ABSTRACT

To generalize the well-known spanwise-oscillating-wall technique for drag reduction, non-sinusoidal oscillations of a solid wall are considered as a means to alter the skin-friction drag in a turbulent channel flow. A series of Direct Numerical Simulations is conducted to evaluate the control performance of nine different waveforms, in addition to the usual sinusoid, systematically changing the maximum wave amplitude and the period for each waveform. The turbulent average spanwise motion is found to coincide with the laminar Stokes solution that can be constructed, for the generic waveform, through harmonic superposition. A newly defined penetration depth of the Stokes layer is then used to build a simple tool that allows predicting turbulent drag reduction and net energy saving rate for any waveform.

Among all the cases considered, the sinusoid at optimal amplitude and period is found to yield the maximum net energy saving rate. However, when the wave amplitude and period deviate from the optimal values, other waves are found to perform better than the sinusoid. This is potentially interesting in view of applications, where a particular actuator limitations might preclude reaching the optimal operating conditions for the sinusoidal wall oscillation. It is demonstrated that the present model can predict the locally optimal waveform for given wave amplitude and period, as well as the globally optimal sinusoidal wave.

INTRODUCTION

The efficient use of energy in systems where a relative motion between a solid wall and a fluid takes place is perhaps the most important driving factor that supports the current research effort into aerodynamic drag reduction. We consider here skin-friction turbulent drag. Existing open-loop techniques provide higher drag reduction than passive methods while being less complex than feedback-control methods. In particular, open-loop techniques that rely on the spanwise forcing of the near-wall turbulent flow have been shown to yield large drag reduction and interestingly positive energy budgets in numerical simulations (Quadrio, 2011), and first laboratory experiments have already been carried out (Auteri et al., 2010; Gouder, 2011; Choi et al., 2011). The present paper deals with the simplest and well-known spanwise oscillating-wall technique.

As a starting point, we select a set of waveforms and comparatively study, via several numerical experiments, how the drag-reduction and energetic performances of the oscillating wall depend on the waveform as well as on the oscillation amplitude and period. Guided by our numerical experiments, we then aim at obtaining results of more general validity, so that a predictive tool for the control performance of non-sinusoidal wall oscillations can eventually be developed. In this process, we take advantage of the laminar solution that exists for the spanwise flow alone (the Stokes oscillating boundary layer), by extending it to a generic (periodic) temporal waveform.
THE NUMERICAL EXPERIMENTS

The performance of non-sinusoidal spanwise wall oscillations is assessed using Direct Numerical Simulation (DNS) of a turbulent channel flow. Discretization is spectral (Fourier) in the homogeneous $x$ and $z$ directions, whereas compact, explicit fourth-order finite differences are used in the wall-normal $y$ direction. Time advancement is carried out with a partially implicit, third-order Crank–Nicolson/Runge–Kutta scheme.

The Reynolds number of the reference simulation without wall oscillation is $Re = U_b h / \nu = 3173$, where $U_b$ is the bulk velocity, $h$ is half the channel gap and $\nu$ is the kinematic viscosity of the fluid. This corresponds to a value of the friction-based Reynolds number $Re_x = 200$. Unless otherwise specified, $h$ and $U_b$ are chosen as length and velocity scales. The computational domain is $9.6h \times 2h \times 4.8h$ along $x$, $y$ and $z$ directions, with $192 \times 128 \times 192$ grid points $l$ modes respectively. Every simulation is started from the same initial condition of fully developed turbulent channel flow without wall oscillation. When the wall moves, drag reduction takes place. Since the flow rate is kept constant, the space-averaged streamwise pressure gradient and the friction drag decrease. The total integration time for each simulation is 95 wash-out time units, where the wash-out time unit is defined as $L_w / U_b$. This value corresponds to 10,000 viscous time units. After the beginning of the oscillating movement of the walls at $t = 0$, a certain time interval is needed for the flow to reach the new equilibrium state. Time average is hence started only after the initial transient of the flow, by simply discarding the first 25% of the entire time integration interval.

