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IS IT POSSIBLE TO DETERMINE SELF-SIMILARITY IN ISOTROPIC TURBULENCE BY THE OBSERVATION OF THE DECAY REGIME CHARACTERISTICS?

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ABSTRACT

The time evolution of initially non-self-similar regimes in isotropic turbulence decay is investigated by both theoretical analysis and EDQNM simulations. The breakdown of self-similarity is recovered by the analysis of a three-range energy spectrum, with two different slopes at scales larger than the integral length scale.

The results of the analysis indicate that, depending on the initial conditions, the solution can bifurcates toward a true self-similar decay regime, or sustain a lasting non-self-similar state. These non-self-similar regimes can not be detected restricting the observation to the time exponent of turbulence statistical properties such as the turbulent kinetic energy or the energy dissipation rate. In fact, it is shown that the decay of the physical quantities is governed by the large scales close to the energy spectrum peak only. In particular, the shape of the energy spectrum near its peak, which may be related to the turbulence production mechanisms, is of pivotal importance. As a conclusion, information about the very large scales of the energy spectrum can not be derived from the observation of the decay regimes characteristics, as those scales have a negligible impact over energy transfer.

Another relevant result is that classical self-similarity theories, which link the asymptotic behaviour the energy spectrum $E(k \to 0)$ and the turbulence decay exponent, are not fully relevant when the large scale spectrum shape exhibits more than one range.

INTRODUCTION

In this work, the analysis of self-similar regimes in incompressible homogeneous isotropic turbulence (HIT) free decay is addressed.

This subject has been extensively analysed in open literature, starting with the seminal studies by Taylor (1935), which were followed by comprehensive reviews (see Batchelor (1953), Hinze (1975), Monin & Yaglom (1975), Davidson (2004), Lesieur (2008) and Sagaut & Cambon (2008)). A number of aspects related to HIT decay, though, is not fully understood at the present time. Among these, the observation of self-similarity is one of the most debated issues. Self-similar solutions in HIT are classically defined as regimes that can be described by the use of a single length scale $l(t)$ and a single velocity scale $u(t)$. This assumption implies that the three-dimensional energy spectrum must comply with the relation $E(k,t) = u^2(t)l(t)F(kl(t))$, where F and k denote a dimensionless shape function and the wavenumber, respectively.

Theoretical analyses (Comte-Bellot & Corrsin, 1966; Lesieur & Schertzer, 1978; George, 1992; Speziale & Bernard, 1992), have shown that the evolution laws of HIT statistical quantities, such as the turbulent kinetic energy $u^2(t)$, the dissipation rate $\varepsilon(t)$ and the integral scale $l(t)$ follow a power-law behaviour. In particular, a relation between the decay exponent and the energy distribution at very large scales is observed, i.e the decay exponent n_{u^2} can be expressed as a function of the slope of the energy spectrum at very small wave numbers, σ.

The two classical values studied in the literature are $\sigma = 2$ and $\sigma = 4$, such that $E(k \to 0) \propto k^2$ and $E(k \to 0) \propto k^4$. The former is related to conservation of linear momentum and, as a consequence, the Birkhoff-Saffman invariant $L = \int \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}+\mathbf{r}) \rangle \, d\mathbf{r} \propto u^2(t) l^3(t)$ (Saffman, 1967). This regime and is usually referred to as Saffman turbulence. The second is associated with the conservation of angular momentum and the Loitsyansky integral $I = \int \mathbf{r}^2 \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}+\mathbf{r}) \rangle d\mathbf{r} \propto u^2(t) l^5(t)$, and is referred to as Batchelor turbulence.

While these two regimes have been extensively analysed in the literature, the question arises for the physical relevance of other values for σ. This issue derives from the fact that, assuming the regularity of $E(k)$ at $k = 0$, the following Taylor series expansion holds $E(k \to 0) = \frac{L}{4\pi^2} k^2 + \frac{I}{24\pi^2} k^4 + \ldots$. Physical and mathematical arguments indicate $1 \leq \sigma \leq 4$, but there is no proof at present time on the occurrence of turbulent solutions with arbitrary real values of σ within the range $[1,4]$.

