SPECTRAL EDDY VISCOSITY OF STRATIFIED TURBULENCE

Sebastian Remmler
Institute of Aerodynamics and Fluid Mechanics, Technische Universität München
85747 Garching bei München, Germany
remmler@tum.de

Stefan Hickel
Institute of Aerodynamics and Fluid Mechanics, Technische Universität München
85747 Garching bei München, Germany
sh@tum.de

ABSTRACT

The analysis of the spectral eddy viscosity is a handy tool to analyze the performance of LES methods. It reflects the cumulated effect of numerical discretization and turbulence subgrid-scale model on the spectral energy transfer. We compute this quantity by filtering and truncating data from direct numerical simulations of neutrally and stably stratified homogeneous turbulence. The results are used for testing of implicit and explicit LES methods. We find indications that for stably stratified turbulence it is necessary to use different subgrid-scale models for the horizontal and vertical velocity components.

INTRODUCTION

In large-eddy simulation (LES) the unresolved part of the turbulent velocity field is modelled by a subgrid-scale (SGS) model. This SGS turbulence model is supposed to modify the flow energy balance in the same way as the small-scale structures of fully resolved turbulence would do. Most SGS models are based on an eddy viscosity hypothesis, i. e., the SGS model dissipates turbulence energy, especially at the smallest resolved scales, but also on larger scales. Heisenberg (1948) introduced the concept of modelling nonlinear interactions in turbulence by a scale-dependent spectral eddy viscosity (SEV). The underlying theory has later been refined by Kraichnan (1976) and others. Although impractical in real-space-based numerical simulations, the SEV as a function of wavenumber can be used to verify the correct behaviour of SGS models in a set-up of homogeneous (but not necessarily isotropic) turbulence.

Algebraic expressions for the SEV have been derived based on the Eddy-Damped Quasi-Normal Markovian (EDQNM) theory (Orszag, 1970) for isotropic turbulence. Furthermore, Domaradzki *et al.* (1987) computed the SEV from direct numerical simulations (DNS) with fully resolved turbulence by truncating the results in spectral space. They found some agreement with the theoretical results of Kraichnan (1976) but also differences due to the finite inertial range in their simulations. Despite these discrepancies, the behaviour of isotropic turbulence is quite well understood. On the other hand, a corresponding study for anisotropic turbulence had not yet been carried out.

Semi-analytical expressions for the eddy-viscosity and eddy-diffusivity spectrum for stratified turbulence are given by Godeferd & Cambon (1994), Staquet & Godeferd (1998), and Godeferd & Staquet (2003) in the framework of the EDQNM approximation. Another form was obtained by Sukoriansky *et al.* (2005) through quasi-normal scale elimination (QNSE). These theoretical results show that turbulence anisotropy can significantly affect SGS energy dissipation in flows dominated by stable stratification, solid body rotation, or shear.

In validating an SGS model for stably stratified flows, we have generated an extensive database of direct numerical simulation results for homogeneous stratified turbulence. The simulations cover a wide range of Froude numbers from the neutrally stratified to the strongly stratified regime (Remmler & Hickel, 2013, 2012). We now analyzed these results with respect to the anisotropic, i. e., direction dependent, SEV. To achieve this, we filtered the DNS results to coarser resolutions in several steps and computed the SGS stress necessary to obtain the same large-scale result on the coarse grid as on the full DNS grid.

In the following section, we will briefly outline the Boussinesq equations which we are solving, review the concept of spectral eddy viscosity and diffusivity and comment on our flow solver. A short overview of the computational set-up follows. The results section presents results for isotropic turbulence in comparison to the work of Kraichnan (1976) and Domaradzki *et al.* (1987) as well as SEV data in a two-dimensional spectral space for stably stratified homogeneous turbulence. Furthermore, we use these newly obtained reference data to evaluate the performance of different existing LES methods. One model follows the implicit LES paradigm, i. e., discretization scheme and SGS model are merged. The other models are more traditional and combine an explicit approximation of the SGS tensor with a non-dissipative central discretization.

COMPUTATIONAL METHODS

Boussinesq equations

The flows to be investigated are characterized by a stable background stratification, so density is not constant.

