SCALING EXPONENTS OF VELOCITY SPECTRA AND STRUCTURE FUNCTIONS IN THE MODERATE RANGE OF TAYLOR REYNOLDS NUMBERS IN A TURBULENT JET

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ABSTRACT

In this paper, the effect of Reynolds number on the scaling exponents of the velocity spectra and structure functions in a round turbulent jet is considered. Hot-wire measurements were taken over a range of exit Reynolds numbers (10,000 $\leq Re_D \leq$ 60,000). The energy spectra in the scaling range demonstrate considerable departure from the $k^{-5/3}$ (Kolmogorov inertial-range scaling) so that the scaling wave number exponent along the jet centreline is in the range -1.56 $\leq m \leq$ -1.43. The magnitude of m exhibits a monotonic change with Reynolds number that agrees with earlier findings for grid generated turbulence. In addition, second and third-order structure functions are investigated. A Reynolds number dependence of the second-order structure function scaling exponent is observed. These exponents rise nearly exponentially to 2/3 as the Reynolds number increases. The peak value of the normalized third-order structure function asymptotically approaches 4/5 as the Reynolds numbers increases. The present round jet data confirms that a proper inertial range is unlikely to be established unless a very high turbulence Reynolds number ($R_\lambda \gg 10^3$) can be reached in jet flows.

INTRODUCTION

Kolmogorov's similarity theory (Kolmogorov, 1941) (known as K41) states that there is no direct interaction between large energy containing eddies and their smaller energy dissipating ones, but rather there is a cascade of energy in the inertial sub-range (IR) from larger to smaller scales ($\eta \ll r \ll L$) . Here L can be defined as the typical size of the energy producing scales (integral scale) and $\eta \equiv (\nu^3/\langle\varepsilon\rangle)^{1/4}$, the Kolmogorov length scale, is the typical size of the energy dissipating scales.

In the inertial sub-range, and for high Reynolds numbers, the variance of longitudinal velocity increments (for the streamwise velocity component u and for a separation r considered along the x direction) is (Pope, 2000)

$$\langle(\Delta u)^2\rangle = C_2(r\langle\varepsilon\rangle)^{2/3}, \tag{1}$$

where $\Delta u \equiv u(x+r) - u(x)$ and C_2 is the Kolmogorov velocity constant and is generally thought to have a value near 2 (Sreenivasan, 1995). In spectral space, (1) takes the form

$$E_{11}(k_1) = C_1\langle\varepsilon\rangle^{2/3}k_1^{-5/3}, \tag{2}$$

where $E_{11}(k_1)$ is the one-dimensional spectrum of u and k_1 is the one-dimensional wave number. The constant C_1 is related to C_2 via the isotropic relation $C_2 = 4.02C_1$ (Monin & Yaglom, 1975).

Kolmogorov also derived a relation from the Navier-Stokes equations between the second- and third-order moments of longitudinal velocity increment using homogeneity and isotropy as

$$-\langle(\Delta u)^3\rangle + 6\nu\frac{\mathrm{d}}{\mathrm{d}r}\langle(\Delta u)^2\rangle = \frac{4}{5}\langle\varepsilon\rangle r, \tag{3}$$

where r is the separation between the longitudinal direction and $\langle\varepsilon\rangle$ is mean dissipation rate of turbulent kinetic energy defined as

$$\langle\varepsilon\rangle = \frac{1}{2}\nu\left\langle\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2\right\rangle. \tag{4}$$

Writing (3) as $A + B = C$, term C, which is proportional to the mean dissipation rate, is associated with the total transfer of energy at any scale r. This equation implies that at a scale r the mean energy is transferred by both turbulent advection (term A) and molecular diffusion (term B).