The existence of self-similar solution in HIT is indicated by Lie-group symmetry analysis (Clark & Zemach (1998); Oberlack (2002)) and observed in experimental results, Direct Numerical Simulations and spectral closure based solutions (e.g. EDQNM results in Lesieur (2008); Sagaut & Cambon (2008)). At the same time, no theoretical argument precludes the existence of other decay regimes which include more independent length scales. Skrbek & Stalp (2000) introduced a cutoff length scale to account for saturation effects due to the finite size of physical domain, while three-range composite energy spectra have been introduced in the works by Frenkel & Levich (1983); Frenkel (1984); Eyink & Thomson (2000); Llor (2011).

A complete theoretical analysis of such three-range solutions is presented in this work, including a possible breakdown of the Permanence of Large Eddies. The theoretical predictions are then compared to numerical results. An Eddy-Damped Quasi-Normal Markovian (EDQNM) model is selected to obtain an accurate investigation of very-high Reynolds number cases with excellent spectral accuracy, for very long evolution times.

The paper is structured as follows. In Section II, the theoretical framework based on a composite three-range energy spectrum is developed and commented. In Section III, the EDQNM model is briefly introduced and the set up of the numerical simulations is described. In Section IV, information about the emergence of pseudo-self-similar regimes and lasting non-self-similar regimes is recovered comparing theoretical and numerical results. In Section V, the conclusions are drawn.

THEORETICAL ANALYSIS FOR COMPOSITE THREE-RANGE SPECTRUM

In this Section, the evolution of an initially non-self-similar regime is investigated by the development a theoretical model, starting from a composite three-range energy spectrum. This analysis allows to investigate the sensitivity of the decay regime to the characteristics of the energy spectrum at the large scales. The composite energy spectrum, which has been addressed in the works by Frenkel & Levich (1983); Frenkel (1984); Llor (2011) is defined by the existence of two ranges at large scales. Thus, the energy spectrum is described by three regions, each of them being defined by a different analytical expression. More precisely, the energy spectrum is given by $E(k) = Ak^{\sigma_1}$ in the very large scale region $0 \leq k \leq k_1$. The large scales region, which is observed in the range $k_1 \leq k \leq k_2$, is instead characterised by a parameter σ_2 so that $E(k) = Ak^{\sigma_2}$. At last, the composite spectrum exhibits an inertial region $k \geq k_2$, in which the inertial range by Kolmogorov (1941) is observed. The two wavenumbers k_1 and k_2, which delimit the transition from one region to another, are linked to two independent length scales. These scales will be referred to as $l_1 = 1/k_1$ and $l_2 = 1/k_2$, respectively. The latter is actually the integral length scale (l in the classical two-range cases discussed in Comte-Bellot & Corrsin (1966)) of the flow, which is associated to the peak of the energy spectrum.

The full analytical functional form is:

$$E(k) = \begin{cases} A\,k^{\sigma_1} & kl_1 \ll 1 \\ B\,k^{\sigma_2} & kl_1 \gg 1,\, kl_2 \ll 1 \\ C_k \varepsilon^{2/3} k^{-5/3} & kl_2 \gg 1 \end{cases} \tag{1}$$

which is completed by the relations

$$A\,l_1^{-\sigma_1} = B\,l_1^{-\sigma_2} \quad \text{or} \quad A\,k_1^{\sigma_1} = B\,k_1^{\sigma_2} \tag{2}$$

$$B\,l_2^{-\sigma_2} = C_k \varepsilon^{2/3} l_2^{5/3} \quad \text{or} \quad B\,k_2^{\sigma_2} = C_k \varepsilon^{2/3} k_2^{-5/3} \tag{3}$$

Equations 2 and 3 grant the continuity of the energy spectrum at the $k_1 = 1/l_1$ and $k_2 = 1/l_2$. Meldi & Sagaut (2012) showed that the coefficients A and B can be represented as $A(t) \propto l_1^{p_1}(t)$, $B(t) \propto l_2^{p_2}(t)$, where $p_i = max[0,\, 0.65\,(\sigma_i - 3.2)]$.

