ON THE ERGODICITY OF GRID TURBULENCE

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ABSTRACT

A long time running direct numerical simulation (DNS) based on the lattice Boltzmann method is carried out in grid turbulence with the view to compare spatially averaged statistical properties in planes perpendicular to the mean flow with their temporal counterparts. The results show that the two averages become equal a short distance downstream of the grid. This equality indicates that the flow has become homogeneous in a plane perpendicular to the mean flow. This is an important result, since it confirms that hot-wire measurements are appropriate for testing theoretical results based on spatially averaged statistics. It is equally important in the context of DNSs of grid turbulence, since it justifies the (lateral) spatial averaging using several realizations, to determine various statistical properties. Finally, the very good agreement between temporal and spatial averages validates the comparison between temporal (experiments) and spatial (DNS) statistical properties.

The results are also interesting because, since the flow is stationary in time and spatially homogeneous in the lateral directions, the equality between the two types of averaging can be seen to provide support for the ergodic hypothesis in grid turbulence in planes perpendicular to the mean flow.

INTRODUCTION

Only recently, it was shown that direct numerical simulations of grid turbulence with the actual grid within the computational domain were possible (e.g. Djenidi, 2006; Ertunc et al., 2010; Djenidi and Tardu, 2012). This permits carrying out meaningful comparisons between DNS and experiments, as well as exploring further grid turbulence and testing theoretical results more adequately. While direct comparisons between DNS and hot-wire data can be carried out by simply performing single-point time averages on the numerical signals as for hot-wire data, this would involve running the simulations over relatively long periods of time. To avoid this, one could perform spatial averaging on relatively few independent velocity fields. However, this supposes that spatial and temporal averages are equivalent, e.g. that the ergodic hypothesis, often used in turbulence, is valid. According to this hypothesis, if the turbulence is both statistically stationary in time and homogeneous in space, then its temporal and spatial statistical properties should be the same. Galanti & Tsinober (2004) performed a DNS of turbulence in a cubic domain with periodic conditions over a long time and showed that temporal and spatial statistical properties are equal. They argued that this equality supported the ergodic hypothesis. Although this hypothesis has been assumed in grid turbulence (Batchelor, 1953, page 16 and 17), it has not yet been tested.

While grid turbulence is statistically stationary in time, it cannot satisfy ergodicity completely since it decays with increasing distance behind the grid. However, one may consider the ergodicity in planes perpendicular to the mean flow direction if the turbulence is spatially homogeneous in these planes. Grant & Nisbet (1957) studied the issue of the lateral inhomogeneity in grid turbulence made of horizontal and vertical bars in a biplane configuration. They reported the presence of lateral inhomogeneity in the turbulence intensity even at x/M as large as 80 (M is the meshlength). They suggested that for the inhomogeneity to be negligible, adjacent "wakes" emanating from the bars should overlap appreciably. This seems to be supported by Ertunc et al. (2010) data who used grids with different porosities. The recent direct numerical simulations of Djenidi and Tardu (2012) of a grid-generated turbulence, where the grid is made up of independent flat square elements and a solidity of about 25% (see Figure 1 below) showed that while the lateral inhomogeneity was high in the region close to the grid ($x/M < 10$), it decreases significantly with increasing distance. However, the downstream extent was only $x/M = 15$, not long enough for the turbulence to reach a complete (lateral) homogenous state. It was observed though that the grid configuration generates a turbulence which tends to reach lateral homogeneity over a smaller distance than that for grids made of bars. This may be because the individual wakes generated by the square blocks interact more strongly and earlier than those generated by the vertical and horizontal bars. Corrsin (1963) proposed at least three criteria that must be satisfied for ensuring homogeneity in grid turbulence (grid made of horizontal and vertical bars): i) large grid porosity, (ii) large L/M (L is the height/diameter of the wind tunnel), and (iii) measurements should be taken at least $40M$ downstream of the grid. While these criteria can be easily satisfied in experiments, they may be problematic for DNS, in particular the last two, which may imply the use of a very large computational domain. We believe that for the present grid, made of square flat blocks with a solidity of 25%, the last two criteria can be relaxed, thus allowing the use of a "minimal" computational domain.