Since it is very difficult to measure all 12 velocity derivatives from (4), additional hypotheses are necessary. Homogeneity leads to $\langle\varepsilon\rangle_{hom} \equiv \nu\ \langle(\frac{\partial u_i}{\partial x_j})^2\rangle$. A relatively crude approximation for this equation is obtained by supposing that the three spatial directions are equivalent so that only derivatives with respect to x_1 appear in the expression namely:

$$\langle\varepsilon\rangle_{hom} = 3\nu \left\langle \left(\frac{\partial u_i}{\partial x_1}\right)^2 \right\rangle. \qquad (5)$$

The local isotropy assumption results in

$$\langle\varepsilon\rangle_{iso} = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \qquad (6)$$

Based on K41 and in the IR, where the effect of viscosity is negligible, (3) reduces to the so-called '4/5-law',

$$\langle(\Delta u)^3\rangle = -\frac{4}{5}\langle\varepsilon\rangle r, \qquad (7)$$

and in the dissipation range

$$\langle(\Delta u)^2\rangle = \frac{1}{15\nu}\langle\varepsilon\rangle r^2. \qquad (8)$$

In spite of the modifications by Kolmogorov himself (Kolmogorov, 1962) (K62), for more than sixty years K41 has been widely discussed and more or less accepted. According to K62, E_{11} can be obtained as

$$E_{11}(k_1) = C_1\varepsilon^{2/3}k_1^{-5/3}(k_1 L)^{-\mu/9}, \qquad (9)$$

where μ is the internal intermittency and L is the integral length scale.

However, there are several reasons why K41 or K62 should be treated and used carefully. The assumption that the Reynolds number should be very large is obviously not realized in most laboratory conditions such as grid turbulence and jets. At small Reynolds numbers, (3) has been shown not to be valid. In fact, Kolmogorov '4/5-law' is strictly valid only if an IR is established. This is expected at very large Taylor Reynolds numbers. Actually, the main problem is that at moderate *Re*, there is not enough separation between dissipative scales and the large scales. The large scales themselves are very rarely isotopic and homogeneous even at very high *Re*. Isotropy at the small dissipative scales is typically good even at small *Re* if there is no production term. Danaila *et al.* (1999) revisited the hypotheses involved in the derivation of (3). Their conclusion was that a new term that reflected the non-stationarity (or the large-scale non-homogeneity in the streamwise direction) must be considered. The generalized equation is then an exact expression, that has been demonstrated directly only for grid turbulence (Lavoie *et al.*, 2007).

Despite the significant attention that has been paid to the study of the effect of Reynolds number on the small-scale structure of grid turbulence, see e.g., Mydlarski & Warhaft (1996) and Zhou & Antonia (2000), less attention has been paid to the round jet. Contrary to grid turbulence, mean shear and production play significant roles in jet flows.

As noted, the effect of turbulence Reynolds number on the scaling exponents of the velocity spectra has been so far explored as the evolution of the grid turbulence with Taylor Reynolds number. In a recent work, Mi & Nathan (2010) claimed that the scaling exponent is m = -1.5 rather than -1.67 along the centreline of turbulent jets and this value is insensitive to the magnitude of the Reynolds number. This disagrees with those previous data presented for grid turbulence. In our opinion, this issue still remains as an open question. Therefore, this paper provides new data to help resolve this issue. One of the objectives of this work, therefore, is to determine the value of the spectral scaling exponents in terms of Re_D and Re_λ in a round free jet. In addition, this paper considers the effects of Reynolds numbers on the second and third-order structure functions in the jet.

EXPERIMENTAL CONDITIONS

The experimental setup and conditions are briefly described in this section, and the reader is referred to Sadeghi & Pollard (2012*b*) and Fellouah *et al.* (2009) for further details. An air jet was generated using a fan mounted on anti-vibration pads. The air then exits a settling chamber via a round duct to the inlet of a smoothly contracting axisymmetric nozzle with exit diameter of D = 73.6 *mm*. The experiments were carried out over the range of Reynolds numbers between 10,000 $\leq Re_D \leq$ 60,000, where Re_D is calculated based on the jet exit mean velocity and the nozzle exit diameter. The measurements were mainly performed at the downstream location of x/D = 10 and along the jet centreline. All measurements of the turbulence statics were obtained using a stationary single hot wire. The hot-wire was smaller than those made in previous investigations. It was made of tungsten wire 2.5 μm in diameter and 0.5 *mm* in length. Constant temperature anemometry was employed. The hot wire probe was carefully mounted on a support and positioned using a motorized traverse mechanism. In the current work, the hot wire was calibrated in the jet core before and after each experiment. The calibration data were fit using King's law. The measurements were taken with a sampling frequency of 30 *kHz* and a sampling time of 2 minutes, which ensured that enough data were taken to achieve statistical convergence at the location farthest from the the nozzle exit. In the present work, the power spectrum of the fluctuating axial velocity was computed by a fast-Fourier-transform algorithm. The wave number, k, was taken to be equal to $2\pi f/U$, where U is the local convection velocity, which was assumed to be equal to the local mean velocity at the measurement point.

