SMALL SCALE UNIVERSALITY AND SPECTRAL CHARACTERISTICS IN TURBULENT FLOWS

Yukio Kaneda
Department of Natural Science
Aichi Institute of Technology
Yachikusa 1247, Toyota, 470-0392
ykaneda@aitech.ac.jp

Koji Morishita
Department of Computational Science
Kobe University
Rokkodai-cho 1-1, Nada-ku, Kobe, 657-8501
morishita@port.kobe-u.ac.jp

Takashi Ishihara
Center for Computational Science
Nagoya University
Furo-cho, Chikusa-ku, Nagoya, 464-8603
ishihara@cse.nagoya-u.ac.jp

ABSTRACT

A review is given on studies of statistics at small scales in turbulent flows from a view point of universality. It is assumed in the view that the statistics at sufficiently small scales in the absence of mean flow are at a certain kind of local equilibrium state, and the influence of the mean flow may be regarded as a perturbation added to the equilibrium state. This idea has been examined by comparison of spectral characteristics derived by the idea with those in turbulent boundary layers, mixing layers and direct numerical simulations (DNS) of homogeneous turbulent shear flow. The applicability of this idea to turbulent channel flows is discussed in the light of the data of the log-law region in DNS of turbulent channel flows with the friction Reynolds number Re_τ up to 5120.

INTRODUCTION

Turbulence is a phenomenon involving a huge number of degrees of dynamical freedom. A paradigm of study dealing with systems consisting of such a huge number of degrees of freedom is the statistical mechanics of systems at or near thermal equilibrium state.

In the statistical mechanics, it is known that although it is difficult to trace the trajectory of each of the molecules or atoms in the physical or phase space, there are certain kinds of simple relations between a few variables, the so-called macroscopic variables, such as the pressure, density and temperature characterizing the equilibrium state. The relations are universal in the sense that they are independent of the detail of the difference in the trajectories of the molecules or atoms. It is also known that there are another kind of universal relations characterizing the response of the thermal equilibrium system to the disturbance added to the system.

It is attractive to assume that the similar idea is applicable to turbulence. In fact, underlying the celebrated Kolmogorov theory (Kolmogorov,1941), referred here as K41, is the idea of existence of universal local equilibriums state, the statistics of which can be characterized by a few variables. In this paper, a review is given on studies along this idea with an emphasis on the spectral characteristics. Discussions are also made on the applicability of this idea to turbulent channel flows in the light of the log-law region in recent DNS of turbulent channel flows with the friction Reynolds number Re_τ up to 5120.

UNIVERSALITY AT LOCAL EQUILIBRIUM STATE

We consider here the motion of incompressible fluid obeying the Navier-Stokes (NS) equation. Although it has not been rigorously proved, nor neither is it trivial that there is universality in the statistics of small scales in high Reynolds number turbulence, evidences supporting the existence have been accumulated.

Among them is the so-called 4/5 law. The NS equation is compatible with the statistical homogeneity and isotropy of turbulent flows. Under the assumption of the homogeneity and isotropy of the turbulence statistics, the NS equation with the incompressibility condition yields a rigorous relation called Kármán-Howarth (KH) equation (Kármán and Howarth, 1938). If (i) the external force is confined to only large scales $\sim L_f$, (ii) the statistics is almost stationary at scales much smaller than the characteristic length scale L_E of the energy containing eddies, and (iii) the viscosity works only at small scales $\sim \eta$, then it is shown from the KH equation that

$$B_3^L(r) = -4/5\,\langle\varepsilon\rangle\, r, \tag{1}$$

for $L_f, L_E \gg r \gg \eta$, where $B_3^L(r)$ is the third order longitudinal velocity structure function, $\langle\varepsilon\rangle$ the average of the rate of energy dissipation ε per unit mass, and η the Kolmogorov micro length scale defined by $\eta \equiv (\nu^3/\langle\varepsilon\rangle)^{1/4}$ with ν being the kinematic viscosity. This 4/5 law has been confirmed by experiments and numerical simulations. Note that the law asserts that (1) holds irrespectively of the details of differences in the forcing at large scales and in the eddy structures at small scales, as far as (i)-(iii) hold.

