NAVIER-STOKES SIMULATION WITH FLUID PARTICLES LOCATION UNCERTAINTY

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ABSTRACT

We present simulation results of a stochastic Navier-Stokes model that incorporates uncertainty on the fluid parcels location. This model ensues from a decomposition of the flow in terms of a differentiable drift component and a time uncorrelated uncertainty random term. The dynamics associated to the drift component, derived from a stochastic version of the Reynolds transport theorem, includes in its general form an uncertainty dependent anisotropic diffusion that cannot be immediately related to usual eddy viscosity assumption. The simulation we present relies on a wavelet numerical scheme and is here experimented on a Green-Taylor vortex.

Introduction

The large scale analysis of complex fluid flows in domains ranging from climate sciences to engineering requires to constitute flow dynamics models that incorporate properly missing contributions. This includes for instance physical phenomenon such as unknown small scale forcing or boundary layers influence, but also partially known inlet or boundary conditions, and numerical errors arising from truncation policy and scale coarsening procedures. An accurate deterministic modeling of the effects of these unknown terms is obviously hardly achievable and we advocate instead the use of a stochastic modeling. Within this prospect, we aim at describing these missing terms as random variables referred to as in the following as flow dynamics uncertainties. The modeling of such uncertainties but also of their evolution is of the utmost importance for data assimilation purposes to analyze accurately past situations from data or for the forecast of a plausible ensemble of realizations. Such a modeling is usually set up through the adjunction of random forcing terms to standard expressions of Navier-Stokes equations (Bensoussan & Temam, 1973; Flandoli, 2008). We explore here a somewhat different strategy. Instead of considering a given – eventually simplified – dynamics and then to supplement it with random forcing terms, we start from a general Lagrangian formulation of the fluid motion. The sought Eulerian dynamics is then deduced from this general stochastic velocity description and standard physical principles or approximations. This construction, reminiscent to the framework proposed by Mikulevicius & Rozovskii (2004), has the great advantage to let naturally emerge deterministic and stochastic uncertainty terms related to the different errors transported by the evolution model.

In order to achieve this aim, we will assume throughout this study that the particles displacement can be separated in two components: a smooth differentiable function and an uncertainty function uncorrelated in time but correlated in space. The whole displacement is defined as an Ito diffusion of the form:

$$d\boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{X}(\boldsymbol{x},t),t)dt + \boldsymbol{\sigma}(\boldsymbol{X}(\boldsymbol{x},t),t)d\tilde{\boldsymbol{B}}_t, \qquad (1)$$

where $\boldsymbol{X}$ represents the trajectory followed by a fluid particle starting at point $\boldsymbol{X}_{|t=0}(\boldsymbol{x}) = \boldsymbol{x}$ of the domain Ω. This constitutes a Lagrangian representation of the fluid flow and $d\boldsymbol{X}(\boldsymbol{x},t)$ figures the Lagrangian displacement map at time t. In this expression, $\boldsymbol{w} = (w_1, w_2, w_3)$, corresponds to the smooth resolved velocity component of the flow. It is assumed to be a deterministic differentiable function (of eventually random arguments). The combination of these velocity fields provides an Eulerian description of the complete velocity fields driving the particles:

$$\boldsymbol{U}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x},t)dt + \boldsymbol{\sigma}(\boldsymbol{x},t)d\tilde{\boldsymbol{B}}_t. \qquad (2)$$

This random field $\boldsymbol{U}(\boldsymbol{x},t)$, which should follow a stochastic linear momentum conservation principle, involves unknown characteristics, $\boldsymbol{\sigma}$, and, $\boldsymbol{w}$, that have to be determined or specified. More precisely, in a similar way to Large Eddies Simulation or to Reynolds Average Numerical Simulation models (see the textbook Sagaut, 2005, for an extended review), we will first derive the resolved drift dynamics considering a given specification of the uncertainty. This dynamics is derived from a stochastic version of the Reynolds transport theorem relying on a specific model of a tempered Brownian motion fields.