Let us now considered the well-known relations:

$$\frac{\partial u^2}{\partial t} \propto -\varepsilon, \qquad u^2(t) = \frac{1}{2}\int_0^{+\infty} E(k,t)dk \tag{4}$$

Integrating the energy spectrum illustrated in Equation 1, we recover:

$$u^2 = \frac{1}{2}\left[\frac{A}{\sigma_1+1}k^{\sigma_1+1}\right]_0^{1/l_1} + \frac{1}{2}\left[\frac{B}{\sigma_2+1}k^{\sigma_2+1}\right]_{1/l_1}^{1/l_2} + \frac{1}{2}\left[\frac{-3C_k}{2}\,\varepsilon^{2/3}\,k^{-2/3}\right]_{1/l_2}^{+\infty} \tag{5}$$

Manipulating Equation 5 and using the two continuity Equations 2 - 3, the resulting equation is:

$$u_e^2 = \left(\frac{3\sigma_2+5}{4(\sigma_2+1)} C_k^{\frac{3+3\sigma_2}{5+3\sigma_2}} B^{\frac{2}{5+3\sigma_2}}\right) \varepsilon^{\frac{2(\sigma_2+1)}{5+3\sigma_2}} \tag{6}$$

where the *decaying energy* u_e^2 is:

$$u_e^2 = u^2 - \frac{\sigma_2-\sigma_1}{2(\sigma_1+1)(\sigma_2+1)} B^{2/3} \left(\frac{A}{B}\right)^{\frac{\sigma_2+1}{\sigma_2-\sigma_1}} \tag{7}$$

This new quantity is the energy that the system would have in the case of a CBC starting energy spectrum with $\sigma_1 = \sigma_2$. The analysis of Equation 6 shows that the decay law depends on σ_2 only. Conversely, the parameter σ_1 appears in the definition of u_e^2. Combining Equation 4 and Equation 6, the decay law recovered is:

$$u_e^2(t) = E\,(t-t_1)^{-\frac{2(\sigma_2+1)}{\sigma_2+3}} \tag{8}$$

This formula has been as well recovered by Comte-Bellot & Corrsin (1966) starting from a two-range energy spectrum. Even if the spectrum exhibits two independent length scales at low wave number, the resulting decay regime is similar to a classical self-similar solution. Therefore, such a state will be referred to as a pseudo-self-similar state.

A striking observation is that the power law coefficient in Equation 8 is governed by σ_2 only, which is related to the shape of the energy spectrum at the energetic scales close to the peak. Conversely σ_1, which is tied to the characteristics of E at the very largest scales, simply modifies the definition of u_e^2. These considerations are in agreement with the numerical result observed by Meldi *et al.* (2011), which recovered a sensitivity of the power law coefficients n_Q when