However, the density differences are small and flow velocities are much smaller than the speed of sound, which justifies the usage of the Boussinesq approximation. The non-dimensional Boussinesq equations for a stably stratified fluid in Cartesian coordinates read

$$\nabla \cdot \mathbf{u} = 0 \tag{1a}$$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p - \frac{\rho}{\mathrm{Fr}_0^2}\mathbf{e}_z + \frac{1}{\mathrm{Re}_0}\nabla^2 \mathbf{u} \tag{1b}$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = \mathbf{u}\mathbf{e}_z + \frac{1}{\mathrm{Pr}\,\mathrm{Re}_0}\nabla^2 \rho \tag{1c}$$

where velocities are made non-dimensional by $\mathscr{U}$, all spatial coordinates by the length scale $\mathscr{L}$, pressure by $\mathscr{U}^2$, time by $\mathscr{L}/\mathscr{U}$, and density fluctuation $\rho = \rho^* - \overline{\rho}$ (ρ^*: local absolute density, $\overline{\rho}$: background density) by the background density gradient $\mathscr{L}|d\overline{\rho}/dz|$. The vertical unit vector is $\mathbf{e}_z$. The non-dimensional flow parameters are

$$\mathrm{Fr}_0 = \frac{\mathscr{U}}{N\mathscr{L}}\,, \quad \mathrm{Re}_0 = \frac{\mathscr{U}\mathscr{L}}{\nu}\,, \quad \mathrm{Pr} = \frac{\nu}{D} \tag{2}$$

We chose a Prandtl number of Pr = 0.7, corresponding to typical values in the atmosphere.

The local dissipation rates ε_k and ε_p of kinetic energy $E_k = \frac{1}{2}u_i u_i$ and available potential energy $E_p = \frac{1}{2}\rho^2/Fr_0^2$ can be computed directly from the velocity and density field:

$$\varepsilon_k = \frac{1}{\mathrm{Re}_0}\left(\partial_{x_j}u_i + \partial_{x_i}u_j\right)\left(\partial_{x_j}u_i + \partial_{x_i}u_j\right) \tag{3}$$

$$\varepsilon_p = \frac{1}{\mathrm{Pr}\,\mathrm{Re}_0\,\mathrm{Fr}_0^2}\left(\partial_{x_i}\rho\right)\left(\partial_{x_i}\rho\right) \tag{4}$$

With the spatial mean values of kinetic energy $\langle E_k \rangle$ and kinetic energy dissipation $\langle \varepsilon_k \rangle$, we define the local Froude and Reynolds number as well as the buoyancy Reynolds number $\mathscr{R}$ (Brethouwer *et al.*, 2007):

$$\mathrm{Fr} = \frac{\mathrm{Fr}_0 \mathscr{L}}{\mathscr{U}}\frac{\langle \varepsilon_k \rangle}{\langle E_k \rangle}\,, \ \mathrm{Re} = \frac{\mathrm{Re}_0}{\mathscr{U}\mathscr{L}}\frac{\langle E_k \rangle^2}{\langle \varepsilon_k \rangle}\,, \ \mathscr{R} = \mathrm{Re}\,\mathrm{Fr}^2 \tag{5}$$

Spectral eddy viscosity

The momentum equation for incompressible homogeneous turbulence in spectral space reads

$$\begin{aligned}\left(\partial_t + \nu k^2\right)\hat{u}_i(\mathbf{k}) &= F_i(\mathbf{k}) - \ldots \\ \imath k_q P_{ij}(\mathbf{k}) &\sum_{\mathbf{m}} \hat{u}_j(\mathbf{m})\hat{u}_q(\mathbf{k}-\mathbf{m}),\end{aligned} \tag{6}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the projection tensor onto a divergence-free velocity field, δ_{ij} is the Kronecker symbol, $k^2 = |\mathbf{k}|^2 = k_i k_j$ is the wave number and F_i contains the forces on the fluid. The kinetic energy of a single mode $\mathbf{k}$ is

$$e(\mathbf{k}) = \frac{1}{2}\hat{u}_i(\mathbf{k})\hat{u}_i(\mathbf{k})^*, \tag{7}$$

where the asterisk * denotes the complex conjugate and summation over repeated indices applies. The temporal evolution of $e(\mathbf{k})$ is described by

$$\begin{aligned}\left(\partial_t + \nu k^2\right) e(\mathbf{k}) - \Re\left\{F_i \hat{u}_i(\mathbf{k})^*\right\} = T(\mathbf{k}) = \ldots \\ k_q P_{ij}(\mathbf{k}) \Im \left\{\sum_{\mathbf{m}} \hat{u}_i(\mathbf{k})^* \hat{u}_j(\mathbf{m}) \hat{u}_q(\mathbf{k}-\mathbf{m})\right\}.\end{aligned} \tag{8}$$