Ergodicity is the equality between ensemble and appropriately defined time or space averages obtained from a single sample. The necessary but not sufficient conditions for ergodicity are stationarity in time and homogeneity in space. Strictly speaking, testing ergodicity implies performing measurements over a very long time in one realization as well as over an infinite number of realizations. Such a procedure is obviously impractical. However, as mentioned earlier, if the system is statistically stationary in time and homogeneous in space then ergodicity implies that the temporal and spatial statistics are equal. Thus, the main purpose of the present work is to compare spatially averaged statistics in planes perpendicular to the mean flow with their temporal counterparts. Agreement between these two types of statistics could be interpreted, albeit indirectly, as providing, support for the ergodic hypothesis.

NUMERICAL DETAILS

The lattice Bolzmann Method

The direct numerical simulation (DNS) is carried out using the lattice Boltzmann method (LBM). Rather than solving the governing fluid equations (Navier-Stokes equations), the LBM solves the Boltzmann equation on a 3D lattice (Frisch Hasslacher & Pomeau, 1986). The method was successfully used to simulate turbulent flows (Djenidi, 2006 and 2008; Burattini et al. 2006). Note that unless otherwise specified all quantities are expressed in lattice units or made non-dimensional. Details of the LBM implementation are given in Djenidi (2006).

Computational domain and boundary conditions The computational uniform Cartesian mesh consists of $1600 \times 240 \times 240$ mesh points with $\Delta x = \Delta y = \Delta z = 1$ (x is the longitudinal direction and y and z the lateral directions). The streamwise size of the computational domain is twice that of Djenidi & Tardu (2012). The turbulence-generating grid (placed at the x-node of 180) is made up of 6×6 floating flat square elements in an aligned arrangement (Figure 1). Each element is represented by $1 \times 20 \times 20$ mesh points and the mesh spacing (M) between the centre of two elements is 40 mesh points (i.e. $2D$), yielding a grid solidity of 0.25. The downstream distance extends to $x/D = 70$ (equivalently $x/M = 35$), where the origin of x is the grid plane and $D = 20$ mesh points is the block side length.

Periodic conditions are applied in the y- and z-directions. At the inlet, a uniform velocity ($U_0 = 0.05$, and $V_0 = W_0 = 0$) is imposed, and a convective boundary condition is applied at the outlet. It was observed that the convective condition affected marginally the simulation results within a distance of less than $1D$ upstream of the outlet. A no-slip condition at the grid elements is implemented with a bounce-back scheme (Succi, 2001). The Reynolds number, R_M, is about 3200. This is a relatively small value, which allows a reasonably good grid resolution which varies from about 2.9η at $x/D = 8$ to 0.78η at $x/D = 68$; η is the Kolmogorov length scale. The steady state solution is obtained after 30,000 iterations. The first velocity field is saved after 10^5 iterations. Subsequently, 100 velocity fields are recorded, each separated by about 10,000 iterations (about $8.5\ \lambda/u'$, u' is the velocity fluctuation rms at $x/D = 50$) to ensure that consecutive fields are uncorrelated. In order to avoid the occurrence of instabilities where the magnitude of the local strain rate could be large, mainly around the grid, a large eddy simulation (LES) scheme with a filter size equal to the mesh resolution was introduced (details of this scheme can be found in Djenidi, 2006).

The Taylor microscale Reynolds number, R_λ, varies from about 60 at $x/D = 8$ to about 18 at $x/D = 70$ (ν is the kinematic viscosity of the fluid). It should be noted that R_λ decreases rapidly from a maximum of about 1300 at $x/D = 0.5$ to 92 at $x/D = 5$.

Results

Figure 1 shows an example of instantaneous contour of the enstrophy ω^2, the square of the vorticity magnitude, in the region $55 \le x/D \le 65$; the contour value shown is that corresponding to $5 < \omega^2 >$ (the brackets denote the spatial average in the volume $55 \le x/D \le 65$, $-12 \le y/D, z/D \le 12$). The figure reveals that despite the low values of R_λ, the turbulence field is made of rather elongated vortical structures, remarkably similar to those observed in the numerical simulation of the 3D periodic box turbulence of Ishihara et al. (2007). These authors found that these fine structures tend to form clusters when the Reynolds number increases. It is not evident that such clusters occur in the present flow, although detailled exploration of the field indicates that there are regions of the domaine where the density of the structures appears larger than in other parts. Interestingly, there appears to be no preferred oriention despite the imposed mean flow in the longitudinal direction. This feature is consistent with the concept of local isotropy. This characteristic feature of turbulence is further observed in Figure 2 showing the Laplacien of the pressure field expressed as:

$$\nabla^2 p/\rho = \frac{1}{2}(\omega^2 - \varepsilon/\nu), \qquad (1)$$

It is commonly argued that the Poisson equation (Eq. 1) can be used to detect high vorticity regions in a turbulent field. This is so because the equation reflects a local excess of enstrophy compared to dissipation. Figure 2 clearly reveals localised regions where ω^2 contribution to $\nabla^2 p/\rho$ exceeds that of ε/ν. The topology of these regions is either structureless flat "blob" or elongated ellipsoid. Villermaux et al. (1995) reported that they observed intermitent intense vortical filements in a stationary turbulence generated by an oscillating grid in a water tank.

Figures 1 and 2 point to a clear similarity between the 3D periodic box turbulence and the grid-generated turbulence despite the streamwise non-homogeneity in the latter case. In particular, On can expect that the statistics of grid turbulence in planes perpendicular to the main flow be equivalent to that obtained in the 3D periodic turbulence. Galanti & Tsinober (2004) performed a DNS of turbulence in a cubic domain with periodic conditions over a long time

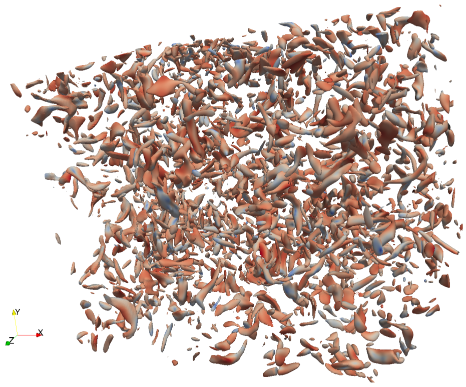

Figure 1. Instantaneous isocontour $\omega^2/<\omega^2>=5$ in the volume $55 \leq x/D \leq 65, -12 \leq y/D, z/D \leq 12$. The x, y and z directions are marked by the red, yellow and green vectors, respectively.

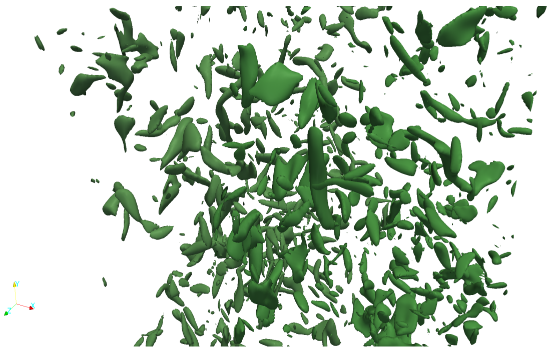

Figure 2. Instantaneous isocontour $\nabla^2 p/<\nabla^2 p>=5$ (with $\nabla^2 p = \frac{1}{2}\rho(\omega^2 - \varepsilon/\nu)$ in the volume $55 \leq x/D \leq 60, -12 \leq y/D, z/D \leq 12$. The x, y and z directions are marked by the red, yellow and green vectors, respectively.

and showed that temporal and spatial statistical properties are equal. They argued that this equality supported the ergodic hypothesis. Although this hypothesis has been assumed in grid turbulence (Batchelor, 1953, page 16 and 17), it has not yet been tested. In the next section we will address this issue

Lateral homogeneity

Before comparing the single-point temporal averages with the spatial averages in planes perpendicular to the mean flow, one must first verify the spatial homogeneity in those planes. For this purpose, long time series (about 773 times λ/u') are recorded at various downstream positions behind the grid. The first set of signals is taken along the grid centreline between four adjacent blocks. The second is taken along a line perpendicular to the centre of a block.