Various waveforms of the temporal oscillation of the wall are examined. We consider spanwise wall velocities $W_w(t)$ varying in time as

$$W_w(t) = W_0 f_\alpha \left( \frac{t}{T} \right),$$

where $f_\alpha$ (with $\alpha = a, \ldots, j$) are ten periodic functions of unit period with values ranging from $-1$ to $1$. All the considered oscillations thus have period $T$ and amplitude $W_0$. It is worth mentioning that, as Quadrio & Ricco (2004) pointed out, in flow control with wall oscillation, a third parameter besides oscillation period and amplitude enters the picture, i.e. the maximum displacement of the wall during the oscillation cycle. If the wall oscillates sinusoidally in time, however, only two of these three parameters are truly independent, and the maximum displacement can be easily deduced once period and amplitude are known. Hence, considering different waveforms, we can open up the third dimension in the parameters space, and investigate the behavior of the oscillating wall when the temporal waveform is considered as a free parameter, and the constraint of sinusoidal oscillation is lifted. The maximum displacement of the wall from the reference position during the oscillation cycle is

$$D_m = \frac{2}{T} \int_{0}^{T/2} W_w(t) \, dt.$$  

The first waveform $f_a$ is the usual sinusoid

$$W_w(t) = W_0 \sin \left( 2\pi \frac{t}{T} \right).$$

for which $D_m = 2W_0 T / \pi$. The other waveforms are sketched in figure 1 (additional waveforms have been considered in this work but are not discussed here). Despite all the possible choices, the periodic functions $f_a$ which have been included in this study have been chosen as being representative of the different features which may characterize non-sinusoidal oscillations in practice. The waveforms $f_a$ allow for discontinuities in velocity and acceleration, large and small accelerations, phase shifts and different fractions of the period with constant velocity and even zero velocity.

For each waveform, the oscillation parameters $W_0$ and $T$ are varied around the values $T_{0}^+$ and $W_0^+$ (the superscript $+$ as customary implies non-dimensionalization in viscous units, where the velocity scale is the friction velocity $u_\tau$ of the reference flow) that yield the maximum net efficiency for the sinusoid. This particular condition was carefully determined by Quadrio & Ricco (2004) and corresponds to $T_{0}^+ = 125$ and $W_0^+ = 4.5$. We consider a parametric set of variations from this basic case, by changing $W_0^+$ and $T^+$ to values twice and one half of the optimal value.

FLOW CONTROL PERFORMANCE

The performance of the oscillating wall as a flow control technique is analyzed following Kasagi et al. (2009) in terms of three dimensionless indicators: the drag reduction rate $R$, the power input due to the applied control, $P_m$ and the net energy saving rate, $S = R - P_m$.

Figure 2 describes how $R$, $P_m$ and $S$ are affected by non-sinusoidal temporal waveforms, for two of the tested waveforms. Notwithstanding the marked quantitative diff-
Figure 2. Power consumption rate \( P_m \) (top), reduction of pumping power \( R \) (center), and net energy saving \( S \) (bottom) as a function of \( W_m \) and \( T^+ \) for two selected waveforms, (c) and (g), see figure 1 for reference. The color coding goes from maximum values, red, to minimum values, blue.

ferences among the considered waveforms, both \( P_m \) and \( R \) qualitatively behave in the \((T, W_m)\)-space like the one reported in Quadrio & Ricco (2004) for the sinusoidal case. The power consumption \( P_m \) for each value of \( T \) increases with \( W_m \). For constant \( W_m \), \( P_m \) decreases with increasing \( T \). The drag reduction rate, \( R \), always presents its maximum at the intermediate period \( T^+ = 125 \), and it increases monotonically with increasing \( W_m \). Overall, the specific waveform \( f_A \) enters the picture by affecting the quantitative values of \( P_m \) and \( R \). For example, waveform \( (b) \) (not shown) leads to much larger \( P_m \) than the sinusoid while yielding a larger \( R \) than the sinusoid. However, the increase in \( P_m \) outweighs the one in \( R \) such that the net saving, \( S \), is reduced in comparison to the sinusoid. Decreasing values for \( P_m \) and \( R \) are observed for the cusp-like waveform \( (f) \) which also results in a worse \( S \) compared to the sinusoid. In fact, all the considered waveforms yield a best \( S \) which is smaller than the \( S \) obtained with sinusoidal oscillations at the optimal conditions \( T^+ = 125 \) and \( W_m^+ = 4.5 \), i.e., \( S_0 = 0.078 \). For nearly all cases considered the balance between \( R \) and \( P_m \) is such that \( S > 0 \) at low \( W_m \), whereas at larger \( W_m \), higher values of \( R \) but even larger values of \( P_m \) are obtained, such that \( S \) is reduced. In general, most of the waveforms show a maximum of \( S \) at intermediate values of \((T, W_m)\) which are close to those of the sinusoidal case.