Random unresolved uncertainty model

This random field, $\tilde{\boldsymbol{B}}_t$, is built from a finite dimensional discrete set of standard Brownian variables as

$$\tilde{\boldsymbol{B}}_t^n(\boldsymbol{x}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{B}_t(\boldsymbol{x}_i)\varphi_\nu(\boldsymbol{x}-\boldsymbol{x}_i), \tag{3}$$

where $\boldsymbol{B}_t = \{\boldsymbol{B}_t(\boldsymbol{x}_i), i = 1,\dots,n\}$ is a set of independent d-dimensional (with $d = 2$ or 3) standard Brownian motions centered on the points of a discrete grid $S = \{\boldsymbol{x}_i, i = 1,\dots,n\} \subset \Omega$ and φ is a Gaussian function. It is immediate to check that $\tilde{\boldsymbol{B}}_t^n$ is a zero mean Gaussian process with uncorrelated increments, which tends in law to a zero mean continuous process with a limiting covariance $\mathbf{Q} = t\varphi_{\sqrt{2}\nu}(\boldsymbol{x}-\boldsymbol{y})\mathbb{I}_d$. This limiting process will be denoted in a formal way through a convolution product $\tilde{\boldsymbol{B}}_t = \boldsymbol{B}_t \star \varphi_\nu$. The covariance operator, $\mathbf{Q}$, is positive definite and of finite trace for non zero Gaussian smoothing standard deviation:

$$\mathbb{E}|\tilde{\boldsymbol{B}}|_2^2 = \frac{td}{n}\sum_i \int \varphi_\nu^2(\boldsymbol{x}-\boldsymbol{x}_i)d\boldsymbol{x} = td(4\pi\nu^2)^{-d/2}. \tag{4}$$

Hence, the energy of the uncertainty term does not depend on the number of grid points used for its construction but only on the Gaussian smoothing standard deviation. The analog of the white noise on the bounded domain Ω is similarly defined from the generalized function $d\tilde{\boldsymbol{B}}_t = d\boldsymbol{B}_t \star \varphi_\nu$ and, $\boldsymbol{\sigma}_t$, a linear bounded deterministic symmetric operator with null value outside the domain interior. The random oscillating component is denoted:

$$\boldsymbol{\sigma}(\boldsymbol{x},t)d\tilde{\boldsymbol{B}}_t = \int \boldsymbol{\sigma}_t(\boldsymbol{x},\boldsymbol{y})d\tilde{\boldsymbol{B}}_t(\boldsymbol{y})d\boldsymbol{y}.$$

Let us remark that divergence free random field necessitates a divergence free tensor.

Toys models pioneered by Kraichnan (1968) and intensively explored for passive scalar turbulence study (Gawedzky & Kupiainen, 1995; Kraichnan, 1968; Majda & Kramer, 1999) can be easily specified with such a model. The Kraichnan model is formally defined from a divergence free projector $\mathbb{P}$ as:

$$d\boldsymbol{\xi}_t^\zeta = \mathbb{P}\star\psi\star f^\zeta \star d\boldsymbol{B}_t,$$

and involves a power law function $f^\zeta(\boldsymbol{x}) = C\|\boldsymbol{x}\|^{\zeta/2}$ with exponent $0 < \zeta < 2$ and a band-pass cut-off function, ψ, covering the inertial range defined between the short dissipative scale ℓ_D and the large integral scale L at which the forcing takes place. The variance (or time derivative of the quadratic variation process) of this isotropic random field is constant and diagonal ($\mathbf{Q}(0) \propto (\zeta^{-1}(L-\ell_D))\delta^{ij}$). We will see that such a random field leads to an intuitive eddy viscosity term defined from the noise variance. However more general random fields will let rise an anisotropic diffusion term that cannot be immediately related to the usual eddy viscosity assumption first formulated by Boussinesq (1877) and that remains intensively used in the Large Eddies Simulation paradigm since the work of Smagorinsky (1963) and Lilly (1966).

