MODELLING OF THE DECAY OF HOMOGENEOUS MAGNETOHYDRODYNAMIC TURBULENCE BY USING COMPACT SCHEMES

Dauren Zhakebayev
Department of Mathematicaln and computer modeling
al-Farabi Kazakh National University
71, av. al-Farabi, Almaty, 050012, Kazakhstan
dauren.zhakebaev@kaznu.kz

Ualikhan Abdibekov
al-Farabi Kazakh National University
71, av. al-Farabi, Almaty, 050012, Kazakhstan
uali13@mail.ru

ABSTRACT

This paper considers the numerical modelling of the homogeneous magnetohydrodynamic turbulence decay based on large eddy simulation. The modelling of the turbulent process is based on the solution of a filtered unsteady three-dimensional Navier-Stokes equation and the equation of magnetic field strength. Dynamic model has been applied to close the main equations. The problem is solved numerically: the equation of motion by modified method of fractional steps using compact schemes, the equation for pressure - by Fourier method with a combination of matrix factorization, the equation for the magnetic field - by the method of fractional steps. Change of the kinetic and magnetic energies of the turbulence obtained over the time depending on the properties conductivity of the medium. A patterns are defined for longitudinal and transverse correlation functions.

INTRODUCTION

An examination of the homogeneous magnetohydrodynamic turbulence decay process, in spite of the large number of publications in this field, is a relevant task for researchers of several generations. The influence of magnetic field on the conducting fluid is studied in various fields of science and used in an engineering and technology. Therefore, studies of magnetohydrodynamic turbulence decay is an important task in the fields of: forming astrophysical and geophysical phenomena, MHD generators, plasma accelerators and engines.

Research problems of the magnetic field influence on the electro conductive fluid is divided into three types:

- examination of the MHD turbulence at a constant value of the magnetic field.
- examination of the self-excitation of magnetic field at a given velocity of the flow.
- examination of the self-excitation of magnetic field and the motion of a conducting fluid at the same time taking into account acting forces.

This work is devoted to study of self-excitation of magnetic field and the motion of the conducting fluid at the same time taking into account acting forces. The idea is to specify in the phase space of initial conditions for the velocity field and magnetic field, which satisfy the condition of continuity. Given initial condition with the phase space is translated into physical space using a Fourier transform. The obtained of velocity field and magnetic field are used as initial conditions for the filtered MHD equations. Further is solved the unsteady three-dimensional equation of magnetohydrodynamics to simulate homogeneous MHD turbulence decay.

The process of the magnetic field influence on developed turbulence was examined by Knaepen et al. (2004) and demonstrated the possibility to apply a quasistationary approximation to solve the second type problem and suggested to use quasi-linear approximations to solve this problem at $Re_m = 20$. Some results on the study of the magnetic field self-excitation at a given flow velocity were reported by Knaepen and Moin (2004); the modelling of diminishing MHD turbulence by LES and DNS methods demonstrated that the magnetic field at the initial time started to decay under the influence of the total kinetic energy. This effect is consistent with Joule dissipation. A similar picture of the decay was not reported by the authors because their main objective was the evaluation of the model adequacy for the LES and DNS methods. Accordingly, there was a justification of the modified dynamic Smagorinsky model for simulation of temporal decaying MHD turbulence.

The results of the study of the third type problem are presented (G.Sahoo et. al., 2011) and give a detailed investigation of pseudospectral direct numerical simulation, with up to 1024^3 nodes, three-dimensional incompressible MHD turbulence, and with no mean magnetic field. The study was carried out considering various statistical properties of both decreasing and statistically steady MHD turbulence on the magnetic Prandtl number Pm taken over a wide range of $0.01 \leq Pm \leq 10$. Turbulent characteristics were obtained at a constant magnetic viscosity for different values of kinetic viscosity.

The numerical modeling of a homogeneous MHD turbulence decay based on the large eddy simulation method depending on the conductive properties of the incompressible fluid is reviewed.