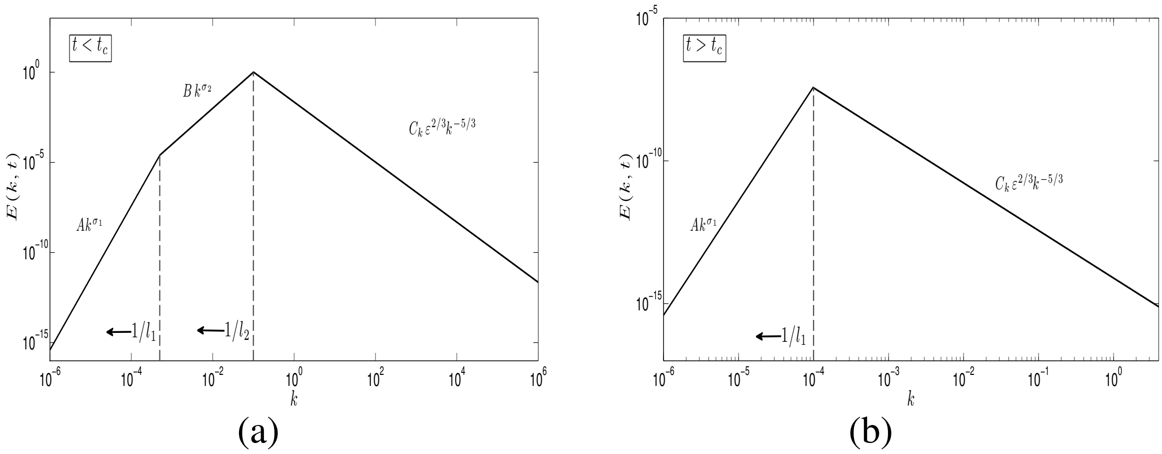

Figure 1. Scheme of the time evolution of the composite three-range energy spectrum (a) before the critical time t_c and (b) after the critical time t_c.

considering uncertainties in the shape of the energy spectrum in correspondence of its peak.

Equation 8 is valid for the early stages of the HIT decay, for which the relation $l_1 \gg l_2$ holds. As a matter of fact, the two length scales l_1 and l_2 evolve in time with different power laws and they can possibly become identical at a critical time t_c. A two-range energy spectrum will be finally observed for $t > t_c$ if the scale l_2 grows faster than l_1, as exemplified in Figure 1. Known the values of σ_1 and σ_2, the use of the CBC formulae along with the continuity Equations 2 and 3 allows to recover a theoretical estimation of the power-law exponents related to l_1 and l_2. In order to have a finite critical time t_c, the following condition must hold:

$$\frac{2p_2}{(\sigma_2-\sigma_1+p_1)(\sigma_2-p_2+3)} < \frac{2}{(\sigma_2-p_2+3)} \tag{9}$$

Through some algebraic manipulation, it is possible to derive as well an estimation of t_c by the formula:

$$t_c = (l_2(0)/l_1(0))^{\alpha_c}\,, \quad \alpha_c = \frac{1}{2}\frac{(\sigma_2-\sigma_1+p_1)(\sigma_2-p_2+3)}{(\sigma_1-p_1)-(\sigma_2-p_2)} \tag{10}$$

The power-law exponent for the pseudo-self-similar regimes with finite t_c are summarised in Table 1 , for three different statistical properties of interest. The transition between the two states will be smooth, since when l_1 and l_2 are close enough scales belonging to both ranges are involved in the energy cascade process. Conversely, if Equation 9 does not hold, the critical time will diverge to infinity, leading to a permanent non-self-similar regime.

EDQNM MODEL & SET UP OF THE SIMULATIONS

The theoretical work developed in Section II will be compared with numerical results recovered by the use of the EDQNM model. A brief description of this model is given in this Section, the reader being referred to Orszag (1970), Lesieur (2008) and Sagaut & Cambon (2008) for an exhaustive discussion.

The EDQNM model is a quasi-normal closure based on the numerical discretisation in the spectral space of the Lin equation:

$$\frac{\partial E(k,t)}{\partial t} + 2\nu k^2 E(k,t) = T(k,t) \tag{11}$$

where $T(k,t)$ is the non linear energy transfer. The model is derived assuming that the statistical moments of velocity field can be correctly evaluated closing the corresponding dynamic equations in wave-number space. The closure takes into account the effects of fourth-order and higher moments by the use of a linear eddy-damping term on non-linear energy transfer. This model is able to accurately predict triadic energy transfer in HIT decay and it has been successfully used to investigate both the statistical properties of the energy spectrum (Meldi & Sagaut, 2012; Tchoufag *et al.*, 2012) and the fluctuations in the pressure spectrum (Meldi & Sagaut, 2013).