Figure 3 shows the probability density function $p(v)$ (odf) defined as:

$$p(v) = pdf(\frac{v-\bar{v}}{\sigma_v}), \quad (2)$$

where the overbar denotes the time average and σ_v the variance of v. For convenience, we will denote

$$\bar{v} =< v(\mathbf{x}) >_T = \frac{1}{T}\int_{t_0}^{t_0+T} v(\mathbf{x},t)dt \quad (3)$$

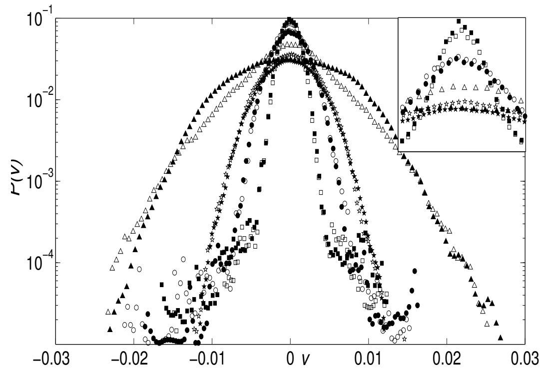

Figure 3. Probability density function $p(v)$ at $x/D = 8.5$ (triangles), 18.5 (stars) 28.5 (circles) and 53.5 (squares) and $y = z = 0$ (opened symbols) and $y = z = D$ (filled symbols). Inset: enlarged region $-0.003 \leq v \leq 0.003$.

where $\mathbf{x} = (x, y, z)$ is a fixed spatial position. Notice first the occurrence of relatively large values of v (about 40% of the mean velocity U at $x/D = 28.5$ and 30% at $x/D = 53.5$) in the tails of the joint pdfs, which suggests that computing the ensemble average for directly testing ergodicity requires a very large number of realizations. Close to the grid ($x/D = 8.5$), the distributions change with the lateral position, highlighting the non-homogeneity of the flow in the near-grid region. For example, notice the flatter shape of the distribution at $y = z = D$ for $-0.01 \leq v \leq 0.01$. As x/D increases, the difference in $p(v)$ between the two lateral locations reduces considerably. At $x/D = 18.5$, the distributions at the two lateral positions are close to each other, whereas they are nearly indistinguishable for $x/D \geq 28.5$. These results indicate that the turbulence becomes spatially homogeneous in the lateral directions at $x/D \sim 20$. This is further reinforced by the (non-normalised) joint probabiliy density function between u and v, or $p(u,v)$, at $x/D = 28.5$ for the thw the two lateral positions (Figures 4). The two distributions are similar and almost perfectly circular.

The lateral homogeneity is further illustrated in Figure 5 showing the variance of the transverse velocity component w, the Taylor microscale, λ $(=< u^2 >^{1/2} / < (\partial u/\partial x)^2 >)$, and the variance of the second-order velocity derivative, $\partial^2 u/\partial x^2$, (the temporal derivative was conveted in to spatial derivative via Taylor's hypothesis $t = U_c x$, where the convective velocity U_c istaken equal t U_0). Notice that there is virtually no difference between the two lateral positions for $x/D \geq 15$.

Temporal and spatial statistics

The previous section indicates that the turbulence downstream of the grid can be considered to be spatially homogeneous in planes perpendicular to the mean flow for $x/D \geq 20$. Since the turbulence is also statistically stationary, we will next test the two types of averages. As stated earlier, if the ergodic hypothesis in the planes perpendicular to the mean flow is satisfied then $< Q(x) >_T =< Q(x,t) >_S$ where $< Q(x,t) >_S$ is the spatial averaging in planes perpendicular to the mean flow and defined as:

$$< Q(x,t) >_S = \frac{1}{S}\int\int_S Q(\mathbf{x},t)dydz \quad (4)$$

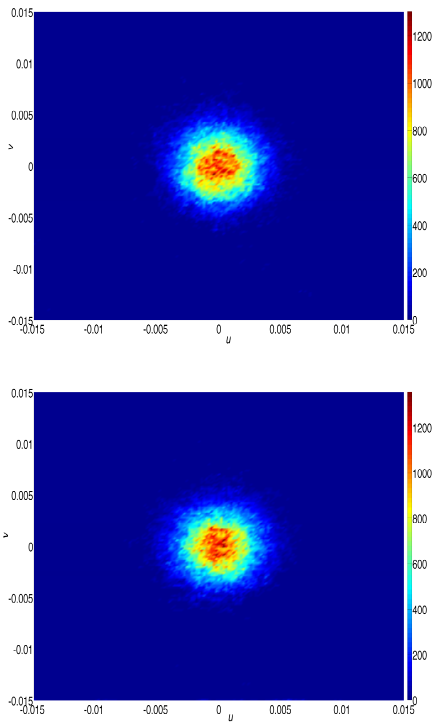

Figure 4. Joint pdf of u and v at $x/D = 28.5$ and $y = z = 0$ (top) and $y = z = D$ (bottom). The coulourmaps are used only as guide.