The numerical modeling of the problem is performed based on solving non-stationary filtered magnetic hydrodynamics equations in conjunction with the continuity equation in the Cartesian coordinate system in a non-dimensional form:

$$\begin{cases} \frac{\partial(\bar{u}_i)}{\partial t}+\frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial(\bar{p})}{\partial x_i}+\frac{1}{Re}\left(\frac{\partial^2\bar{u}_i}{\partial x_j^2}\right)- \\ \qquad -\frac{\partial\tau_{ij}^u}{\partial x_j}+A\frac{\partial}{\partial x_j}\left(\bar{H}_i\bar{H}_j\right), \\ \frac{\partial\bar{u}_j}{\partial x_j}=0, \\ \frac{\partial\bar{H}_i}{\partial t}+\frac{\partial}{\partial x_j}(\bar{u}_j\bar{H}_i-\bar{H}_j\bar{u}_i)=\frac{1}{Re_m}\frac{\partial^2\bar{H}_i}{\partial x_j^2}-\frac{\partial\tau_{ij}^H}{\partial x_j}, \\ \frac{\partial\bar{H}_j}{\partial x_j}=0, \\ \tau_{ij}^u=\left((\overline{u_iu_j})-(\bar{u}_i\bar{u}_j)\right)-\left((\overline{H_iH_j})-(\bar{H}_i\bar{H}_j)\right), \\ \tau_{ij}^H=\left((\overline{u_iH_j})-(\bar{u}_i\bar{H}_j)\right)-\left((\overline{H_iu_j})-(\bar{H}_i\bar{u}_j)\right), \end{cases} \tag{1}$$

where $\bar{u}_i(i=1,2,3)$ are the velocity components, $\bar{H}_1,\bar{H}_2,\bar{H}_3$ are the magnetic field strength components, $A=H^2/\left(4\pi\rho V^2\right)=\Pi/Re_m^2$ is the Alfvén number, H is the characteristic value of the magnetic field strength, V is the typical velocity, $\Pi=\left(V_AL/\nu_m\right)^2$ is a dimensionless value (on which the value Π depends in the equation for $\bar{H}_i$). If $\Pi << 1$, then $\partial\bar{H}_i/\partial t=0$. The publicatio n by Ievlev (1975) discussed in detail the physics of phenomena related to the ability to disregard the summand $\partial\bar{H}_i/\partial t$. $(V_A)^2=H^2/4\pi\rho$ is the Alfvén velocity, $\bar{p}=p+\bar{H}^2A/2$ is the full pressure, t is the time, $Re=LV/\nu$ is the Reynolds number, $Re_m=VL/\nu_m$ is the magnetic Reynolds number, L is the typical length, ν is the kinematic viscosity coefficient, ν_m is the magnetic viscosity coefficient, ρ is the density of electrically conducting incompressible fluid, and τ_{ij}^u, τ_{ij}^H is the subgrid-scale tensors responsible for small-scale structures to be modeled.

To model a subgrid-scale tensor, a viscosity model is presented as $\tau_{ij}^u=-2\nu_T\bar{S}_{ij}$, where $\nu_T=C_S\Delta^2\left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{1}{2}}$ is the turbulent viscosity, $\bar{S}_{ij}=(\partial\bar{u}_i/\partial x_j+\partial\bar{u}_j/\partial x_i)/2$ is the deformation velocity tensor value. To model a magnetic subgrid-scale tensor, a viscosity model is used: $\tau_{ij}^H=-2\eta_t\bar{J}_{ij}$, where $\eta_t=D_S\Delta^2\left(\bar{J}_{ij}\bar{J}_{ij}\right)^{\frac{1}{2}}$ is the turbulent magnetic diffusion, the coefficients C_S, D_S are calculated for each determined time layer, and $\bar{J}_{ij}=(\partial\bar{H}_i/\partial x_j-\partial\bar{H}_j/\partial x_i)/2$ is the magnetic rotation tensor reviewed by Muller and Carati (2002).

Periodic boundary conditions are selected at all borders of the reviewed area of the velocity components and the magnetic field strength. The initial values for each velocity component and strength are defined in the form of a function that depends on the wave numbers in the phase space:

$$u_i(k_i,0)=k_i^{\frac{b-2}{2}}e^{-\frac{b}{4}\left(\frac{k_i}{k_{\max}}\right)^2};H_i(k_i,0)=k_i^{\frac{b-2}{2}}e^{-\frac{b}{4}\left(\frac{k_i}{k_{\max}}\right)^2},$$