The numerical implementation of the EDQNM model is based on a logarithmic discretisation in wavenumber space and the three-points velocity correlations are computed on the elements of the triads $[k, p, q]$. This wavenumber discretisation becomes progressively less efficient when non-local very elongated triads $k \ll p \sim q$ are considered. This drawback can have a non-negligible impact on the accuracy of the model, in particular if the two-points velocity correlation decays fast. Therefore, the original model proposed by Orszag (1970) is extended by the addition of a non-local transfer term, which exactly determines the non-local triadic interactions (Lesieur, 2008).

	n_{u^2}	n_l	n_ε
$t<t_c$	$2\frac{\sigma_2-p_2+1}{\sigma_2-p_2+3}$	$\frac{2}{\sigma_2-p_2+3}$	$\frac{3(\sigma_2-p_2)+5}{\sigma_2-p_2+3}$
$t>t_c$	$2\frac{\sigma_1-p_1+1}{\sigma_1-p_1+3}$	$\frac{2}{\sigma_1-p_1+3}$	$\frac{3(\sigma_1-p_1)+5}{\sigma_1-p_1+3}$

Table 1. Theoretical formulae derived from the analysis of the three-range energy spectrum, in the case of a finite value of the critical time t_c.

The extended model, which has been assessed for both Saffman and Batchelor turbulence by the comparison with the original model and theoretical results, exhibits significant improvement in the prediction of the slope of the energy spectrum.

A database of EDQNM simulations is created for a number of combinations of the parameters $[\sigma_1, \sigma_2]$, imposing a composite three-range initial energy spectrum at $Re_\lambda(0) = 10^5$. The initial energy spectrum is a simplified version of the functional form proposed by Pope (2000):

$$E(k) = \begin{cases} A\,k^{\sigma_1} & kl_1 \ll 1 \\ C_k\,\varepsilon^{2/3}k^{-5/3}f_l(kl_2) & kl_1 \gg 1 \end{cases} \tag{12}$$

with

$$f_l(kl_2) = \left(\frac{kl_2}{[(kl_2)^\alpha + \beta]^{1/\alpha}}\right)^{5/3+\sigma_2} \tag{13}$$

This functional form has been chosen in order to connect smoothly the large scales range with the Kolmogorov inertial range. The smoothness of this transition is governed by the parameter α, which is set to $\alpha = 1.5$. The parameter β is chosen to recover the initial condition $l_2(0) = 1$ for all the cases investigated. The results are referred to the normalised time scale $\tau = t/t_0$, $t_0 = u^2(0)/\varepsilon(0)$. The length scales l_1 and l_2 are initially separated by 3 decades, i.e. $l_1(0) = 10^3\,l_2(0)$. The large scales maximum resolution l_0 is chosen so that $l_0 = 10^6\,l_1(0)$. On the other hand, the minimum resolution is set to $l_N = 0.1\,\eta(0)$, where η is the Kolmogorov scale. This very high resolution implies that about 17 decades in the spectral space have been considered for three-range spectrum cases. This space has been discretised in 290 modes.

In order to preclude possible corruption of the results by spurious saturation/confinement and low-Reynolds number effects, it was checked that at final time t_f the integral scale complies with the relation $l(t_f) < 300\,l_0$ and that $Re_\lambda \geq 170$, for all the simulations performed.

EDQNM RESULTS FOR FINITE AND INFINITE CRITICAL TIME

In this Section, EDQNM results starting from a composite three-range spectra are discussed and compared with the theoretical model derived in Section II. We first consider $[\sigma_1, \sigma_2]$ values which lead to a finite t_c. More specifically, the case $\sigma_1 = 3$ and $\sigma_2 = 2$ is selected. Using Equation 10, we can expect that the transition between the two regimes will occur at a critical time $t_c = 10^{7.5}t_0$.