If the numerical discretization acts as a perfect low-pass filter, only wavenumbers $|\mathbf{k}| < k_c$ are resolved and we can split the transfer term $T(\mathbf{k})$ into

$$T(\mathbf{k}) = T^-(\mathbf{k}, k_c) + T^+(\mathbf{k}, k_c)\,; \quad |\mathbf{k}| < k_c, \tag{9}$$

where $T^-(\mathbf{k}, k_c)$ involves only interactions of wavenumbers $|\mathbf{k}| < k_c$ and is thus resolved by the numerical grid. The SGS transfer $T^+(\mathbf{k}, k_c)$ represents all unresolved interactions and has to be modelled in an LES.

We can model the average SGS transfer by using the spectral eddy viscosity hypothesis

$$\nu_t(\mathbf{k}, k_c) = \frac{\langle T^+(\mathbf{k}, k_c)\rangle}{k^2 \langle e(\mathbf{k})\rangle}. \tag{10}$$

The average $\langle \ldots \rangle$ is taken over time and on thin spherical shells with radius $|\mathbf{k}|$ for isotropic turbulence. For flows with spectra symmetric about the k_z-axis, such as rotating or stratified turbulence, we average over thin cylindrical shells with radius $|\mathbf{k}_h| = \sqrt{k_x^2 + k_y^2}$.

For isotropic turbulence the SEV is generally normalized by the cut-off wavenumber and the kinetic energy at this wavenumber

$$\nu_t^+(k/k_c) = \nu_t(k, k_c)\sqrt{\frac{k_c}{E(k_c)}}, \tag{11}$$

where the integral kinetic energy is $E(k) = 4\pi k^2 \langle e(\mathbf{k})\rangle$. This is only a useful definition if the energy spectrum is known, (e. g. $E(k) = C_K \varepsilon^{2/3} k^{-5/3}$) at the cut-off wavenumber. Otherwise, it is helpful to use the original formulation of Kraichnan (1976)

$$\nu_t^*(\mathbf{k}/k_c) = \nu_t(\mathbf{k}, k_c)\varepsilon_k^{-1/3} k_c^{4/3}. \tag{12}$$

For isotropic turbulence with an infinite inertial range, ν_t^+ and ν_t^* are simply related by $\nu_t^* = \sqrt{C_K}\nu_t^+$, where C_K is the Kolmogorov constant. A parametrization for $\nu_t^+(k/k_c)$ in isotropic turbulence is given by Chollet (1984).

Having a fully resolved simulation of homogeneous turbulence, we can extract the full transfer term $T(\mathbf{k})$. By filtering the solution to a coarser test grid, we find the resolved term $T^-(\mathbf{k}, k_c)$ for the test grid resolution and we compute the SGS transfer $T^+(\mathbf{k}, k_c)$ from equation (9).

We can derive an expression for the spectral eddy diffusivity (SED) of any conserved scalar, such as the density.

$$D_t(\mathbf{k}, k_c) = \frac{\langle T_p^+(\mathbf{k}, k_c)\rangle}{k^2 \langle e_p(\mathbf{k})\rangle}, \tag{13}$$

where T_p^+ is the SGS transfer term in the density equation and

$$e_p(\mathbf{k}) = \frac{1}{2}\hat{\rho}(\mathbf{k})\hat{\rho}(\mathbf{k})^* \tag{14}$$

is the spectral potential energy density. We normalize the SED by

$$D_t^*(\mathbf{k}/k_c) = D_t(\mathbf{k}, k_c)\varepsilon_k^{-1/3}k_c^{4/3} \tag{15}$$

using the kinetic energy dissipation rate ε_k as for the SEV.

Flow solver

With our flow solver INCA, the Boussinesq equations are discretized by a fractional step method on a staggered Cartesian mesh. The code is parallelized both for shared and distributed memory systems and it offers different discretization schemes depending on the application. For time advancement the explicit third-order Runge-Kutta scheme of Shu (1988) is used. The time-step is dynamically adapted to satisfy a Courant-Friedrichs-Lewy condition with $CFL \leq 1.0$.