**GENERALIZED WAVEFORM**

The alternating boundary layer that is created by a sinusoidally oscillating wall in a quiescent fluid is described by the solution of the so-called Stokes’ second problem (Schlichting & Gersten, 2000). It has been shown that this solution also describes well the spanwise component of a turbulent channel flow modified by the oscillating wall, when properly averaged in space and considered as a function of the oscillation phase. Moreover the analytical expression \( w(y,t) \) of the laminar Stokes layer has proven useful for the prediction of practically important quantities, like \( P_m \) and \( R \), for turbulent flows over oscillating walls (provided that the oscillating period does not largely exceed its optimal value). The equivalence between \( w_S(y,t) \) and the space-mean, phase-averaged turbulent profile \( \langle w \rangle(y, t) \), first found for the pipe flow by Quadrio & Sibilla (2000), implies that the spanwise space-averaged momentum equation reduces to a diffusion equation analogous to that of the laminar Stokes problem, i.e.

\[
\frac{\partial w}{\partial t} = \frac{1}{Re} \frac{\partial^2 w}{\partial y^2},
\]

and the Reynolds stress term \( \partial (w'v') / \partial y \) is negligible (Ricco & Quadrio, 2008).

Our preliminary step becomes that of verifying whether this property applies to the general, non-sinusoidal case too, since it might help interpreting and generalizing the results reported in the previous section. Since Eq. (4) is linear, the obvious starting point is to consider a harmonic decomposition of the waveform, and to build the solution as linear superposition of the various Stokes components. For a single sinusoidal mode, i.e. the sinusoidal oscillation \((3)\), the analytical Stokes solution for laminar flows reads

\[
w_S(y,t) = W_m e^{-y/\delta} \sin\left(\frac{2\pi}{T} t - \frac{y}{\delta}\right)
\]

where the wall-normal lengthscale \( \delta \) is defined as

\[
\delta = \sqrt{\frac{T}{\pi Re}}.
\]

In the general case, the time-dependent boundary condition (1) for the diffusion equation (4) can be expressed via the following Fourier series

\[
w_y(t) = W_m \sum_{n=1}^{\infty} A_n e^{i(2\pi n/T)t} + c.c.
\]

where \( j \) is the imaginary unit, \( A_n \) is the complex coefficient of the \( n \)-th Fourier component and \( c.c. \) stands for complex conjugate. The resulting expression for the waveform- and generalized spanwise Stokes layer, obtained by superposition of the elementary solutions, reads as

\[
w_S(y,t) = W_m \sum_{n=1}^{\infty} A_n e^{-\sqrt{n}/\delta} e^{i[(2\pi n/T)t - \sqrt{n}/\delta]} + c.c.
\]

Figure 3 demonstrates the close agreement between the laminar solution expressed by the superposition (8) and the turbulent space-averaged profile for a non-sinusoidal waveform. It should be noted that the figure plots the turbulent profiles for waveform \( (f) \), that most significantly deviates from the sinusoid, to emphasize how Eq.(8) provides a rather robust description of the transverse boundary layer created by the wall movement. The knowledge of the spanwise velocity profile can now be exploited to derive a predictive tool for the assessment of the control performance for wall oscillations of arbitrary waveform. This is described in the following.
Figure 3. Comparison between the laminar solution expressed by Eq. 8 (symbols), with summation truncated at 128 coefficients, and the turbulent averaged spanwise velocity, $|w|(y, \tau)$ at different oscillation phases for waveform $(f)$ (lines). Left: $(T_0, W_{th0})$. Right: $(2T_0, W_{th0})$.