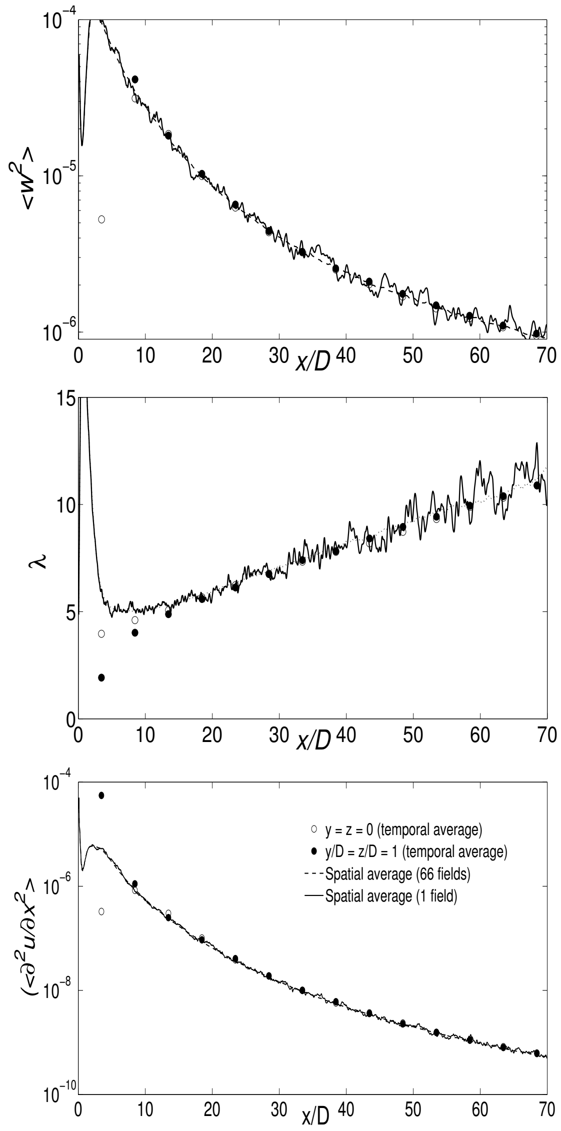

Figure 5. Streamwise variations of temporal (symbols) and spatial (lines) averages of $< w^2 >$, λ and $< (\partial^2 u/\partial x^2) >$. The temporal averages are calculated at two lateral locations. A linear-log scale is used for $< w^2 >$ and $< (\partial^2 u/\partial x^2) >$.

with S representing the plane (y,z) perpendicular to the mean flow. In Eq. (4), the spatial integration is carried out over one randomly selected realization (one velocity field). Figure 5 compares $< Q(x) >_T$ with $< Q(x,t) >_S$. The figure also shows the lateral spatial averaging over 66 realizations. The good correspondance between one and 66 realizations points to the ergodicity nature of the quantities investigated, despite the inevitable noise associated with the single realization (due to the finite domain, which limits the lateral extent of the spatial averaging). Clearly, the square root of the "noise" between one realization and converged statistics goes to zero as the size of the sample is increased. That is precisely the definition of mean-ergodic processes. Whether or not all possible measures needed to describe the turbulence behind a grid are ergodic is of course an entirely open question.

There is a perfect match in figure 5 between the spatial averages ($< . >_S$) of the various quantities and their temporal counterparts ($< . >_T$), demonstrating the equivalence between the two types of averaging. It is not quite clear why the correspondence between the spatial averaging for one realization and temporal averaging is better for velocity derivatives than velocity fluctuations. Galanti & Tsinober (2004), who reported similar differences, speculate that the better agreement between the temporal and (one-realization) spatial statistics associated with the field of velocity derivatives is the phenomenon of self-amplification of the velocity field in three-dimensional turbulence, which is not affected by the large scale motion.