Another evidence supporting the existence of the universality at small scales is concerned with the energy spectra of the fluctuating velocity fields. The longitudinal energy spectra $E_{11}(k_1)$ of various turbulent fields under different flow conditions are known to overlap well at large longitudinal wave number k_1, if the spectra are plotted against the normalized wavenumber $k_1\eta$. In particular, in the wavenumber range $1/L_E \ll k_1 \ll 1/\eta$, the spectra at high Reynolds number fit well to the $-5/3$ power law spectrum

$$E_{11}(k_1) = K_o \langle \varepsilon \rangle^{2/3} (k_1\eta)^{-5/3}, \tag{2}$$

where K_o is a non-dimensional universal constant. Equation (2) is consistent with K41. Although the overlap is in general not perfect, it strongly suggests that there is a certain kind of universality or common features in the statistics of small scales in high Reynolds number turbulence.

For some more details supporting the existence of universality at small scale statistics, readers may refer to, e.g., Kaneda and Morishita (2012).

TURBULENT SHEAR FLOW

In considering turbulent shear flows, it is a common practice to decompose the velocity field $\mathbf{v}$ into the mean and fluctuating parts such as $\mathbf{v} = \mathbf{U} + \mathbf{u}$, where $\mathbf{U} = \langle \mathbf{v} \rangle$ with $\langle \mathbf{v} \rangle$ being the mean of $\mathbf{v}$. Then the NS equation yields

$$\frac{\partial}{\partial t}\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - [(\mathbf{U}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{U}] + \nu\nabla^2\mathbf{u} + \ldots, \tag{3}$$

where the fluid density is assumed to be unity, and we have omitted writing terms representing the effects of the pressure, external force and $\langle(\mathbf{u}\cdot\nabla)\mathbf{u}\rangle$. The first, second and the third terms on the right-hand-side of (3) represent (a) the non-linear coupling between the fluctuating velocity $\mathbf{u}$, (b) the coupling between the mean and fluctuating fields $\mathbf{U}$ and $\mathbf{u}$, and (c) the viscous term, respectively.

In the following, we consider the statistics at the small scales, that are much smaller than the characteristic length scale of the mean flow $\mathbf{U}$. This implies that at the scales, $\partial U_i/\partial x_j \equiv S_{ij}$ may be approximated to be constant. Let u_ℓ be the characteristic velocity of small eddies of scale ℓ in such a scale range, and $\tau_N(\ell)$ and $\tau_S(\ell)$ be the characteristic time scales associated with the nonlinear coupling in (a) between the small scale eddies and the coupling in (b) between the mean and fluctuating fields, respectively. Then a simple estimate gives

$$\tau_N(\ell) \sim \ell/u_\ell, \quad \tau_S(\ell) \sim 1/S, \tag{4}$$

where $S \equiv \max_{ij}|S_{ij}|$, and the so-called random sweeping effect has been removed in the estimate of (b). According to K41, $u_\ell \sim (\langle\varepsilon\rangle\ell)^{1/3}$, so that $\tau_N \sim (\ell^2/\langle\varepsilon\rangle)^{1/3}$. Then (4) gives

$$\delta(\ell) \equiv \tau_N(\ell)/\tau_S(\ell) \sim S\ell^{2/3}/\langle\varepsilon\rangle^{1/3} \ll 1 \tag{5}$$

for $\ell \ll L_S \equiv \langle\varepsilon\rangle^{1/2}/S^{3/2}$. This suggests that at small enough scales, the effect of the coupling between the mean flow and the small eddies are small as compared with the nonlinear coupling between the small eddies, so that in considering the small scale statistics the effect of the former may be treated as a disturbance added to the dynamics governed by the latter. (see, e.g., Ishihara *et al.*, 2002; Kaneda and Ishihara, 2009; Kaneda and Morishita, 2012)