Stochastic Reynolds transport theorem

In a similar way as in the deterministic case, our derivation relies essentially on a stochastic version of the Reynolds transport theorem, which states the rate of change of a scalar function, q, within a material fluid volume $\mathscr{V}(t)$ transported by (1):

$$d\int_{\mathscr{V}(t)} q(\boldsymbol{x},t)d\boldsymbol{x} = \int_{\mathscr{V}(t)} \{dq_t + [\boldsymbol{\nabla}\cdot(q\boldsymbol{w}) + \frac{1}{2}\|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2 q \\ - \sum_{i,j}\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}q)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0}]dt + \boldsymbol{\nabla}\cdot(q\boldsymbol{\sigma}d\tilde{\boldsymbol{B}}_t)\}d\boldsymbol{x}. \tag{5}$$

In this expression the first term is a time difference at fixed coordinates, $\boldsymbol{x}$, and the third term must be computed considering the diffusion tensor is divergence free. The tensor $\mathbf{a}(\boldsymbol{x})$ is the uncertainty variance. It is defined as:

$$a^{ij}(\boldsymbol{x},t) = \sum_k \sigma_\nu^{ik}(\boldsymbol{x},t)\sigma_\nu^{kj}(\boldsymbol{x},t), \tag{6}$$

where $\boldsymbol{\sigma}_\nu(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{\sigma}(\boldsymbol{x},\bullet,t)\star\varphi_\nu(\boldsymbol{y})$ denotes a filtered version of the diffusion tensor along its second component. This rate of change is obtained from the Ito-Wentzell differentiation of a function tending to the material volume characteristic function and through an integration by part (Mémin, 2013). This relation allows us stating a mass conservation principle that accounts for an uncertainty on the fluid particles location. Applying the previous transport theorem to the fluid density $\rho(\boldsymbol{x},t)$ and canceling this expression for arbitrary volumes, we get a general mass conservation constraint:

$$d\rho_t + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{w})dt = \frac{1}{2}(\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\rho)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} \\ - \frac{1}{2}\|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2\rho)dt - \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}d\tilde{\boldsymbol{B}}_t). \tag{7}$$

For an incompressible fluid with constant density, canceling separately the slow deterministic terms and the rapid oscillating stochastic terms, and imposing to the whole deformation field (2) to be volume preserving, this system simplifies in a set of incompressibility relations:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}d\tilde{\boldsymbol{B}}_t) = 0,\ \ \boldsymbol{\nabla}\cdot\boldsymbol{w} = 0,\ \ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\mathbf{a}) = 0, \tag{8}$$

composed of two standard volume preserving constraints accompanied with a less intuitive additional constraint on the quadratic variation tensor. For the Kraichnan model (or for any divergence free homogeneous random fields) this last constraint is naturally satisfied as its quadratic variation is constant. The system reduces hence to the standard incompressibility constraint.

For isochoric flow with varying density we get a mass conservation constraint of the form:

$$d\rho_t + \boldsymbol{\nabla}\rho\boldsymbol{w}dt - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i x_j}(\rho\mathbf{a}^{ij})dt = \boldsymbol{\nabla}\rho\sigma d\tilde{\boldsymbol{B}}_t. \tag{9}$$

In the case of the Kraichnan model the density variation involves a Laplacian diffusion and for a mean field dynamics (*i.e.* when $\boldsymbol{w} = \mathbb{E}d\mathbf{X}_t$) the density expectation evolution comes to an intuitive advection diffusion equation. Let us note that the same kind of deterministic advection equation with anisotropic diffusion is also obtained if the noise lies in the tangent plane of isodensity surfaces. This type of diffusion is considered in geophysics to encode large scale mixing of stratified fluids. They are called isopycnal or isoneutral diffusion in this context (Gent & McWilliams, 1990).