Further illustration of the equivalence between the two averages is seen in the good agreement between the temporal and spatial pdfs of v (Figure 6; the same is observed for the other velocity components). The differences seen at large values of v are due to an insufficient separation between the integral length scale and the size of the computational domain. This is more pronounced at $x/D = 53.5$ than at $x/D = 28.5$; the integral length scale is larger at the latter station, while the dimensions of the lateral sides are kept constant. Galanti and Tsinober (2004) observed a similar feature in their three-dimensional box turbulence simulation.

Batchelor (1953) pointed out that it is the objective of the ergodic theory to show that the spatial average is the same for all realizations and is identical to the ensemble average (or probability average) and simply assumed this to be the case for grid turbulence. Androulakis & Dostoglou (2004) provided rigorous proofs that in homogeneous flow not only does the space average exist almost always, but that it is equal to the ensemble average, if ergodicity is valid. The brief ergotic theory outlined in the introduction and the data of figures 5 and 6 suggest that the spatial average in a plane perpendicular to the mean flow is equal to the en-

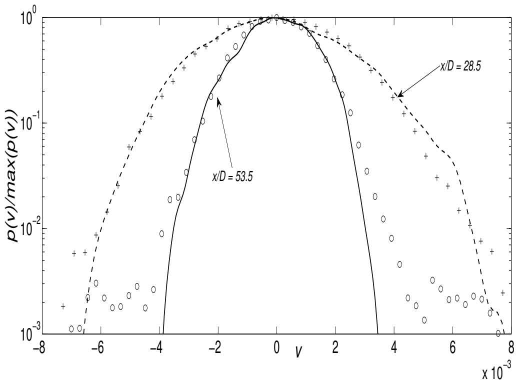

Figure 6. Probability density function $p(v)$ at $x/D = 28.5$ and 53.5 and $y = z = 0$. Temporal PDF: symbols; spatial PDF: lines

semble average, and provide strong support for the ergodic hypothesis, at least for the variables presented.

CONCLUSIONS

A long time running DNS of grid turbulence, implemented with the lattice Boltzmann method, shows that the turbulence becomes homogeneous in planes perpendicular to the mean flow at a relatively short distance downstream of the grid. For the present geometry, with a grid made of flat square blocks and a solidity of 25%, this distance is about $20D$ (or $\sim 10M$). The results also indicate that temporal and (lateral) spatial statistical properties are equal, lending strong support for the validity of ergodicity in grid turbulence in planes perpendicular to the mean flow. This is an important result since it confirms that hot-wire measurements are appropriate for testing theoretical results based on spatially averaged statistics. It is equally important for DNSs of grid turbulence, since it validates carrying out spatial lateral averaging using several realizations, to determine various statistical properties. Finally, the very good agreement between the temporal and spatial averages validates the comparison between the temporal (experiments) and spatial (DNS) statistical properties

A parallel can be drawn with a DNS of turbulent channel flow, where temporal and spatial statistics in planes parallel to the wall are assumed equal. Since comparisons between DNSs and experiments support this assumption, ergodicity, without ever being tested or verified, is validated implicitly because the flow is stationary in time and spatially homogeneous in (x,z) planes (z is the transverse direction) at fixed positions from the wall. Based on the present results, one may conjecture that the ergodic hypothesis can be valid in a turbulent channel flow at fixed positions from the wall. An obvious extension of this conjecture, is that ergodicity in a turbulent boundary layer along a line perpendicular to the mean flow and parallel to the wall (i.e. at fixed points from the wall in the transverse direction) can also be valid. Galanti & Tsinober argued that the ergotic hypothesis can be expected to be valid along homogeneous coordinates of nonhomogeneous flows, or that one can at least expect that temporal and spatial statistical properties to be equal.

Finally, the results indicate that systematic checks on the homogeneity of the flow must be carried out before undertaking any statistical analysis for which homogeneity is required. For example, in the case of grid turbulence, it is important to determine the distance downstream of the grid where the flow becomes approximately homogeneous. Failing to perform such checks may lead, for instance, to an incorrect estimate of the power law decay n for q ($q \sim x^n$, where q is twice the turbulent kinetic energy) or inadequate tests for local isotropy since one may inadvertantly include data within a non-homogeneous region where ergodicity is not valid.

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