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THE ROLE OF THE HAIRPIN VORTEX SOLUTION ON LAMINAR-TURBULENT TRANSITION OF PLANE COUETTE FLOW AT MODERATE REYNOLDS NUMBER

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ABSTRACT

The shooting method is employed to outline a structural aspect of phase space of plane Couette flow, associated with laminar-turbulent transition at moderate Reynolds numbers ($Re \sim 10^4$). The basin boundary separating the laminar and the turbulent attractors in the phase space is emerged out of trajectories starting from initial conditions constituted by superposition of three distinct exact steady solutions of PCF, including the Hairpin Vortex Solution (Itano & Generalis (2009); Gibson et al. (2009)) and the so-called Nagata's solution (Nagata (1990)). The result implies that HVS is on the basin boundary even at moderate Reynolds numbers, and that one of the unstable manifolds of HVS heads towards the NBW, which lies on the basin boundary.

INTRODUCTION

Hairpin vortex structure has been known as one of predominant vortex structures in turbulent shear flows. A candidate of corresponding exact solution, Hairpin Vortex Solution (HVS) of plane Couette flow (PCF), was solved at $Re \sim 200$ recently by Itano & Generalis (2009), Gibson et al. (2009). The solution is composed of the upper and lower branches arising from a saddle-node bifurcation at the turning point $Re = 139.2$, where Reynolds number is defined as $Re = Uh/\nu$ in terms of the kinematic viscosity ν, the channel half width h, and the moving-wall velocity $\pm U$. By contrast, the so-called Nagata's solution of PCF (hereafter we used the acronym "NBW" formed from discoverers, Nagata (1990); Clever & Busse (1997); Waleffe (1998)) is bifurcated from the lower Reynolds number, $Re = 127.7$, in spite the fact that NBW is a derivative rather than the counterpart of the HVS.

The upper branch of the HVS contains vortex structures with the shape of a hairpin observed ubiquitously in turbulent shear flows, satisfying the reflection symmetry in the spanwise direction. At a couple of legs of a vortex structure, the localised vorticity lift up the fluid near the boundary so as to form a couple of streaky regions near the boundary, and the head of the vortex structure induces coalescence of these streak structures, which is visualised as a bulge beneath the head of the vortex structure at $Re \leq 300$. Though the solution is steady, nevertheless drastic change of topology of the velocity field with increase of Re ($Re > 400$) gives rise to fully complicated deformation of streaky structures, which leads to enhancement of the fluid mixing between the boundaries. While the upper branch of HVS has recently attracted researcher's interest due to such a characterisation of its spatial structure (Generalis & Itano (2010), Deguchi & Nagata (2010)), the lower branch of HVS has been left out of the analysis of the HVS solutions.

Wang et al. (2007) verified numerically that the lower branch of NBW exists on the basin boundary (BB) of PCF, which is the boundary with codimension-1 separating the laminar and the turbulent attractors (basins) of PCF. They argued that the lower branch of NBW has the only unstable manifold, which escapes from the BB, and that it is an attractor of the dynamical system restricted on the BB at high Reynolds numbers. On the other hand, it remains open to verify whether the more primitive exact solution, HVS, satisfies the same condition. In the present study, focusing on the lower branch of the HVS, we will reveal its significance in PCF at moderate Reynolds number ($Re \sim 10^4$). To put it concretely, employing the shooting method, which was established originally as a tool in order to find out an unstable steady solution of subcritical shear flows (Itano & Toh (2001)), we will outline a structural aspect of phase space associated with laminar-turbulent transition, that is, the relation of the lower branch of HVS and the BB of PCF at moderate Reynolds numbers. The numerical result obtained by the shooting method suggests that HVS remains on the BB at moderate Reynolds number, and moreover that one of the unstable manifolds of HVS connects to NBW, which lies on the BB.