The numerical results recovered show an agreement with the theoretical background proposed. To exemplify that, the time evolution of the energy spectrum is reported in Figure 2 (a). It is possible to observe that the length scale l_1 does not vary in time as the turbulent flow decays and the three-range-shape of E is conserved until the magnitudes of the two length scales l_2 and l_1 are of the same order.

This case corresponds to the transition from a pseudo-self-similar regime ($t < t_c$) to a self-similar regime ($t > t_c$).

The agreement with the theoretical analysis is observed as well if the power law exponents of the main HIT statistical quantities are considered, as shown in Figure 2 (b) for u^2. The transition between the two regimes is mainly smooth, even if a kink is locally observed before the regime governed by σ_1 is fully established. Moreover, it is possible to observe that the estimation of t_c by the analytical Equation 10, which is represented as a vertical dash-dot line, is in very good agreement with the EDQNM results.

An interesting result is that during the first evolution time, the spectrum is defined by two independent length scales, and therefore the solution is not self-similar. This can be clearly observed in Figure 3, where the scaling law proposed by Clark & Zemach (1998) is applied to the composite three-range spectra and to a classical case of Saffman turbulence. Nevertheless, the decay of the statistical properties is identical to a self-similar regime for a classical two-range spectrum with slope σ_2 at very large scales. Therefore, the decay regime is governed by the large scales located near the energy spectrum peak and the features of $E(k)$ for $k \simeq 1/l_2$, and not by asymptotically large scales $k \to 0$.

The case of an infinite critical time t_c is now addressed. This case corresponds to a permanent non-self-similar behaviour, as two different length scales exist at all times. To observe this regime, we enforce as initial condition $\sigma_1 = 4$ and $\sigma_2 = 3.7$.

The energy spectrum evolution, which is shown in Figure 4 (a), indicates that the slope of the energy spectrum changes in time. Moreover, the energy spectrum between l_1 and l_2 gradually assumes an intermediate slope between the two enforced initial values. This means that the regime will not asymptotically converge to the theoretical self-similar regime associated to σ_1. A confirmation is given by the observation of the predicted power-law coefficient for u^2, which is reported in Figure 4 (b). In fact, n_{u^2} is close to the theoretical value associated to the regime $\sigma_2 = 3.7$ after the transient has faded. The predicted power law coefficient then evolves in time, reaching a value which is included in the range of the regimes associated to $\sigma_2 = 3.7$ and $\sigma_1 = 4$. In this case the turbulent decay, even if not self similar, can be roughly approximated as being self similar, and governed by a parameter σ_∞ which is estimated to be in the range $\sigma_2 < \sigma_\infty < \sigma_1$.

CONCLUSIONS

The initial conditions leading to the breakdown of self-similarity in HIT decay at high Re_λ number have been investigated by the use of a simplified theoretical model and EDQNM simulations. For both the approaches, and initial composite three-range energy spectrum has been considered.

The results indicate that if the two initial parameters σ_1 and σ_2 are chosen so that the critical time t_c is finite, two different regimes are observed in HIT long time decay. Two dynamically active independent length scales are observed during the first regime, which implies a breakdown of the classical definition of self-similarity. Nevertheless, the time evolution of the HIT statistical quantities is identical to a self-similar solution of a classical two-range spectrum, whose large-scale shape is equal to the parameter σ_2 related to the energetic large scales in the composite spectrum. Therefore, this regime can be considered as a pseudo-self-similar regime. For $t > t_c(\sigma_1, \sigma_2, l_1(0)/l_2(0))$, this regime turns into a classical self-similar decay.