The use of Taylor's hypothesis for the stationary hot-wire data can lead to large errors. It has been demonstrated that Taylor's hypothesis and the assumption of local isotropy may be acceptable for small turbulence intensities and may cause large uncertainties in flows with "high" turbulent intensity. Therefore, in the present work, the modified Taylor hypothesis similar to Antonia *et al.* (1982) was used to convert time into a spatial series.

The errors introduced from the use of Taylor hypothesis are compounded by the high-frequency noise in the velocity signals. Moreover, the effect of the finite spatial resolution of the hot-wire probes can be important when analyzing the small-scale vortex motion. Here, a fast-convergent iteration filtering scheme similar to Xu *et al.* (2013) is used to remove the effect of high frequency noise and correct stationary hot wire data.

It is known that the effect of finite spatial resolution of the hot-wire probes can be important in the analysis of the small scale data. If the scale η of the smallest eddies existing in the flow becomes smaller than the wire length l_w, then their experimentally determined contribution to the turbulence energy dissipation deviates considerable from that which would be expected in true measurements, introducing in this way a systematic error into the experimental results. In the current paper, in order to present the small scale data (Kolmogorov and Taylor scales), the dissipation and mean-square fluctuation derivatives were corrected using a spectral correction method, e.g., (Hearst *et al.*, 2012). Data first were taken using the modified Taylor hypothesis and corrected for high frequency noise.

RESULTS AND DISCUSSION

The axial velocity spectra, $E_{11}(k_1)$, for various exit Reynolds numbers (Re_D = 10,000-60,000) at x/D = 10 along the jet axis (y/D = 0) are presented in figure 1. At this location, the mean shear rate is zero. Over this range of Reynolds numbers, a least squares fit to the spectrum in the scaling range yields to slopes -1.55 $\leq m \leq$ -1.43, compared to -1.67 for the inertial sub-range of K41. The results are summarized in Table 1. In addition to values for m, the Taylor Reynolds number (Re_λ) and internal intermittency, μ, are also listed in Table 1. Here, the Taylor Reynolds number is defined as $Re_\lambda = \sqrt{\langle u^2 \rangle}\lambda/\nu$, where $\lambda = \sqrt{\langle u^2 \rangle}/\left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle$.

The internal intermittency is also estimated based on the following relation

$$R_{\varepsilon\varepsilon}(r) = \left\langle \left[\frac{\partial u(x)}{\partial(x)} \frac{\partial u(x+r)}{\partial(x)} \right]^2 \right\rangle \approx r^{-\mu}. \qquad (10)$$

The autocorrelation functions $R_{\varepsilon\varepsilon}$ versus r at $x/D = 10$ along the jet axis (y/D = 0) are plotted in figure 2. All curves reveal a power law behavior in the scaling range defined in (10) at different Reynolds numbers that is illustrated by solid lines.

From Table 1, the magnitude of m is observed to decrease with Re_λ. As noted above, there have been some attempts to modify the power law exponent in the inertial sub-range, first suggested by Kolmogorov in 1962 (K62), mostly to account for the internal (fine scale) intermittency. However, the observed deviations from the $k^{-5/3}$ behaviour cannot be only due to the effect of the internal intermittency. In fact, the internal intermittency may decrease the slope of the spectrum in the scaling range, e.g., m = -5/3 - μ/9, (Monin & Yaglom, 1975). For example, for internal intermittencies listed in Table 1 (0.096 $\leq \mu \leq$ 0.166), the scaling exponents would be in the range of -1.696 $\leq m \leq$ -1.682

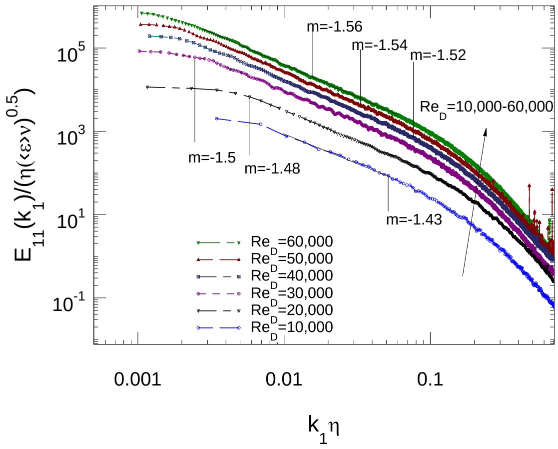

Figure 1. Normalized spectra of u at $(x/D, y/D)$ = (10, 0) for Re_D = 10,000 to 60,000. Spectra shifted vertically by multiples of Re_D/10,000.