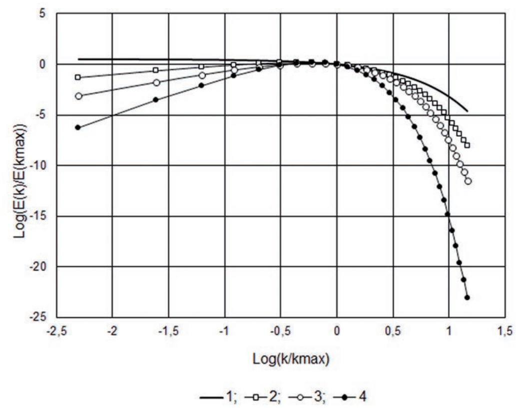


Figure 1. The equation of initial level turbulence, depending on the fixed wave number and the variational parameter b: 1) $b=2$; 2) $b=4$; 3) $b=6$; 4) $b=8$.

where $\bar{u}_i$ is the one-dimensional velocity spectrum, $i=1$ refers to the longitudinal spectrum, $i=2$ and $i=3$ refer to the transverse spectrum, $\bar{H}_i$ is the one-dimensional magnetic field strength spectrum, m is the spectrum power, and k_1,k_2,k_3 are the wave numbers.

For this problem, we selected a variational parameter b and a wave number k_{max}, which determine the type of turbulence. In Fig. 1, the parameter b varies when $k_{max}=10$. To model homogeneous MHD turbulence, the parameters k_{max} and b can be set corresponding to the experimental data, which was shown by Sirovich, Smith and Yakhot (1994).

To solve the problem of homogeneous incompressible MHD turbulence, a scheme of splitting by physical parameters is used: At the first stage, the Navier-Stokes equation is solved with no pressure consideration. For the approximation of convective and diffusion equation members, a compact scheme of a higher order of accuracy is used by Abdibekov et al. (2013). At the second stage, the Poisson equation is solved, which is derived from the continuity equation by considering the velocity fields of the first stage. For the 3D Poisson equation, an original solution algorithm has been developed: a spectral transform in combination with the matrix run. At the third stage, the obtained pressure field is used to recalculate the final velocity field. At the fourth stage, the obtained velocity field is used to solve an equation in order to obtain the components of the magnetic field strength, which are included in the initial equation.

NUMERICAL MODELLING RESULTS.

Numerical model allowed to describe the homogeneous magnetohydrodynamic turbulence decay based on large eddy simulation. For this task, the kinematic viscosity $\nu=10^{-4}$ was taken constant and the magnetic viscosity were set in the range of $\nu_m=10^{-3}\div10^{-4}$. The characteristic values of the velocity, length, magnetic field strength were taken equal to: $U_{CH}=1$, $L_{CH}=1$, $H_{CH}=1$ respectively. Reynolds number is $Re=10^4$, the magnetic Reynolds number varied depending on the magnetic viscosity coefficient. The Alfven number characterizing the motion of conductive fluid for various numbers of magnetic Reynolds: $A=Ha^2/Re_m$, where Hartmann number is $Ha=1$. For the calculations used grid size 128x128x128. The time step was taken equal $\Delta\tau=0.001$.

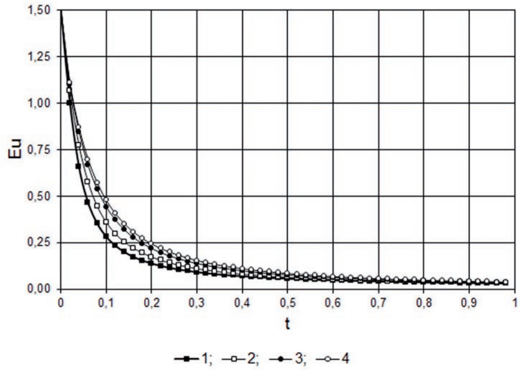


Figure 2. Variaion of kinetic turbulent energy vs. magnetic Reynolds numbers at different points in time: 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$.

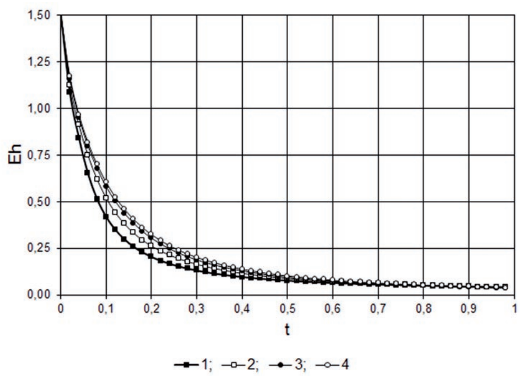


Figure 3. Variation of magnetic energy vs. magnetic Reynolds numbers at different points in time: 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$.