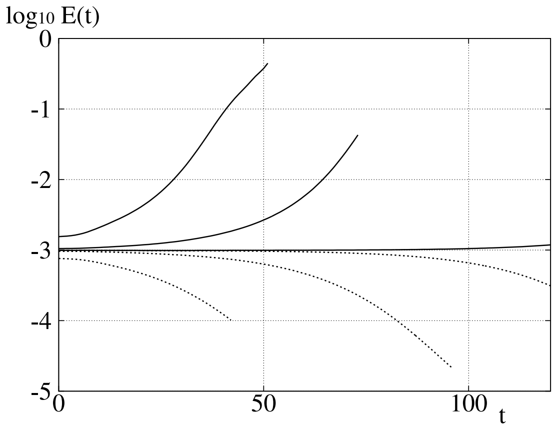

Figure 1. Trajectories starting around $u_{\rm H}$ separates towards either towards the laminar (dotted) or the turbulent (solid) states due to a little additive perturbation to the initial state, where the additive perturbation satisfies the reflection symmetry the HVS does. $E(t)$ is the half of norm of the wall-normal component of the perturbation at time t, which is nondimensionalised by U and h.

COMPUTATIONAL METHOD

A canonical turbulent shear flow, PCF, has two attractors in the phase space corresponding to the laminar and the turbulent states, both of which are kept to be attractors even in high Reynolds numbers, which is partly proven by the linear stability analysis. In general, such a bistable system as PCF, the BB (boundary of basin of attraction) can be defined. The BB is a singular set consisting of the (unstable) points in phase space which are initial conditions falling into neither of these attractors within the infinite time interval. Therefore, it requires a formidable task to fully identify the BB. On the other hand, the BB is a key to understand the laminar-turbulent transition, as it is the theoretically exact criterion to predict the critical perturbation arising transition.

By the following procedure in case that a few assumptions are satisfied, the BB may emerge partly in the phase space. Suppose that a distance between two states in phase space is well defined, and also that the time interval which it takes for a state starting from an arbitrary initial condition to approach to either of these attractors is certainly shorter than an sufficiently long characteristic time interval, T. Then, the BB is numerically identified just as an assembly of the initial conditions which approach neither to the laminar state nor to the turbulent state in T. If we refer to a state of flow at the time t as $u(t)$, which is governed by $\frac{du}{dt} = F(u,t)$, then

$$|u(t) - u_{\rm L}| > \delta_{\rm L} \quad \text{and} \quad |u(t) - u_{\rm T}| > \delta_{\rm T} \quad \text{for} \quad 0 \le t < T \\ \rightarrow \quad u(0) \quad \text{on BB}$$

under a given finite distance $\delta_i > 0$ for each attractor u_i, where the index i indicates either the laminar (L) or the turbulent (T) states. To figure out completely the BB of PCF even based on the above definition requires to solve trajectories starting from an infinite number of initial conditions in infinite dimensional phase space under the full nonlinear governing equation, which is still beyond the current standard computer ability, unfortunately.

Additionally, it should be noted that the BB is an unstable singular set in phase space as itself, on which there moreover exist singular exact solutions, so called "edge states". (It could take infinite time to reach them though.) An edge state may be referred to as an attractor of the dynamical system restricted on the BB. Hereby, when one tries to find such a point on the BB as to satisfy the above definition by carrying out the simulation starting from initial conditions on the line intersecting the BB based on a bisection scheme, he tends to obtain a part of the boundary which is close to the edge state. The procedure to trace the unstable boundary by the bisection scheme is performed as if a golf player tries to win a hole, which may be an edge state, on the tip of a swelled-up mound by a putter, so it was termed originally as "shooting method" (Itano & Toh (2001), Schneider et al. (2008)). Although edge states are no more than a part of the BB, the BB may be constituted by several stable or unstable manifolds of these singular exact states as if a tent is spanned by a frame of poles and a supporting rope.