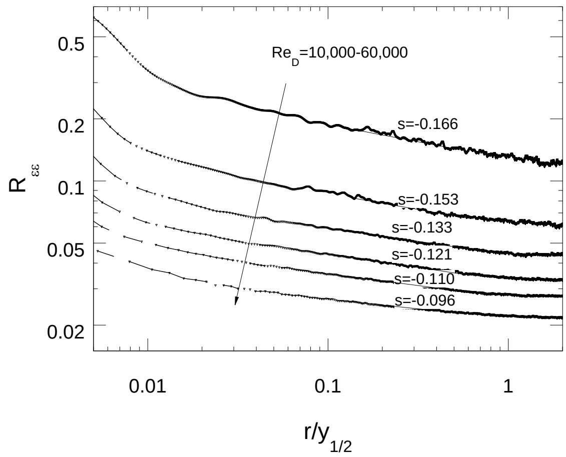

Figure 2. Correlation function of the energy dissipation of u at $(x/D, y/D)$ = (10, 0) for Re_D = 10,000 to 60,000. Note that data of $R_{\varepsilon\varepsilon}$ are arbitrary values and r is normalized by jet half radius.

which are very close to m = -5/3 and far from the observed slopes listed in Table 1.

This difference in the scaling region slope was previously observed by Mydlarski & Warhaft (1996) and Gamard & George (1999) who related this difference in the value of m to the finite value of Re_λ. Similar data for turbulent jets has not been published. The present results along the jet centreline confirm those observed in grid turbulence data. The scaling range exponents versus Taylor Reynolds numbers for the above-mentioned location on the jet centreline are plotted in figure 3 and compared with $m - Re_\lambda$ relationship obtained by Mydlarski & Warhaft (1996) and Gamard & George (1999) for grid generated turbulence. The current data on the jet centreline are in good agreement with those grid generated turbulence data of Mydlarski & Warhaft (1996) and with the analysis of Gamard & George (1999). Only small departures at lower Taylor Reynolds numbers can be observed, which could be due to energy production in turbulent jets that is not found in grid turbulence. The adherence of the present jet results to the previous finding in grid turbulence suggests that this

Table 1. Summary on the values of m and μ at $(x/D, y/D)$ = (10, 0).

Re_D	Re_λ	m	μ
10000	122	-1.43	0.166
20000	174	-1.48	0.153
30000	224	-1.50	0.133
40000	256	-1.52	0.121
50000	291	-1.54	0.110
60000	316	-1.56	0.096

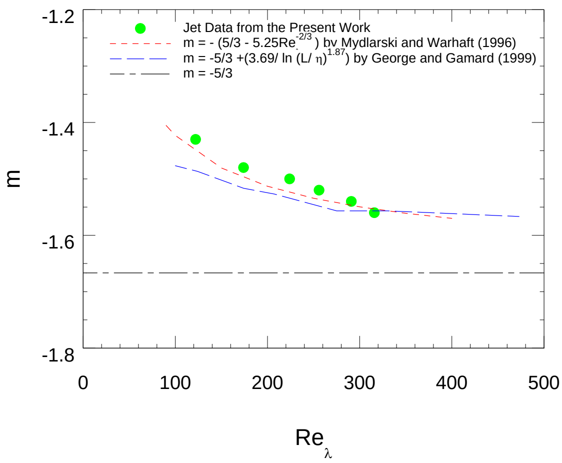

Figure 3. The scaling range exponent versus Taylor Reynolds number.