Linear momentum conservation

The mass conservation constraint and the stochastic version of the Reynolds theorem allows us expressing the balance between the momentum and the forces:

$$d\int_{\mathscr{V}(t)} \rho(\boldsymbol{w}(\boldsymbol{x},t)dt + \boldsymbol{\sigma}(\boldsymbol{x},t)d\tilde{\boldsymbol{B}}_t)d\boldsymbol{x} = \int_{\mathscr{V}(t)} \boldsymbol{F}(\boldsymbol{x},t)d\boldsymbol{x}.$$

The acceleration is highly irregular and has to be interpreted in the sense of distribution. For every $h \in C_0^\infty(\mathbb{R})$:

$$\int h(t) \int_{\mathscr{V}(t)} \boldsymbol{F}(\boldsymbol{x},t)d\boldsymbol{x}dt = -\int h'(t) \int_{\mathscr{V}(t)} \boldsymbol{\sigma}(\boldsymbol{x},t)d\tilde{\boldsymbol{B}}_t d\boldsymbol{x}dt + \int h(t)\, d\int_{\mathscr{V}(t)} \rho \boldsymbol{w}(t,\boldsymbol{x})d\boldsymbol{x}dt. \quad (10)$$

Since both sides of this equation must have the same structure, the forces can be written as:

$$\int h(t) \int_{\mathscr{V}(t)} \boldsymbol{F}(\boldsymbol{x},t)d\boldsymbol{x}dt = -\int h'(t) \int_{\mathscr{V}(t)} \boldsymbol{\sigma}(t,\boldsymbol{x})d\tilde{\boldsymbol{B}}_t d\boldsymbol{x}dt + \int h(t) \int_{\mathscr{V}(t)} (\boldsymbol{f}(t,\boldsymbol{x})d\boldsymbol{x}dt + \boldsymbol{\theta}(t,\boldsymbol{x})d\tilde{\boldsymbol{B}}_t)d\boldsymbol{x}. \quad (11)$$

The right hand first terms of (10) and (11) are identical and cancel out. The second term of equation (10) corresponds to the momentum derivative associated to the resolved velocity component. The second term of (11) provides us the structure of the forces under localization uncertainty. We will consider that only body forces and surface forces are involved (*i.e.* there is no external force except gravity). As a direct extension of the deterministic case, the surface forces are given as

$$\boldsymbol{\Sigma} = \int_{\mathscr{V}} -\boldsymbol{\nabla}(pdt + d\tilde{p}_t) + \mu(\Delta \boldsymbol{U} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{U})),$$

where μ is the dynamic viscosity, $p(\boldsymbol{x},t)$ denotes the deterministic contribution of the pressure and $d\tilde{p}_t$ is a zero mean stochastic pressure fluctuation. The gravity force is deterministic and standard. Expressing the forces balance with momentum derivative from (5), incorporating the mass preservation principle (7), and finally equating separately the slow temporal components and the highly oscillating Brownian terms, we get a general form of the Navier Stokes equations under location uncertainty (12). The first equation of this system describes the evolution of the deterministic resolved component and similarly to Reynolds formulation it includes an additional stress term that depends here on the resolved velocity component and on the uncertainty variance. The subsequent equations denote a stochastic balance on the diffusion tensor and a mass conservation constraint. This system simplifies greatly in several particular cases. For the divergence free Kraichnan model with a fluid of constant density, we get a system of Navier-Stokes equations:

$$(\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\boldsymbol{\nabla}^T\boldsymbol{w} - \gamma\frac{1}{2}\Delta\boldsymbol{w})\rho = \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\Delta\boldsymbol{w},\ \boldsymbol{\nabla}\cdot\boldsymbol{w} = 0, \quad (13)$$

where the diffusion is augmented by the Kraichnan noise variance. This corresponds to the simplest Boussinesq eddy viscosity assumption with a constant diffusivity coefficient. For a general divergence free random component the drift evolution reads:

$$(\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\boldsymbol{\nabla}^T\boldsymbol{w} - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i x_j}(a^{ij}\boldsymbol{w}))\rho = \rho\boldsymbol{g} - \boldsymbol{\nabla}p + \mu\Delta\boldsymbol{w}, \quad (14)$$

together with the volume preserving constraint (8). This model includes now a more general diffusion term that can be no more directly related to the Boussinesq eddy viscosity formulation. We observe that for divergence free uncertainty this term is globally dissipative as its energy is

$$\int_\Omega \boldsymbol{w}^T \sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\boldsymbol{w})d\boldsymbol{x} = -\int_\Omega \|\nabla\boldsymbol{w}\|^2_{\mathbf{a}} d\boldsymbol{x}.$$