Linear Response Theory

Let us briefly review here the idea of the linear response theory in the statistical mechanics for systems near thermal equilibrium. Suppose that an external force or disturbance, say X, is added to an equilibrium state, whose distribution function or the density matrix ρ in the absence of the force X, is given by ρ_e. In response to the disturbance, ρ changes to

$$\rho = \rho_e + \Delta\rho + \ldots, \tag{6}$$

where $\Delta\rho$ is the change of ρ due to X and first order in X. The changes in ρ results in the changes of observable, say $\mathscr{B}$, as

$$\langle\mathscr{B}\rangle = \langle\mathscr{B}\rangle_e + \Delta\langle\mathscr{B}\rangle + \ldots, \tag{7}$$

where $\langle\mathscr{B}\rangle_e$ is the average over the equilibrium distribution ρ_e, and

$$\Delta\langle\mathscr{B}\rangle = \mathscr{C}X, \tag{8}$$

in which $\mathscr{C}$ is a constant, determined by the equilibrium sate and independent of X. Here we omit the time factors. (see, e.g., Kubo, 1966; Kaneda and Morishita, 2012)

Although, in contrast to the thermal equilibrium state, we do not know how to accurately specify the "equilibrium" state of turbulence, or something corresponding the density matrix ρ_e, it is attractive to assume that there is a certain kind of universal local equilibrium state at small scales, and consider the response of the state to disturbance added to the system.

Velocity Correlations and Spectra

Consider a small space domain $\mathscr{D}$, whose scale is much smaller than L_f, L_E and the characteristic length scale of the mean shear, so that the mean shear rate S_{ij} may be regarded to be constant in $\mathscr{D}$. Let ρ be the probability distribution of the velocity difference $\delta\mathbf{u}(\mathbf{r}) \equiv \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ where both $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ are in $\mathscr{D}$. In this case, the above consideration yields ρ as (6), where ρ_e stands for the distribution in the absence of the mean shear, and $\Delta\rho$ represents the change of ρ in response to the shear and is linear in S_{ij}. Corresponding to (7) and (8), we have

$$\begin{aligned}\left\langle\delta u_i(\mathbf{r})\delta u_j(\mathbf{r})\right\rangle &= \left\langle\delta u_i(\mathbf{r})\delta u_j(\mathbf{r})\right\rangle_e \\ &\quad + \Delta\left\langle\delta u_i(\mathbf{r})\delta u_j(\mathbf{r})\right\rangle + \cdots, \qquad (9)\\ \Delta\left\langle\delta u_i(\mathbf{r})\delta u_j(\mathbf{r})\right\rangle &= C'_{ijmn}(\mathbf{r})S_{mn}, \qquad (10)\end{aligned}$$

where $\langle..\rangle_e$ is the average over the equilibrium distribution ρ_e, and C'_{ijmn} is a fourth order tensor satisfying $C'_{ijmn} =$

C'_{jimn}. Here we use the summation convention for repeated indices, unless otherwise stated, and consider only one-time correlations so that we omit writing the time argument t at will.

In the following we assume that the statistics represented by the equilibrium density ρ_e are homogenous and isotropic. Then the tensor C_{ijmn} must be a fourth order isotropic tensor independent of $\mathbf{x}$. It must be compatible with the incompressibility condition of $\mathbf{v}$.