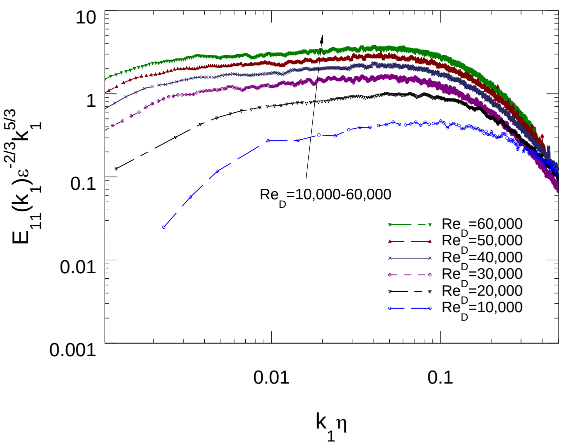

Figure 4. Compensated velocity spectra of u at $(x/D, y/D)$ = (10, 0) for Re_D = 10,000 to 60,000 based m = -5/3. Note: Spectra shifted vertically by multiples of Re_D/10,000.

relationship between m and Re_λ may apply more broadly than first thought, at least in the moderate range of Taylor Reynolds numbers. However, it should be pointed out that some other measurements taken in the radial direction suggest that mean shear rate together with large scale intermittency influence the scaling-range exponents (Mi & Antonia, 2001), (Sadeghi & Pollard, 2012*a*). Therefore, the Taylor Reynolds number could be responsible for the departure from $k^{-5/3}$ for any flow if the effect of mean shear is ignored.

A more straightforward test for Kolmogorov scaling is to display the data in compensated form. Figure 4 provides the compensated form of velocity spectra, which is defined as $E_{11}(k_1)$ multiplied by $\langle\varepsilon\rangle^{-2/3}k^{5/3}$. In the inertial sub-range, the Kolmogorov hypothesis predicts that the compensated spectra should be nearly independent of $k\eta$. In other words, the inertial sub-range should be a constant, $C_1 = E_{11}(k_1)\langle\varepsilon\rangle^{-2/3}k_1^{5/3}$. Monin & Yaglom (1975), Saddoughi & Veeravalli (1994) and Sreenivasan (1995) suggested that this constant is universal and equal to approximately $C_1 \approx 0.5$ although they noted that different methods of estimating these values using the same data (e.g., accuracy of determining dissipation) could result in uncertainty in order of 10-15%. As shown in Table 1, the present data do not support m = -5/3 and a different exponent should be used in the energy spectra equation (2). The one dimensional longitudinal spectra equation maybe recast with $E_{11}(k_1) = C_1^*\langle\varepsilon\rangle^{2/3}k_1^{-5/3}(k_1\eta)^{m+5/3}$, where m is a power law exponent and C_1^* is the modified Kolmogorov variable (this is a modified similarity form similar in approach to Sreenivasan (1991) for the spectral density of temperature in the inertial range). Figure 5 presents the compensated spectra plotted according to the values of m listed in Table 1. As can be observed, the slope of the curves observed in the intermediate range of the spectra in the figure 4 disappears when a different m is applied for the compensated spectra, figure 5. The difference in the width of the scaling range (over $k\eta$) increases with Reynolds number. Additionally, the spectra do not display the spectral 'bump' at the high wave numbers near the end of scaling range when the compensated spectra are scaled using the measured m, see e.g., Fellouah & Pollard (2010) and Coantic & Lasserre (1999). It should be pointed out that the real reason for the occurrence of this bump, when m = -5/3 is used for the compensated form of the spectra, is not entirely clear. As shown here, one possible explanation could be from the finite Reynolds numbers that have the effect of producing a spectral roll-off rate that is less steep than the Kolmogorov form. However, Sreenivasan (1995) noted that "it has been argued that the combination of the two facts- namely the existence of a constant energy flux across the wavenumber and the rapid damping due to viscosity-leads naturally to this energy pileup near the crossover between inertial and dissipative regions, and has been called the 'bottleneck' effect". In addition, the appearance or lack of the spectral bump at higher wave numbers could be due to measurement resolution, as recently noted by Hutchins *et al.* (2009) in wall bounded flow. This is the topic of on going research efforts.

The second-order structure function of u normalized by the mean turbulent fluctuation, $\langle u^2\rangle$, is presented in figure 6. It can be seen that at higher r, all data collapse around a value of 2. Also, the obvious departure of the slope from 2/3 and its Reynolds number dependence can be noticed.