**PREDICTION OF CONTROL PERFORMANCE**

The input power required by the sinusoidal oscillation is written analytically via the Stokes' solution (5), and reads

$$P_n = \frac{W_n^2}{2} \sqrt{\frac{\pi}{TRe}} \sum_{n=1}^{\infty} 2|A_n|^2 \sqrt{n}.$$  \hspace{1cm} (9)

Normalization of this quantity with the time-averaged pumping power per unit channel wall area in the fixed-wall case corresponds to the performance indicator $P_{in}$ introduced above. For a generic waveform, expressed through the Fourier series (7), the same quantity becomes

$$P_{in} = \frac{W_n^2}{2} \sqrt{\frac{\pi}{TRe}} \sum_{n=1}^{\infty} 2|A_n|^2 \sqrt{n}.$$  \hspace{1cm} (10)

As shown in Fig. 4 (top left), the power consumption computed with Eq. (10) is in excellent agreement with the simulation results for the entire dataset. The inset highlights how the percentage error remains small even when the absolute value of $P_{in}$ approaches zero. Eq. (10) can thus be used to predict $P_{in}$ for arbitrary values of $T$ and $W_n$, as well as for arbitrary waveforms. Moreover, the same equation highlights that $P_{in} \propto W_n^2 \sqrt{\pi/ReT}$ such that the qualitative dependency of $P_{in}$ on $T$ and $W_n$ is independent of the waveform, as already observed in figure 2.

The prediction of the turbulent drag reduction rate $R$ is much less trivial, since $R$ does not simply derive from the laminar solution (8) but results from the complex non-linear interaction between the oscillation of the wall and near-wall turbulence. Nonetheless, several proposals are available in the literature to link properties of the transverse layer with $R$. In particular, it has been suggested, for example by Choi et al. (2002) and Quadrio & Ricco (2004), that $R$ scales with a parameter that combines a length and an acceleration scale of the spanwise alternating layer.

The wall-normal length scale, $\tilde{\ell}$, is related to the penetration of the Stokes layer into the channel, and is defined as the largest distance from the wall where the maximum wall-induced spanwise velocity exceeds a threshold velocity, $W_{th}$. For the sinusoidal waveform, the analytical solution (5) yields

$$\tilde{\ell}(W_{th}) = \delta \ln \left( \frac{W_m}{W_{th}} \right),$$  \hspace{1cm} (11)

where $\delta$ is defined by (6).
the various harmonic components, so that the correct velocity scale to define the penetration depth cannot simply be $W_m$ anymore. A good candidate for the definition of an effective penetration length for non-sinusoidal oscillations is the mean square value (variance) of the oscillating spanwise velocity, $\overline{w^2(y,t)}$, which for the sinusoidal case reads

$$\overline{w^2(y)} = \frac{W_m^2}{2} e^{-2y/\delta}. \tag{13}$$

In analogy to the classical penetration length of the Stokes layer, $\ell$, a new penetration length, $\ell_\text{th}$, can thus be defined as the distance from the wall where the induced variance of the velocity drops below a certain threshold value $\sigma^2_\text{th}$. The previous Eq. (13) thus yields for the sinusoidal case

$$\ell(\sigma^2_\text{th}) = 1 \frac{\delta}{2} \ln \left( \frac{W_m^2}{2\sigma^2_\text{th}} \right). \tag{14}$$

For the generic waveform, the variance of the oscillating velocity is given by

$$\overline{w^2(y)} = W_m^2 \sum_{n=1}^{\infty} 2|A_n|^2 e^{-2\gamma_n y/\delta}, \tag{15}$$

and this highlights how each mode contributes to the variance with a weighing factor which decays exponentially with increasing $n$. This observation is important for the following derivations. We also remark that, in general, the penetration length, $\ell$, cannot be expressed analytically, but must be computed numerically from Eq. (15). In computing $\ell$, the value $W_m^2 = 1.2$ is converted into the equivalent $\sigma^2_\text{th} = 0.8$ which follows form Eq. (14) for the sinusoidal waveform.