Setting the uncertainty or its variance tensor allows defining directly the subgrid diffusion term that has to be incorporated in the resolved drift component. For instance, considering uncertainties along iso-density surfaces provides immediately a clear justification of the isopycnal diffusion employed in oceanic circulation models. The use also of constant eddy viscosity is also justified as the direct consequence of an isotropic homogeneous uncertainty component. Such an approach opens new perspectives for flow modeling that goes from *a priori* uncertainty specification to data based strategies. This framework, which does not rely neither on Reynolds averaging nor on spatial filtering concept, might be of great interest when uncertainties are prevalent as it is the case in geophysical flows or climate modeling. As another practical consequence, if one consider measurements as supplied by particle image velocimetry methods and related to the true flow kinematics only up to a Gaussian uncertainty, then those measurements does not follow exactly the actual flow dynamics. Their physical interpretation should then be carried out with some care.

Numerical implementation and results

In this section, we describe the principal steps of the wavelet numerical scheme we used to simulate the deterministic drift component, $\boldsymbol{w}$, with periodic boundary conditions. Then, we present numerical results obtained on Taylor-Green vortex (Brachet et *al*., 1983) simulation at moderate Reynolds number. We will consider here only the case of incompressible fluids with a general divergence free uncertainty model.

$$\begin{cases} (\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\boldsymbol{\nabla}^T \boldsymbol{w})\rho - \frac{1}{2}\sum_{i,j} a_{ij}\rho\frac{\partial^2 \boldsymbol{w}}{\partial x_i \partial x_j} - \sum_{i,j}\frac{\partial(a_{ij}\rho)}{\partial x_j}|_{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0}\frac{\partial \boldsymbol{w}}{\partial x_i} = \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu(\Delta \boldsymbol{w} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{w})), \\ \boldsymbol{\nabla} d\tilde{p}_t = -\boldsymbol{w}\boldsymbol{\nabla}^T\rho\boldsymbol{\sigma} d\tilde{\boldsymbol{B}}_t + \mu(\Delta(\boldsymbol{\sigma} d\tilde{B}_t) + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma} d\tilde{B}_t))), \\ d\rho_t + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{w}) - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\rho)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} + \frac{1}{2}\|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2\rho) = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma} d\tilde{\boldsymbol{B}}_t). \end{cases} \quad (12)$$

As previously described, the resolved component is in that case driven by the usual Navier-Stokes equations (14) with an additional "subgrid" anisotropic diffusion provided by $\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\boldsymbol{w})$.

Since the component $\boldsymbol{w}$ satisfies the incompressibility constraint $\nabla\cdot\boldsymbol{w}=0$, projecting (14) onto to the divergence-free function space allows us eliminating the pressure gradient:

$$\frac{\partial \boldsymbol{w}}{\partial t} - \nu\Delta\boldsymbol{w} = \mathbb{P}[\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\boldsymbol{w}) - \boldsymbol{w}\boldsymbol{\nabla}^T\boldsymbol{w}], \quad (15)$$

where $\mathbb{P}$ denotes the orthogonal projector from $L^2(\mathbb{R}^d)^d$ onto the divergence-free function space. This projector, referred to as the Leray projector, is usually defined in the Fourier domain. In this work, it is numerically specified through a projection onto a divergence-free wavelet basis $\{\Psi^{\text{div}}_{\boldsymbol{j},\boldsymbol{k}}\}$, see (Deriaz & Perrier , 2006; Kadri-Harouna & Perrier , 2011) for details.