The spatial discretization is a finite-volume method. We use a non-dissipative central difference scheme with 4th order accuracy for the convective terms and 2nd order central differences for the diffusive terms and the pressure Poisson solver. The Poisson equation for the pressure is solved at every Runge-Kutta sub step. The Poisson solver employs fast Fourier-transform in the vertical direction and a Stabilized Bi-Conjugate Gradient (BiCGSTAB) solver van der Vorst (1992) in the horizontal plane. By the FFT, the three-dimensional problem is transformed into a set of independent two-dimensional problems, which can be solved in parallel.

NUMERICAL SET-UP

We simulated homogeneous stratified turbulence in a triply-periodic box with side-length $2\pi\mathscr{L}$ and a resolution of 512^3 cells. A fluctuating large scale horizontal volume force is applied to the fluid that injects a constant forcing power into the domain. The time- and space-dependent forcing term reads (Aspden *et al.*, 2008)

$$\mathbf{F}(\mathbf{x},t) = \sum_{k_i,k_j=1}^{2} \mathbf{a}_{i,j}\cos(2\pi k_i x + p_{i,j}) \cdot \ldots \\ \cos(2\pi k_j y + q_{i,j})\,. \tag{16}$$

The random amplitudes $\mathbf{a}_{i,j}$ and phases $p_{i,j}$ and $q_{i,j}$ are recomputed at every time step. After an initial transient phase, the turbulence kinetic energy remains at a constant level, where the forcing power is balanced by molecular dissipation. A more detailed description of the simulations is provided by Remmler & Hickel (2013). We sampled the SEV and SED in time intervals sufficiently large to ensure decorrelated velocity and density fields. To limit computational costs, we restricted ourselves to 20 samples per simulation. All figures presented below are averages of these samples.

A list of the simulations can be found in table 1, where we provide the non-dimensional parameters for each case

#	Re	Fr	$\mathscr{R}$	Re_λ	ηk_{max}
1	20800	∞	∞	372	0.95
2	23150	0.089	184.0	393	0.97
3	28250	0.025	17.2	434	0.83
4	33480	0.008	2.1	472	0.71

Table 1. List of the presented direct numerical simulations ordered by the strength of the stable stratification. #1 is neutrally stratified, #4 is strongly stratified.

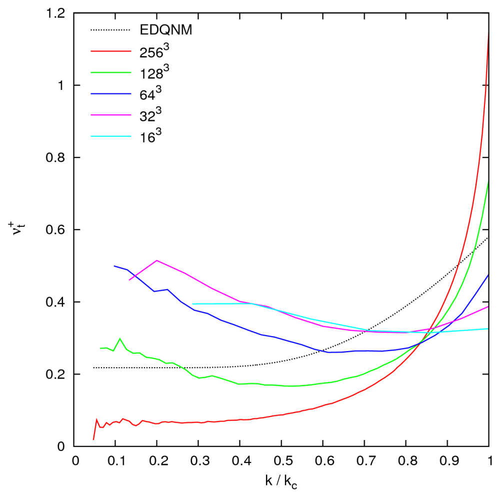


Figure 1. Spectral eddy viscosity of neutrally stratified turbulence at different test grid levels. For comparison the EDQNM prediction (Kraichnan, 1976)

as well as the Taylor-scale Reynolds number, which can be estimated from the integral Reynolds number (Pope, 2000, pg. 200) by

$$\mathrm{Re}_\lambda = \sqrt{\frac{20}{3}\mathrm{Re}}. \tag{17}$$

RESULTS

Neutrally stratified turbulence

For neutrally stratified turbulence we can compare our results directly to the EDQNM prediction. In figure 1 we show the results of spherically averaged SEV for five different coarse test grids together with the algebraic law of Kraichnan. It turns out that in our simulations

- the values of ν_t^+ are similar to the theoretical ones,
- the plateau-cusp shape of the curve is reproduced.

However,

- the cusp is sharper than in the theoretical curve and its maximum value increases with the test grid resolution,
- the “plateau” at low wavenumbers is tilted, its level rises with decreasing test grid resolution and saturates for the test grid with 64^3 cells and the coarser grids.