In order to take into account of the incompressibility condition, it is convenient to work in the wave vector space. It is then shown that (9) and (10) give

$$Q_{ij}(\mathbf{k}) = Q^e_{ij}(\mathbf{k}) + \Delta Q_{ij}(\mathbf{k}) + \cdots, \qquad (11)$$

where $Q_{ij}(\mathbf{k}), Q^e_{ij}(\mathbf{k})$ and $\Delta Q_{ij}(\mathbf{k}) + \cdots$ are the Fourier transforms with respect to $\mathbf{r}$ of $\langle u_i(\mathbf{x}+\mathbf{r})_j u_j(\mathbf{x})\rangle$, $\langle u_i(\mathbf{x}+\mathbf{r})u_j(\mathbf{r})\rangle_e$ and $\Delta\langle u_i(\mathbf{x}+\mathbf{r})u_j(\mathbf{r})\rangle$, respectively. $Q^e_{ij}(\mathbf{k})$ and $\Delta Q_{ij}(\mathbf{k})$ represent the equilibrium spectrum in the absence of the mean flow and the response of the equilibrium state to the mean shear, respectively. The latter is linear in S_{mn}, and may be written as

$$\Delta Q_{ij}(\mathbf{k}) = C_{ijmn}(\mathbf{k})S_{mn}, \qquad (12)$$

where C_{ijmn} is a fourth-order isotropic tensor depending on $\mathbf{k}$, and satisfies $k_i C_{ijmn} = k_j C_{ijmn} = 0$ and $C_{ijmn} = C_{jimn}$.

A simple symmetry consideration then gives

$$C_{ijmn}(\mathbf{k}) = \frac{1}{2}q^{(1)}(k)\left[P_{im}(\mathbf{k})P_{jn}(\mathbf{k}) + P_{jm}(\mathbf{k})P_{in}(\mathbf{k})\right] + q^{(2)}(k)P_{ij}(\mathbf{k})\frac{k_m k_n}{k^2} + q^{(3)}(k)P_{ij}(\mathbf{k})\delta_{mn}, \qquad (13)$$

where $P_{ij} = \delta_{ij} - (k_i k_j)/k^2$, and $q^{(\alpha)}(k)(\alpha = 1,2,3)$ depends on $\mathbf{k}$ only through $k = |\mathbf{k}|$. The $q^{(3)}$-term may be neglected without loss of generality, because $S_{mm} = 0$.

If we assume that the equilibrium state is characterized in accordance with K41 then a dimensional analysis yields

$$Q^e_{ij}(\mathbf{k}) = (K_o/4\pi)\langle\varepsilon\rangle^{2/3}k^{-11/3}P_{ij}(\mathbf{k}), \qquad (14)$$

$$q^{(1)}(k) = A\langle\varepsilon\rangle^{1/3}k^{-13/3}, \qquad (15)$$

$$q^{(2)}(k) = B\langle\varepsilon\rangle^{1/3}k^{-13/3}, \qquad (16)$$

in the wavenumber range $1/L_S \ll k \ll 1/\eta$. For simple mean shear flow given by $S_{ij} = \delta_{i1}\delta_{j2}S$, we have

$$Q_{12}(k) = (4\pi/15)(7A - B)\langle\varepsilon\rangle^{1/3}k^{-7/3}S, \qquad (17)$$

$$Q^{12}_{ii}(k) = (8\pi/15)(-A + B)\langle\varepsilon\rangle^{1/3}k^{-7/3}S, \qquad (18)$$

where K_o, A and B are non-dimensional universal constants, and

$$Q_{ij}(k) \equiv \langle Q_{ij}(\mathbf{q})\rangle_k, \quad Q^{ab}_{ij}(k) \equiv \left\langle \frac{k_a k_b}{k^2}Q_{ij}(\mathbf{q})\right\rangle_k,$$

in which $\langle\cdots\rangle_k$ denotes the integral over the spherical surface $|\mathbf{q}| = k$. As regards one-dimensional spectra, we have

$$E_{12}(k_1) = a\langle\varepsilon\rangle^{1/3}k_1^{-7/3}S, \qquad (19)$$

$$E^{12}_{ii}(k_1) = b\langle\varepsilon\rangle^{1/3}k_1^{-1/3}S, \qquad (20)$$

$$a = -\frac{18\pi}{1729}(-33A + 7B),\ b = \frac{216\pi}{1729}(-2A + B), \qquad (21)$$

where the one-dimensional spectrum $E_{12}(k_1)$ is defined as

$$E_{12}(k_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_{12}(\mathbf{k})dk_2 dk_3 \qquad (22)$$

and $E^{12}_{ii}(k_1)$ is defined similarly by $k_1 k_2 Q_{ii}(\mathbf{k})$.