In the case of infinite critical time, a permanent non-self-similar regime is triggered. Moreover, the decay

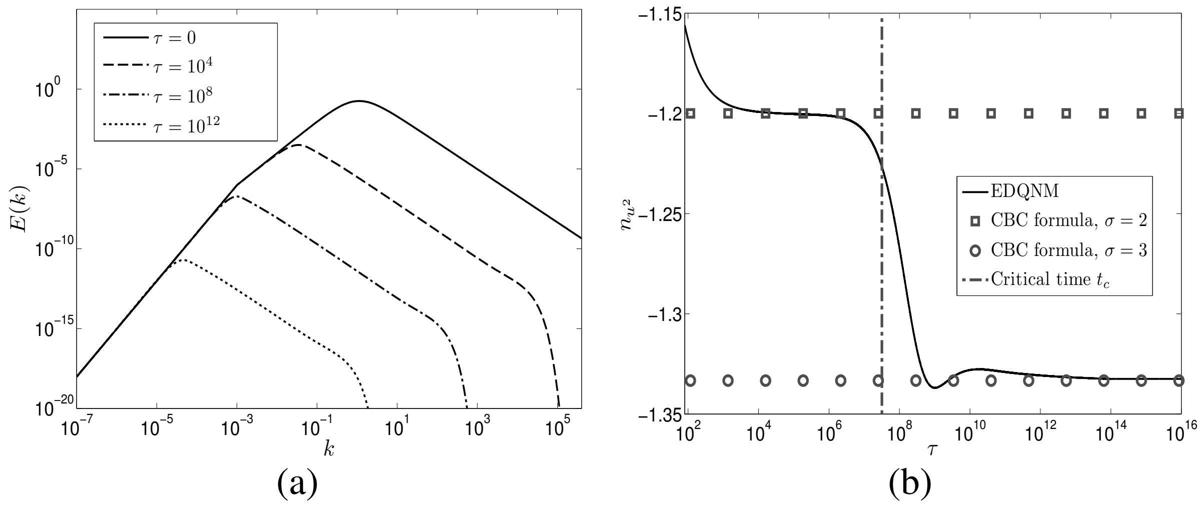

Figure 2. EDQNM results for (a) the decay of the energy spectrum $E(k,t)$ and (b) the time evolution of the power law exponent n_{u^2}, considering a composite three-range initial energy spectrum at $Re_\lambda(0) = 10^5$. The case of finite critical time, with $\sigma_1 = 3$ and $\sigma_2 = 2$, is investigated.

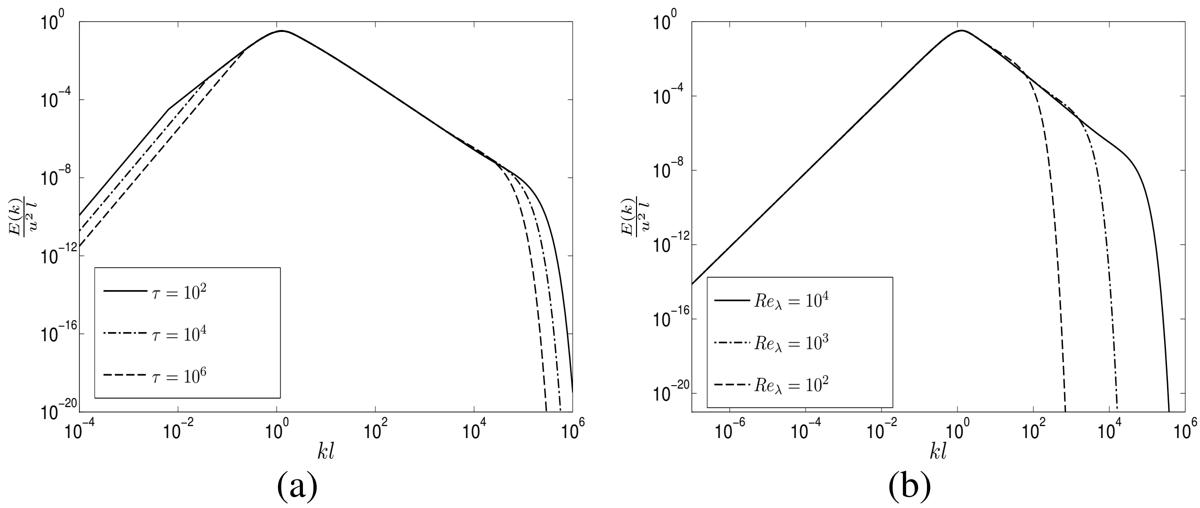

Figure 3. Normalised spectra by the scaling law based on the integral length scale l (Clark & Zemach, 1998). The case of (a) composite three-range initial energy spectrum ($\sigma_1 = 3$ and $\sigma_2 = 2$) and (b) classical Saffman turbulence decay are reported.