Equation (15) corresponds to a heat equation with a source term $\mathbb{P}[\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\boldsymbol{w}) - \boldsymbol{w}\boldsymbol{\nabla}^T\boldsymbol{w}]$. Classical methods of heat kernel discretization can hence be used to describe its discrete time evolution. For a chosen time step δt and setting $\boldsymbol{w}^n(\boldsymbol{x}) \simeq \boldsymbol{w}(n\delta t,\boldsymbol{x})$ with $n\in\mathbb{N}$, an implicit Euler scheme applied to the diffusion term leads to:

$$(I-\nu\delta t\Delta)\boldsymbol{w}^{n+1} = \boldsymbol{w}^n - \delta t\mathbb{P}[\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\boldsymbol{w}^n) - \boldsymbol{w}^n\boldsymbol{\nabla}^T\boldsymbol{w}^n]. \quad (16)$$

The pressure p is recovered from the Helmholtz decomposition of the advection term with the anisotropic diffusion term:

$$\boldsymbol{w}\boldsymbol{\nabla}^T\boldsymbol{w} - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\boldsymbol{w}) = \mathbb{P}[\boldsymbol{w}\boldsymbol{\nabla}^T\boldsymbol{w} - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\boldsymbol{w})] - \nabla p.$$

The component $\boldsymbol{w}(t,\boldsymbol{x})$ is spatially specified in terms of its divergence-free wavelet series (Deriaz & Perrier , 2006; Kadri-Harouna & Perrier , 2011):

$$\boldsymbol{w}(t,\boldsymbol{x}) = \sum_{\boldsymbol{j},\boldsymbol{k}} d_{\boldsymbol{j},\boldsymbol{k}}(t)\ \Psi^{div}_{\boldsymbol{j},\boldsymbol{k}}(\boldsymbol{x}),\ \ \boldsymbol{j},\boldsymbol{k}\in\mathbb{Z}^2. \quad (17)$$

This decomposition provides a good time and space scales separation due to wavelets space localization. In this framework, the coarse scale j is fixed as the coarse resolution of the wavelet basis. This corresponds to 2^j mesh grid points per direction on a cartesian grid.

As the divergence-free wavelet basis remains fixed the unknowns in (17) correspond to the set of coefficients $d_{\boldsymbol{j},\boldsymbol{k}}(t)$. Incorporating (17) in (16) and taking $\Psi^{div}_{\boldsymbol{j},\boldsymbol{k}}$ as test function, we get:

$$\mathbb{R}\left(d^{n+1}_{\boldsymbol{j},\boldsymbol{k}}\right) = \mathbb{M}\left(d^n_{\boldsymbol{j},\boldsymbol{k}}\right) - \delta t\mathbb{M}\left(f^n_{\boldsymbol{j},\boldsymbol{k}}\right), \quad (18)$$

with $d^n_{\boldsymbol{j},\boldsymbol{k}} = d_{\boldsymbol{j},\boldsymbol{k}}(n\delta t)$, $f^n_{\boldsymbol{j},\boldsymbol{k}}$ the divergence-free wavelet coefficients of $\mathbb{P}[\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\boldsymbol{w}^n) - \boldsymbol{w}^n\boldsymbol{\nabla}^T\boldsymbol{w}^n]$, $\mathbb{M}$ the Gram matrix of the basis $\{\Psi^{div}_{\boldsymbol{j},\boldsymbol{k}}\}$ and $\mathbb{R}$ the matrix of the operator $(I-\nu\delta t\Delta)$ on this basis. For the computation and the inversion of theses matrices, see (Kadri-Harouna , 2010; Kadri-Harouna & Perrier , 2011). One of the main advantages of the method lies in its great flexibility of the anisotropic diffusion coefficients specification. The tensor $\boldsymbol{a}$ can be fixed from the *a priori* knowledges we have either on the uncertainty's variance or on the uncertainty diffusion tensor. The uncertainty variance could as well be learned from small scale measurements or specified from empirical local statistics of the resolved component. Let us point out nevertheless, that whatever the choice carried out, the explicit discretization of anisotropic diffusion term in the temporal discrete scheme of (15) induces a viscosity dependent CFL type condition that must be carefully taken into account. In this study we implement a very simple strategy that consists in defining $\boldsymbol{a}(n\delta t,\boldsymbol{x})$ as local empirical covariances of the resolved velocity fields $\boldsymbol{w}(x)^n$:

$$a_{ij}(x,t) = < (w_i(y)-\mu_i(x,t))(w_j(y)-\mu_j(x,t)) >_{y\in\mathscr{W}(x,t)},$$

where $\mu_i(x,t)$ is the empirical mean on a spatial or temporal window $\mathscr{W}(x,t)$. The empirical averaging is computed either spatially over a small $(3\times3\times3)$ window centered around point $(\boldsymbol{x},t)$ or temporally at point $\boldsymbol{x}$, over the time interval $[(n-2)\delta t, n\delta t]$. In the following, they are referred to as the spatial and temporal uncertainty covariances respectively. To evaluate the numerical accuracy and effectiveness of our method for those two solutions, we take as benchmark the simulation of Taylor-Green vortex at $Re=1600$. This flow becomes rapidly turbulent with creation of small scales structures, followed by a decay phase similar to a decaying homogeneous turbulence. Figure 1 shows the total energy time evolution of the solutions computed with the

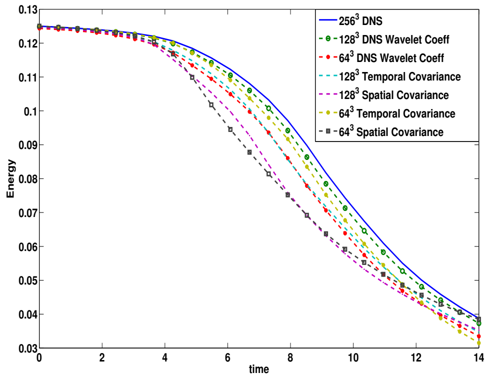

Figure 1. Evolution of the dimensionless energy as a function of the dimensionless time.

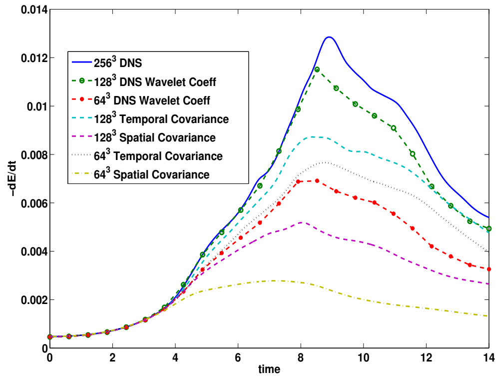

Figure 2. Evolution of the dimensionless energy dissipation rate as a function of the dimensionless time.

previous two uncertainty covariance models, together with the reference solution computed by a direct numerical simulation (DNS) of the incompressible Navier-Stokes equations on a 256^3 grid. This corresponds to $j = 8$ wavelet space resolution. The reference DNS solution on 128^3 and 64^3 grids corresponds to wavelet multiresolution projection onto those grids of the reference solution. They are hence computed from a spatial cut-off. Let us point out that as it does not correspond to a spectral filtering, the energy of this projection intrinsically depends on the wavelet generator. Wavelets offer from that point of view an optimal choice with respect to a scale space energy representation due to their fine space-frequency localization property. On Figure 2 we show the energy dissipation rates associated to the different solutions. As can be observed from these two curves, the spatial covariance is associated to a higher dissipation rates than the temporal covariance. The temporal covariance clearly performs better. Compared to state of the art results of the literature (Brachet et *al*., 1983; Fauconnier et *al*., 2009), those first results are very encouraging. The temporal covariance exhibits a lower dissipation and does not yield any local dissipation when the flow is locally stationary. On the other hand it seems to require a higher CFL condition. On Figure 3, we show the plot of Q iso-surfaces for the dimensionless time $t \approx 8.5$, with:

$$Q = -\frac{1}{2}\sum_{i,j}\frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i}.$$

The sub spatial domain of Figure 3 corresponds to $(0,\pi)^3$. It can be readily observed that the solution computed using the temporal covariance exhibits smaller vortex structures in comparison to the spatial covariance uncertainty model. Let us note that for both models the solutions resemble very much to the corresponding projected DNS.