Figure 5 (bottom left) shows that the relationship between $R$ and $\ell$ is indeed similar for all the different waveforms considered. As already remarked by several authors, for cases with large oscillation periods drag reduction drops and the interaction between the streamwise turbulent flow and the slowly oscillating Stokes layer changes nature and trivially becomes a cyclic reorientation of the former by the latter. In figure 4 (bottom) the same graphical representation as employed by Quadrio & Ricco (2011) is adopted, and the data corresponding to slow oscillations are plotted with open symbols.

For a given forcing period at $T^+ < 150$ (i.e. for datasets represented with black- and gray-filled symbols), data for all the waveforms collapse onto one line, well fitted by a power law $\ell^2 T^{-1/2}$. As shown in figure 4 (bottom right), a linear scaling is obtained when $R$ is plotted against $\ell^2 T^{-1/2}$, as in Eq. (12), accounts for the physical process of diffusion. The term $T^{-1/2}$ carries the relevant contribution of the acceleration term $a_m$ in the expression (12) for $V_\text{th}$, which however is affected not only by $T$ but also by the harmonic distribution in the waveform. Neglecting the latter dependency upon the specific waveform yields the correct scaling parameter that is valid for any waveform.

Hence, the turbulent drag reduction rate is well predicted by the expression

$$R = h_1 \ell^{1.3/2} T^{-1/2} + h_2 \tag{16}$$

where $h_1$ and $h_2$ are coefficients for which a linear fit of the present data at $Re_T = 200$ and $T^+ < 150$ yields $h_1 = 0.0738$ and $h_2 = 0.02$. As expected, the proposed scaling is indeed not valid for $T^+ > 150$.

The strong link between the penetration length $\ell$ and the drag reduction $R$ for $T^+ < 150$ can be further exploited to provide an analytical a priori estimate of the drag reduction capabilities of a generic waveform, when $W_m$ and $T$ are given. Owing to the $n$ modulation, the sum in (15) can be approximated with its first term as

$$\sum_{n=1}^{\infty} 2|A_n|^2 e^{-2\gamma_n y/\delta} \approx 2|A_1|^2 e^{-2y/\delta}. \tag{17}$$

This approximation allows for an analytical estimate of $\ell$ as

$$\ell(\sigma^2_\text{th}) = 1 \frac{\delta}{2} \ln \left( \frac{2W_m^2|A_1|^2}{\sigma^2_\text{th}} \right). \tag{18}$$

By plugging actual numerical values of the spectral components in (15), one can easily realize that for the second Fourier component to produce a contribution to the
The dependency of the power consumption rate $P_m$ (left) and reduction of friction power $R$ (right) on the maximum displacement $D_m$ for five different pairs of $(T, W_m)$.

Figure 7. Dependency of the power consumption rate $P_m$ (left) and reduction of friction power $R$ (right) on the maximum displacement $D_m$ for five different pairs of $(T, W_m)$.

where $p_m$ is a complex coefficient which results from the integration of the Fourier polynomials and is independent on the waveforms. Equation (20), once compared with equations (10) and (15) for $P_m$ and $\overline{w_T}(y)$ respectively, clearly shows that the spectral distribution of the different waveforms influences in completely different ways these three observables. Namely, an infinite number of waveforms, corresponding to different sets of coefficients $A_n$, can lead to the same displacement. Parallelly, waveforms characterized by the same $P_m$ or $R$, and hence by the same $S$, can have completely different values of $D_m$.

This lack of correlation between the maximum displacement and $P_m$, $R$ and $S$ is verified by the present data as shown in Fig. 7.

CONCLUSION

In short, it is found that the waveform actually matters, and that the sinusoidal waveform is the best when one is interested in the net energy saving. However, what is true for the absolute optimum in parameter space is not true anymore locally, so that for certain combinations of oscillation periods and amplitudes temporal waveforms exist that outperform the sinusoid.

REFERENCES