regime will behave *as if* it was self similar and driven by a parameter $\sigma_\infty \in [\sigma_{min(1,2)}, \sigma_{max(1,2)}]$.

The most important conclusion that can be drawn by the present work is that the decay rate of kinetic energy, as well as for the other statistical quantities, is not tied to the asymptotic behaviour of large scales, i.e. $E(k \to 0)$ in the general case of a three-range initial energy spectrum. Both theoretical analysis and detailed investigation of non-linear transfers by the EDQNM model show that the very large scales are not active, in the sense that their associated energy transfers are almost negligible. Conversely, the large scales close to the peak of the energy spectrum are of major importance in the energy cascade. Moreover, the results indicate that it is not possible to deduce information about the shape of the spectrum at large scales, and in particular about self-similarity, by the sole analysis of the decay of HIT statistical quantities.

From the physical point of view, the dependency on detailed features of the spectrum at large scale deserves certainly further investigation, and may, at least partially, explain the observed discrepancies between experimental data and theoretical predictions, since the energy peak detailed features may be intimately related to the turbulence production mechanisms, which are not universal.

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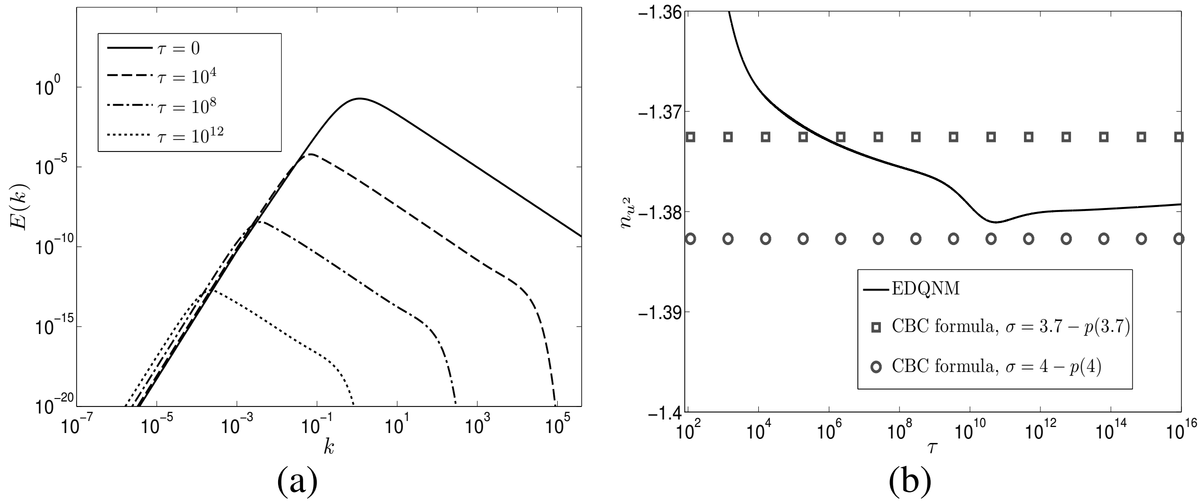

Figure 4. EDQNM results for (a) the decay of the energy spectrum $E(k,t)$ and (b) the time evolution of the power law exponent n_{u^2}, considering a composite three-range initial energy spectrum at $Re_\lambda(0) = 10^5$. The case of infinite critical time, with $\sigma_1 = 4$ and $\sigma_2 = 3.7$, is investigated.

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