Equation (11) with (12) - (21) is consistent with (2). The scaling $\propto k_1^{-7/3}$ in (19) is consistent with theoretical predictions including those by Lumley (1967), Leslie (1973), Yoshizawa (1998) and Cambon and Rubinstein (2006).

The spectra are also consistent with those of DNS of homogeneous turbulent shear with the Taylor scale Reynolds number $Re_\lambda \approx 284$ by Ishihara *et al.* (2002), according to which $A = -0.16 \pm 0.03$, $B = -0.40 \pm 0.06$. This yields $2a \approx -0.16$, which is close to the estimate $2a \approx -0.15$ by measurements of turbulent boundary layer by Saddoughi & Veeravalli (1994).

Recently, by measurements of a mixing layer generated by a jet with the Taylor scale Reynolds number up to $R_\lambda \approx 700$, Tsuji and Kaneda (2012) obtained $A \approx -0.16$, $B \approx -0.45$, which yields $2a \approx -0.14$, in fairly good agreements with the DNS estimate $2a \approx -0.16$ by Ishihara *et al.*(2002).

A spectral closure analysis by (Yoshida *et al.*, 2003) suggests that for accurate estimation by DNS or experiments of the universal constants A, B, the Reynolds number must be much higher than that required for the estimation of K_o in (2) or (14), because the slope of $k^{-13/3}$ in (15) and (16) is steeper than that of $k^{-11/3}$ in (14), i.e., $k^{-13/3}$ is much larger than $k^{-13/3}$ at low k.

The idea described above can be applied not only to the second order correlation of the velocity field but also to higher order correlations, for example, to the correlation $B_{ijk}(\mathbf{r}) \equiv \langle u_i(\mathbf{x})u_j(\mathbf{x})u_k(\mathbf{x}+\mathbf{r})\rangle$. Its Fourier transform $Q_{ijk}(\mathbf{k})$ with respect to $\mathbf{r}$ may be written as

$$Q_{ijk}(\mathbf{k}) = Q^e_{ijk}(\mathbf{k}) + \Delta Q_{ijk}(\mathbf{k}) + \cdots, \qquad (23)$$

$$\Delta\langle Q_{ijk}(\mathbf{k})\rangle = C_{ijkmn}(\mathbf{k})S_{mn}. \qquad (24)$$

If the equilibriums state is given *a la* K41, then $Q^e_{ijk}(\mathbf{k})$ and $C_{ijkmn}(\mathbf{k})$ must be isotropic third order and fifth order tensors depending only on $\mathbf{k}$ and $\langle\varepsilon\rangle$, and compatible with $k_k Q_{ijk}(\mathbf{k}) = 0$. Since $C_{ijkmn}(\mathbf{k})$ is a fifth order tensor, it consists of too many terms to be written down here. However, as regards the contraction $Q_{iik}(\mathbf{k})$, it can be written in a simple form as seen below. First, it is shown that $Q^e_{iik}(\mathbf{k}) = 0$, because $Q^e_{iik}(\mathbf{k})$ must be third order isotropic tensor. By substituting the inverse Fourier transforms of (11) and (23) into

$$\frac{\partial}{\partial t}B_{ii} = -\frac{\partial}{\partial r_j}(B_{iji} + B_{iji}) + \nu\frac{\partial^2}{\partial r_j\partial r_j}B_{ij} + r_i S_{ij}\frac{\partial}{\partial r_j}B_{ij} + S_{ij}B_{ij}, \qquad (25)$$