RESULTS

Following the earlier work by Wang et al. (2007), which verified that the lower branch of NBW exists as an edge state on the BB of PCF, we investigate a structural aspect of the BB around HVS, the more primitive solution than NBW, by the shooting method. We take a plane in phase space, on which the three distinct exact steady solutions of PCF lie, the laminar state, the lower branch of HVS, the lower branch of NBW. The associated states are respectively depicted as $u_{\rm L}$, $u_{\rm H}$, and $u_{\rm N}$, hereafter. Using a couple of parameters (a, b), a state $u(0)$ on the plane is expressed as superposition of these solutions as fol-

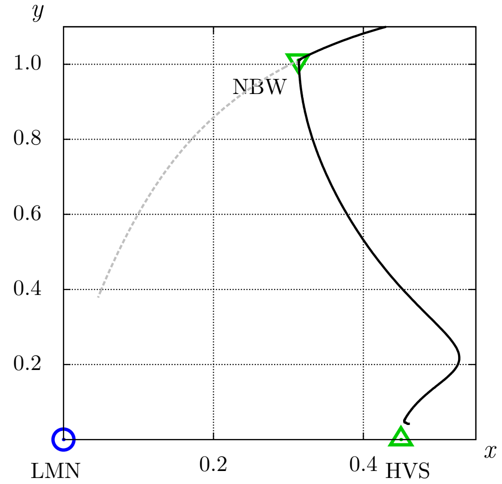

Figure 2. Trajectories starting around $u_{\rm H}$ separates towards either towards the laminar or the turbulent states due to a little additive perturbation to the initial state, via NBW, which are obtained by the shooting method at $Re = 300$. The initial state consists of $u_{\rm H}$ with the little perturbation, which does not satisfy the reflection symmetry.

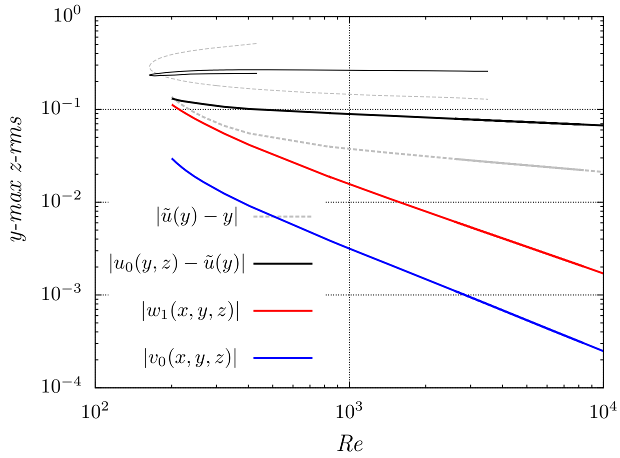

Figure 3. Amplitude of Fourier modes for HVS in PCF are plotted against Re for $(L_x, L_z) = (\pi, 2\pi)$. Streak $u_0(y,z) - \bar{u}(y)$ is indicated as order 1 for the reference, which was verified in Wang et al. (2007). All the amplitudes of HVS decreases with increase of Re. This fact suggests the HVS, which could bifurcate from infinity, may be a criterion of transition.

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$$u(0) = bu_{\rm L} + (1-b)u' \ , \quad u' = (1-a)u_{\rm H} + au_{\rm N} \ .$$

It should be noted that $u(0)$ satisfies the incompressible condition, but is not necessarily equivalent to a steady state of the governing equation. By adopting the state $u(0)$ as an initial state, a number of trial calculations of time development of the state based on the shooting method are carried out to outline a part of the BB. Firstly, here fixing $a = 0$, we demonstrates the shooting method (Fig. 1). It is shown that the final state $u(t)$ goes down to $u_{\rm L}$ in case of $b < 0$ but it develops to some turbulent state in case of $b > 0$, which provides an evidence that the HVS is on the BB.

Hereafter, let us plot trajectories in the phase space on the x-y plane as follows. The distance between two different flow states is defined as the half of norm of the difference of the associated velocity field,

$$\Delta E(u_1, u_2) = \int_V |u_1 - u_2|^2 dv,$$

where V is the total volume of channel. For instance, the distance of HVS from the laminar state is measured as $\sqrt{\Delta E(u_{\rm H}, u_{\rm L})}$. If one plots the laminar state at the origin, and the HVS on the x axis with the distance, then, in a natural sense, NBW may be plotted on the x-y plane so as to be consistent with the corresponding distances from the laminar state and the HVS. NBW bifurcated from HVS with breaking a symmetry, so another reflected image of NBW would be plotted at the half plane $y < 0$. Generally speaking, since the original phase space has much higher dimensions than that of the x-y plane shown here, any other state of flow cannot be plotted with keeping its identity in the projection.