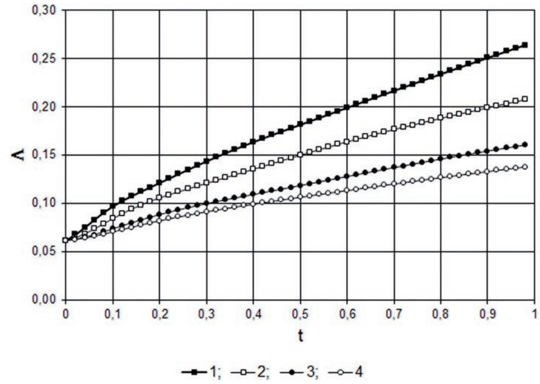


Figure 4. Change of the integral turbulence scale calculated at different magnetic Reynolds numbers: 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$.

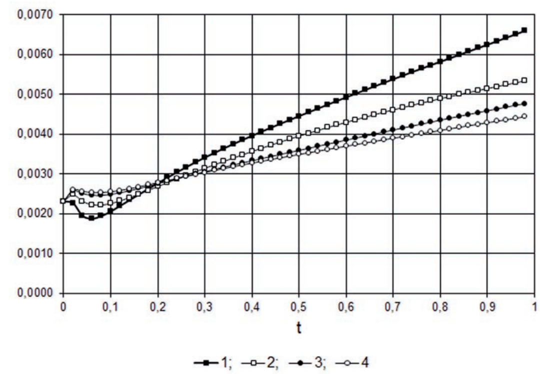


Figure 5. Change of the Taylor scale calculated at different magnetic Reynolds numbers: 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$.

As a result of simulation at different magnetic Reynolds numbers were obtained the following turbulence characteristics: kinetic energy, magnetic energy, integral scale, Taylor scale, transverse and longitudinal correlation functions.

The results displayed in Fig.2, show the influence of the magnetic viscosity on the decay of kinetic and magnetic energies calculated at different magnetic Reynolds numbers.

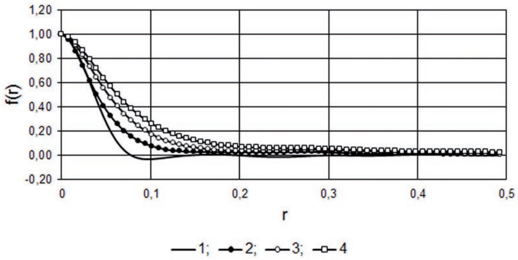
Fig.2-3 show the dynamics of the mutual influence of magnetic energy on kinetic energy at the different time instants: at the initial time the kinetic and magnetic energies are given the same,at the next instant,when the fluid is studied with high conductivity (Re_m- small), the decay of the MHD turbulence occurs faster, than when Re_m starts to rise, which specifies fluid with less conductivity, and at $Re_m = 10^4$ the decay of magnetohydrodynamic turbulence practically corresponds to the decay of isotropic turbulence.

According to semi-empirical theory of turbulence integral scale should grow with time. The results presented in Fig.4 illustrates the effect of magnetic viscosity on the internal structure of the MHD turbulence. Variation of the coefficient of magnetic viscosity leads to a proportional change in the integral scale. Fig.4 shows that the size of large eddies rapidly increases at small number of magnetic Reynolds $Re_m = 10^3$, than in the case, when $Re_m = 10^4$ which leads to fast energy dissipation. Fig.5 shows the change in the micro scale - calculated at different numbers of magnetic Reynolds 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$.

