and $J_\alpha/J = 3$ but with three typical $J$ corresponding to figure 6. At $J = 0.1$, the central line of amplitude of $w'$ is located slightly above the center of the initial shear layer ($z = 0$), as a result of weak convection flow due to a small $J_\alpha$. At $J = 0.2$, the strong variation section of $w'$ starts from the bottom of the domain for both the ‘SS’ and ‘RB’ modes.
For the ‘RB’ mode, as shown in figure 6(b), the profile of the amplitude of $w'$ changes remarkably at $\bar{z} = -5 \sim -1$, indicating that most of $w'$ originate from the bottom convective flow rather than from the initial shear layer which extends from $\bar{z} = -1 \sim 1$. For the ‘SS’ mode as shown in figure 6(c), although the amplitude of $w'$ starts to vary from the bottom of the domain, the maximum $w'$ still occurs within the initial shear layer where $\bar{z} = -1 \sim 1$. At $J = 1.0$ where the ‘RB’ completely dominates, $w'$ shows strong propagative features as the magnitude of phase is larger than the magnitude of amplitude.

Figure 8 shows the calculated eigenfunctions for the buoyancy flux perturbation $B' = b'w'$ at $\bar{\alpha} = 0.5$ and $J_0 / J = 3$ with three typical $J$ corresponding to figures 6 and 7. At $J = 0.1$, the buoyancy flux perturbation is formed on the initial shear layer at the center of the domain. As $J$ increases to 0.2, significant differences are found between the ‘SS’ and ‘RB’ modes. For the ‘RB’ mode, as shown in figure 8(b), strong buoyancy flux perturbation occurs near the interface at $\bar{z} = -2$, with intense and positive magnitude of amplitude compared to figure 8(a). For the ‘SS’ mode as shown in figure 8(c), although the variations of $B'$ occur mainly in the initial shear layer, such variations appear very weak as the magnitude of the amplitude and phase significantly decrease compared to figure 8(a) and 8(b). At $J = 1.0$, the buoyancy flux perturbations are almost negligible. The transitional behaviors of $B'$ from the ‘SS’ mode to the ‘RB’ mode suggests that as unstable stratification increases, the buoyancy flux generated from the bottom thermal boundary will gradually smooth out the buoyancy flux perturbations generated by the central shear stratified layer and eventually dominates in the entire domain.

CONCLUSIONS

The influences of the bottom thermal convection region, in terms of an unstable stratification factor $J_0$, are added to the Taylor-Goldstein equations to describe the stability features of the sheared convective boundary layer (SCBL) flow. As $J_0$ is introduced into the Taylor-Goldstein equation system, new unstable regions indicating thermal instability are found above the critical stratification factor $J = 0.25$, for the shear stratified flow, in the $J - \bar{\alpha}$ plane, in addition to the semicircle instability region of the classic shear stratified flow at $J < 0.25$. As the stratification ratio $J_0 / J$ further increases, the thermal unstable region expands and gradually merges with the shear stratified unstable regions. In the $J - \bar{\alpha}$ and $\bar{\alpha} - \bar{\sigma}$ planes, increasing $J_0$ expands the thermal unstable regions and accordingly leads to domination of the thermal unstable mode over the shear stratified unstable mode. It is further found that the temporal growth rate of the thermal unstable mode increases in a parabolic fashion with the stratification ratio $J_0 / J$ and gradually approaches, intersects and at last overwhelms the temporal growth rate of the shear stratified mode, leading to a transition from shear stratified dominant SCBL mode to thermal convection dominant SCBL mode. The critical transition condition for SCBL flow is found at $J_0 / J = 3$ and $\bar{\alpha} = 0.5$. The analysis of the eigenfunctions of the buoyancy perturbation $b'$, vertical velocity perturbation $w'$ and buoyancy flux perturbation $B' = b'w'$ shows distinctively different spatial perturbation structures for the shear stratified dominant mode, transitional mode and the thermal dominant mode.

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