Domaradzki *et al.* (1987) already observed a lower level of the SEV at low wavenumbers compared to theory, when they analyzed DNS of isotropic turbulence at very low Reynolds number. So the low level plateau is probably due to the high cut-off wavenumbers that are close to the dissipative range.

In figure 2 we present the SEV in a two-dimensional spectral space. Averaging was performed on circles with constant distance from the vertical axis. The analysis was done separately for the horizontal and vertical kinetic energy components. It turns out that, as expected, the horizontal eddy viscosity spectrum $\nu_t(E_h)$ shows an isotropic distribution. The SEV of the vertical kinetic energy component $\nu_t(E_v)$ is not isotropic, which is due to the anisotropic spectrum of any single direction kinetic energy in a divergence free velocity field.

Stably stratified turbulence

We applied the same analysis to the simulations with stable stratification, see figure 3. In this case, a third type of energy has to be considered: available potential energy and hence the SED $D_t(E_p)$. In the following, it is sometimes helpful to discuss the results not in Cartesian spectral coordinates but in terms of absolute wavenumber k and the angle ϕ, which has the range $0 \leq \phi \leq \pi/2$ for the horizontal and vertical direction.

The SEV of horizontal kinetic energy is still almost isotropic in case #2 with $\mathscr{R} = 184$. At lower Froude numbers the cusp does not appear any more in all directions, but only at medium angles ϕ. At the lowest Froude number investigated, it almost completely vanishes.

For the vertical kinetic energy, there is no visible difference between the neutral and the weakly stratified case. With increasing stable stratification, the cusp vanishes and the plateau level is decreased, being almost zero in the strongest stratified case.

The SED of available potential energy differs quite strongly from the SEV described above. In the weakly stratified case, there is a clear plateau-cusp behaviour, but the plateau level depends on the spectral direction. It strongly decreases when ϕ is increased. The cusp level, on the contrary, is almost unaffected by the spectral direction. It decreases only slightly at $\phi \approx \pi/2$. Case #3 ($\mathscr{R} = 17.2$), looks very similar, just the plateau level is decreased and the drop of the cusp level at high ϕ is more pronounced than in the previous case. For the strongest stratification, the picture changes significantly. There is a peak at high horizontal wavenumbers and no plateau region as is the cases before.

Analysis of LES schemes

The reference data obtained from filtering the DNS can now be used to analyze LES methods. In LES, the effective SEV and SED are both influenced by the numerical discretization and the turbulence SGS model. Both interfere with each other and cannot be judged independently, which motivates the idea of implicit LES where discretization and SGS model are fully merged. Since quantitative comparison of two-dimensional plots as in figures 2 and 3 is difficult, we show the SEV and SED of different LES methods in figure 4 in a one-dimensional graph that is a cut through the spectral space at $\phi = \pi/4$ (the "diagonal" modes). As a test case we selected case #3 with a medium stable stratification. Together with the EDQNM prediction and the DNS reference result, we show the results of the following methods:

- Adaptive Local Deconvolution Method (ALDM), an implicit LES method developed by Hickel *et al.* (2006, 2007) and Remmler & Hickel (2012)
- A conservative 4th order central discretization scheme without any SGS model (CDS4)
- CDS4 with the standard Smagorinsky model (SSM) with a model coefficient of $C_s = 0.18$
- CDS4 with a dynamic Smagorinsky model (DSM) with averaging in all spatial directions, i. e. just choosing the most suitable model coefficient for the homogeneous problem
- CDS4 with a dynamic Smagorinsky model that allows for spatial variations of the model coefficient (DSM2)

It turns out that none of the tested methods is able to correctly reproduce all three SEV and SED spectra at the same time. ALDM does on average a good job, which is remarkable since the method was optimized to reproduce the EDQNM curve as close as possible. The averaged DSM gives good results for horizontal kinetic energy and available potential energy, but fails for the vertical kinetic energy. The vertical kinetic energy, on the other hand, is well predicted by the pure CDS4 discretization without turbulence SGS model. The DSM2 model which allows for local variations in the model coefficient does not improve the result over the averaged DSM, but rather makes it worse.

SUMMARY AND CONCLUSIONS

We have computed the spectral eddy viscosity and diffusivity of homogeneous turbulence with and without stable stratification. This was achieved by filtering fully resolved DNS results and by computing the additional spectral energy flux that is necessary to obtain in the coarse-grained flow field the same total flux as in the fully resolved case.