With a variety of a couple of values (a, b), a plenty of trial calculations are carried out to outline a part of the BB. In Fig.2, we plot only a could of trajectories starting from two different initial states using b with a subtle difference ($a = 0.04$ in both cases), on the plot. This implies that the BB must exist between two trajectories; one of the unstable manifolds of HVS heads towards NBW, which lies on the BB. This shooting method carried out at $Re = 300$, but even if Reynolds number is increased $Re \sim 1000$, we could obtain similar behaviour as shown on the same plot.

According to Wang et al. (2007), again, it was shown that the NBW remains a streak structure at high Reynolds numbers. Here, the streak structure of NBW is referred to as streamwise velocity component with a finite amplitude perturbation $\bar{u}(y)$ from the laminar state y, which is maintained by a vortex sheet of NBW at high Reynolds numbers. This vortex sheet is a kind of critical layer getting to be thinner as increase of Reynolds number. This fact means that the distance of NBW from the laminar state, $\Delta E(u_{\rm N}, u_{\rm L})$, is not vanishing but kept finite at the infinite Reynolds limit. It is also supported even by another fact that NBW is a derivative of the HVS, which bifurcates from the laminar state via secondary flow (Itano & Generalis (2009)). Unless the bifurcation point of the derivative branch (NBW) becomes that of the primitive branch (HVS), which happens hardly in a natural sense, NBW does not connects directly to the laminar state at the infinite Reynolds number. These two facts imply the NBW does "not bifurcate from infinity", on the contrary of the title of the first report on the exact solution in PCF by Nagata (1990), which was also previously suggested by Wang et al. (2007).

In contrast with NBW, as has been shown in Generalis & Itano (2010), the bifurcation point of HVS (tertiary branch) from the secondary branch is quite close to that of the secondary branch from the laminar state. Additionally, we performed the tracing of HVS here up to as high Reynolds number as possible, and showed the several representative components of the HVS to compare it to that of NBW in Fig.3. For reference with Fig.1 of Wang et al. (2007), the streamwise component of perturbation $\bar{u}(y) - y$ is plotted in the Fig.3 for the both states, NBW and HVS. The figure indicates that HVS asymptotically approaches to the laminar state, as it were, bifurcates from the laminar state at the infinite Reynolds number. Thus, the lower branch of HVS, rather than that of NBW, gives us a more practical criteria of laminar-turbulent transition of PCF at the relatively high Reynolds number, which provides the importance of HVS in the laminar-turbulent transition of PCF at the moderate Reynolds number. The comparison with result obtained by Duguet et al. (2010) still remains open for the time being.

CONCLUDING REMARKS

We will summarise briefly our results. Firstly, the lower branch of HVS is on the BB at a higher Reynolds number. Secondly, while one of unstable manifolds of HVS connects to the laminar state, another unstable manifold of HVS connects to the lower branch NBW, that is, a hetero-clinic orbit of these solutions constitutes the BB. Thirdly, NBW is a robust attractor in BB even at high Reynolds numbers, which is a conclusion given by Wang et al. (2007). Fourthly, HVS asymptotically approaches to the laminar state.

Taking all four results into account, one would reach a scenario of typical turbulent transition from the dynamical view of point. In experiments of the turbulent transition, the norm of disturbance in the stream is annihilated at the initial stage. In case that the flow occurs transition, though, the adopted smallest perturbation necessarily satisfies the reflection symmetry, which is inferred from the first and the forth results. From the second result, we can concludes that the trajectory tends to approaches towards NBW along the hetero-clinic orbit on BB. In the downstream, NBW-type structures are prevalent, while HVS is observed rarely, which is suggested from the third result. This scenario gives us a clue to the question, why the hairpin-like structure with spanwise reflection symmetry is often observed in experiments of the turbulent transition, where all the perturbation from the laminar state are kept to be as small as possible, while meandering streaky structure like NBW is ubiquitous in fully developed turbulent at further downstream of the experiments (cf.Schlatter et al. (2011)).

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