Conclusion

In this paper we have described a decomposition of the Navier-Stokes equation in terms of a resolved deterministic component and a random uncertainty component figuring the unresolved flow component. This decomposition leads to a new large scale simulation paradigms. Such a scheme has been assessed for a Taylor-Green vortex flow and an accurate wavelet based numerical scheme. The results obtained for a local empirical uncertainty model computed on a temporal window of the resolved component leads to very encouraging results when compared to state of the art large scale simulation of this flow. Within the continuation of the study, we will investigate the derivation of similar model for geophysical flow equations and its use in the case of physical boundary conditions. This situation is more realist and it is known that many sub grid models do not succeed in the presence of a wall. Currently, we analyze the consistency and stability of the associated wavelet numerical schemes. We wish to investigate the use of variational data assimilation technique to determine the sub grid tensor from image data observation operator. This study will be the subject of a new forthcoming paper.

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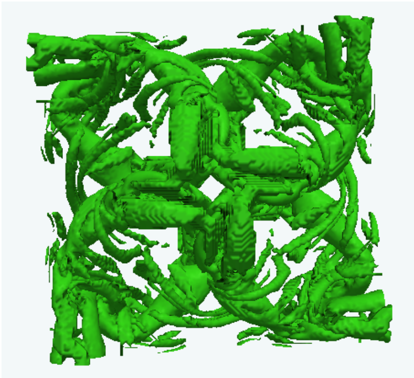
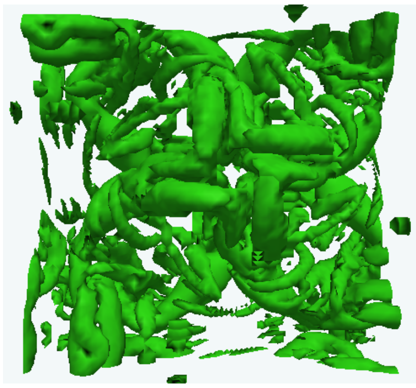
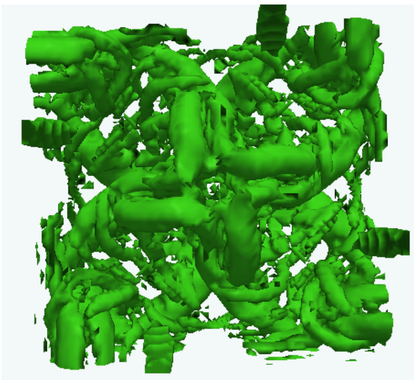


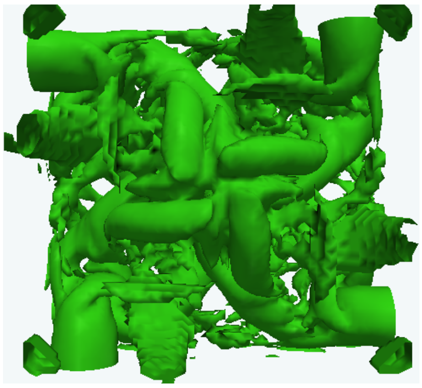
(a) DNS solution



(b) DNS projected



(c) Temporal correlation



(d) Spatial correlation

Figure 3. Iso-surfaces $Q = 1$ for the dimensionless time $t \approx 8.5$: (a) 256^3 grid points DNS solution, (b) DNS solution projected onto 128^3 grid points, (c) solution computed with temporal covariance on 128^3 grid points and (d) solution computed with spatial covariance on 128^3 grid points.

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