For neutrally stratified turbulence we found eddy viscosity spectra that are, in general, similar to the EDQNM prediction of Kraichnan (1976) showing the well known plateau-cusp behaviour. On the other hand, the results are not completely independent of the chosen test filter size; especially the amplitude of the cusp at the cut-off wavenumber is decreased for coarser test grids.

If stable stratification is increased, the SEV and SED spectra get more and more anisotropic. For the most stable case investigated the plateau-cusp topology has almost completely vanished. This illustrates that the characteristics of the flow change significantly, as soon as the buoyancy Reynolds number approaches $\mathscr{R} \approx 1$. The treatment of SGS stresses in such cases must generally be different than in fully turbulent flows with higher values of $\mathscr{R}$.

We used the results from the filtered DNS to test the implicit SGS model ALDM and a central discretization scheme with and without Smagorinsky model, either in the standard or in the dynamic form. We found that ALDM, despite being calibrated for the SEV from EDQNM theory, yields acceptable results for all three forms of flow energy. The dynamic Smagorinsky model does a good job except for the vertical kinetic energy, which is best matched by the central discretization without any SGS model. These results suggest that a potentially better model could be obtained by applying the dynamic Smagorinsky model only to the horizontal velocity components and leaving the vertical velocity component unmodified. This will be subject of our future work.

ACKNOWLEDGEMENTS

This work was funded by the German Research Foundation (DFG) under the grant HI 1273-1 in the framework

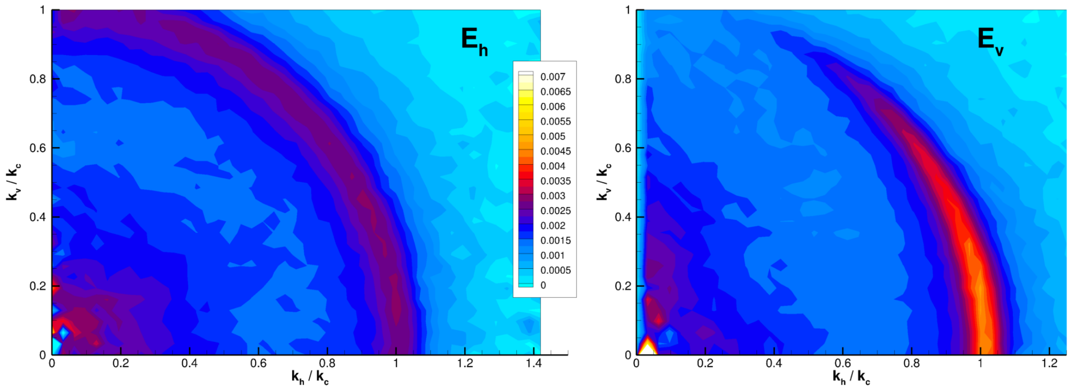
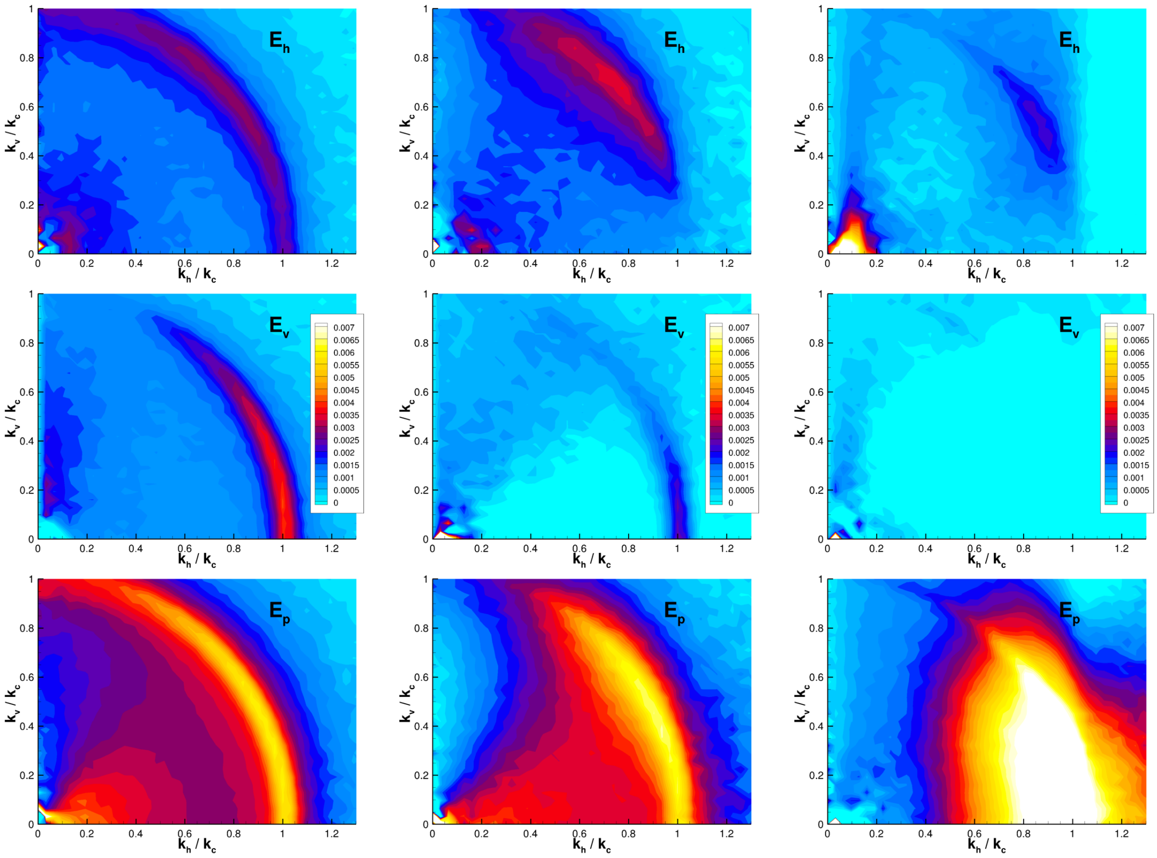