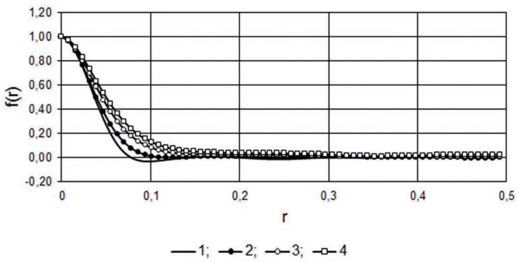
Fig.5 shows the change of the Taylor microscale at different magnetic Reynolds numbers. It can be seen that in the case $Re_m = 10^3$ when the magnetic viscosity coefficient is large then the dissipation rate increases. In the case when the magnetic viscosity coefficient is smaller then the scale gradually increases, and the small scale structure of the turbulence tends to slowly isotropy. This also indicates that with small numbers Re_m the decay of isotropic turbulence occurs faster than in the case when Re_m is high.

Fig.6 shows the changes of the longitudinal correlation function calculated at $Re_m = 10^3$ and $Re_m = 10^4$. The Fig.7 illustrates the changes in the transverse correlation function calculated at $Re_m = 10^3$ and $Re_m = 10^4$. These illustrations also show that there are an influence of the magnetic field on the isotropic turbulence decay, as these figures are fixed the result of changes in the correlation functions at different Re_m.

The correlation function is expressed the average by volume the correlation ratio between the components of the velocity at various points, the farther points are located between the various components of the velocity, the smaller should be the correlation coefficients, i.e. they should be close to zero. Figure 6a shows the change in the longitudinal correlation function $f(r)$ in time and calculated at $Re = 10^4$, $Re_m = 10^3$. It is seen that with increasing value r of the function tends to zero. Character of the correlations change corresponds to the change of the correlation functions given (Abdibekov and Zhakebayev, 2011).

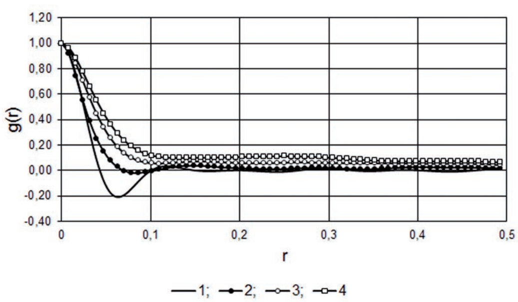


(a)

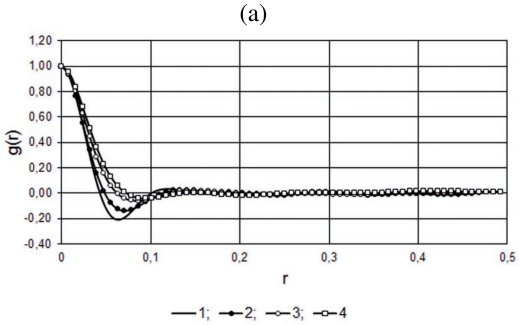


(b)

Figure 6. Change the longitudinal correlation function $f(r)$ when (a) $Re_m = 10^3$ and (b) $Re_m = 10^4$ at different points in time: 1) $t = 0$; 2) $t = 0.2$; 3) $t = 0.3$; 4) $t = 0.5$.



(a)



(b)

Figure 7. Change the transverse correlation function $g(r)$ when (a) $Re_m = 10^3$ and (b) $Re_m = 10^4$ at different points in time: 1) $t = 0$; 2) $t = 0.2$; 3) $t = 0.3$; 4) $t = 0.5$.

Basing on the results of the study, it has been found that the first part of the turbulent kinetic energy is used for turbulent mixing, the second part for creating the magnetic field, and the third part for the forces of resistance between the components of velocity and magnetic tension.

Conclusions

Based on the LES method with using compact scheme, the influence of magnetic viscosity on the decay of uniform magnetohydrodynamic turbulence has been numerically modelled. The modified LES method in combination of compact scheme is allowed to obtain a compact approximation for the convective terms of the motion equations of the third, and for the diffusion terms of the fourth, order of accuracy. The obtained results allow to sufficiently accurately calculate the variations of the characteristics of uniform MHD turbulence with time at large Reynolds and magnetic Reynolds numbers. A numerical algorithm has been developed to solve unsteady three-dimensional magnetohydrodynamic equations as well as to model the MHD turbulence decay at different magnetic Reynolds numbers. The analysis of the simulation results makes possible the following conclusion: the magnetic viscosity of the flow has a significant influence on MHD turbulence and, therefore, can be used for the process control at the production of semiconductor single crystals. Physical processes and phenomena of uniform MHD turbulence were identified in the numerical simulation. The proposed method can be used to solve MHD turbulence without significant changes.

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