The distributions of $-\langle(\Delta u)^3\rangle/\langle\varepsilon\rangle r$ are shown in figure 7. The peak value of this quantity increases systematically as Re asymptotically approaches 4/5. This approach suggests that it is unlikely that the scaling range approaches the 4/5 before reaching a very high Reynolds. To the best of our knowledge, the highest Taylor Reynolds numbers observed in the literature for laboratory studies of turbulent jets are less than $Re_\lambda \sim 10^4$ which was suggested as the appropriate Re_λ in grid turbulence studies to reach a proper IR (Mydlarski & Warhaft, 1996).

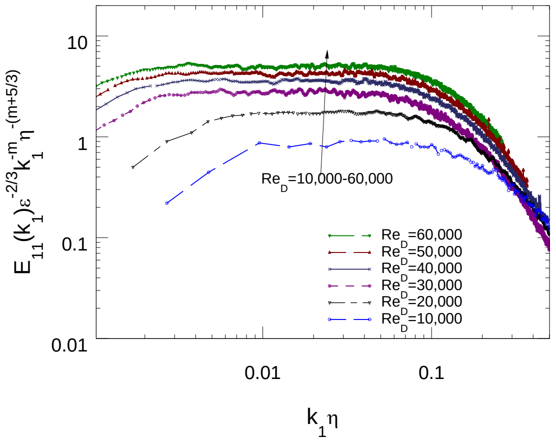

Figure 5. Compensated velocity spectra of u at $(x/D, y/D) = (10, 0)$ for Re_D = 10,000 to 60,000 based on the measured m as listed in Table 1. Spectra shifted vertically by multiples of Re_D/10,000.

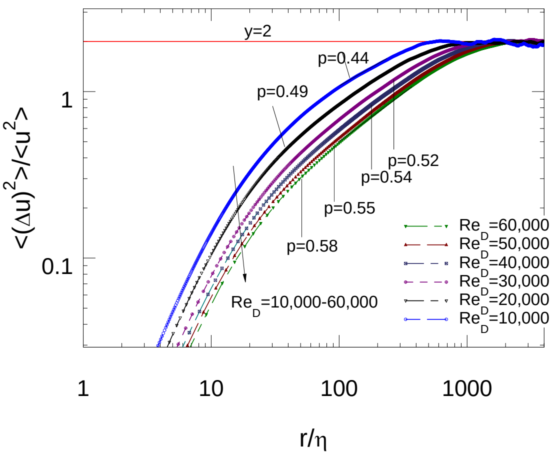

Figure 6. Normalized second-order structure function of u at $(x/D, y/D) = (10, 0)$ for Re_D = 10,000 to 60,000.

CONCLUSION

In this work, the spectral scaling range exponents in a round turbulent jet were investigated. Velocity spectra along the jet centreline revealed that the exponents in the scaling region are slightly greater than m = -5/3 (i.e., -1.56 $\leq m \leq$ -1.43). This was related to the finite value of Taylor Reynolds number. As the Reynolds number increases, m reduces in value, but still remains far from m = -5/3 in the current experiments. The adherence of the present jet results to the previous finding in grid turbulence suggest that this relationship between m and Re_λ may apply more broadly than first thought, at least in the moderate range of Taylor Reynolds numbers. The Reynolds number dependence of second- and third-order structure functions was also observed.

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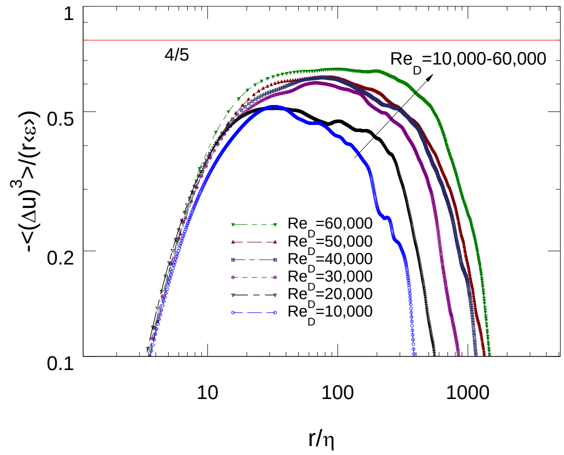

Figure 7. Normalized third-order structure function of u at $(x/D, y/D) = (10, 0)$ for Re_D = 10,000 to 60,000.

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