Figure 2. 2D spectral eddy viscosity of neutrally stratified horizontally forced turbulence (test grid with 32^3 cells). Horizontal (left) and vertical (right) kinetic energy



(a) $\mathscr{R} = 184,\ \mathrm{Fr} = 0.089$ (b) $\mathscr{R} = 17.2,\ \mathrm{Fr} = 0.025$ (c) $\mathscr{R} = 2.1,\ \mathrm{Fr} = 0.008$

Figure 3. Spectral eddy viscosity ν_t of stably stratified turbulence (test grid with 32^3 cells) at different Froude numbers (corresponding to weak, medium and strong stratification). Horizontal (top) and vertical (middle) kinetic energy as well as potential energy (bottom)

of the MetStröm priority programme (SPP 1276). Computational resources were provided by the High Performance Computing Center Stuttgart (HLRS) under the grant TIGRA.

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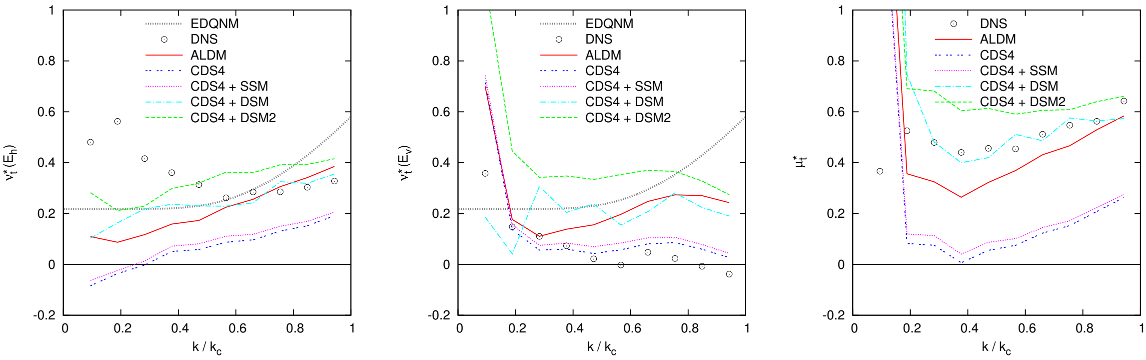

(a) $\nu_t^*(E_h)$ (b) $\nu_t^*(E_v)$ (c) $D_t^*(E_p)$

Figure 4. Diagonal SEV and SED of stably stratified turbulence at $\mathscr{R} = 17.2$ (32^3 cells) computed with different discretization schemes and turbulence SGS models

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