and comparing the terms linear in S, it is shown after some algebra that

$$Q_{iij}(\mathbf{k}) = e\frac{k_j k_m k_n}{k^3} S_{mn} \langle\varepsilon\rangle^{2/3} k^{-14/3}, \tag{26}$$

in the wavenumber range $1/L_S \ll k \ll 1/\eta$, where e is a non-dimensional universal constant, $B_{ij} = \langle u_i u'_j\rangle, B_{ijk} = \langle u_i u_j u'_k\rangle$ with $u_i = u_i(\mathbf{x},t), u'_j = u_i(\mathbf{x}+\mathbf{r})$, and (25) is verified by the NS equation. The relation (26) remains to be examined by experiments and/or DNS.

Spectra Involving Pressure

The idea discussed above can be applied also to correlations involving pressure (Tsuji and Kaneda, 2012). It is shown that

$$Q_{pp}(\mathbf{k}) = Q^e_{pp}(\mathbf{k}) + \Delta Q_{pp}(\mathbf{k}) + \cdots, \tag{27}$$
$$Q_{pi}(\mathbf{k}) = Q^e_{pi}(\mathbf{k}) + \Delta Q_{pi}(\mathbf{k}) + \cdots, \tag{28}$$

where $Q_{pp}(\mathbf{k})$ and $Q_{pi}(\mathbf{k})$ are the the Fourier transforms with respect to $\mathbf{r}$ of the one-time two-point pressure correlation $\langle p(\mathbf{x}+\mathbf{r})p(\mathbf{x})\rangle$, and pressure-velocity correlation $\langle p(\mathbf{x}+\mathbf{r})u_i(\mathbf{x})\rangle$, respectively. Here $Q^e_{pp}(\mathbf{k})$ and $Q^e_{pi}(\mathbf{k})$ are the spectra at the equilibrium state, and $\Delta Q_{pp}(\mathbf{k}) + \cdots$ and $\Delta Q_{pi}(\mathbf{k}) + \cdots$ represents the response to the perturbation, i.e., the mean shear added to the equilibrium state. Corresponding to (8), we have

$$\Delta Q_{pp}(\mathbf{k}) = C_{mn}(\mathbf{k}) S_{mn}, \tag{29}$$
$$\Delta Q_{pi}(\mathbf{k}) = C_{imn}(\mathbf{k}) S_{mn}, \tag{30}$$

where $C_{mn}(\mathbf{k})$ and $C_{mni}(\mathbf{k})$ are second and third order tensors, respectively. They are determined by the statistics of the equilibrium state, independently of the mean shear.

If the equilibriums state is characterized in accordance with K41, then $Q^e_{pi}(\mathbf{k}) = 0$ because of the isotropy of the equilibrium state, and

$$Q^e_{pp}(\mathbf{k}) = \frac{K_p}{4\pi}\langle\varepsilon\rangle^{4/3} k^{-13/3}, \tag{31}$$
$$\Delta Q_{pp}(\mathbf{k}) = a_1 \frac{k_m k_n}{k^2}\langle\varepsilon\rangle k^{-5} S_{mn}, \tag{32}$$
$$\Delta Q_{pi}(\mathbf{k}) = \left(b_1 \frac{k_i k_m k_n}{k^3} + b_2 \frac{k_m}{k}\delta_{ni} + b_3 \frac{k_n}{k}\delta_{mi}\right) \times \langle\varepsilon\rangle^{2/3} k^{-14/3} S_{mn},$$

in the wavenumber range $1/L_S \ll k \ll 1/\eta$, where K_p, a_1, b_1, b_2, b_3 are non-dimensional universal constants, and $b_1 + b_2 + b_3 = 0$ because $k_i C_{imn}(\mathbf{k}) = 0$ for any $\mathbf{k}$.

Equation (31) implies that the one-dimensional spectra $E^e_{pp}(k_1)$ and $\Delta E_{pp}(k_1)$ defined similarly to (19) from $Q^e_{pp}(\mathbf{k})$, and $\Delta Q_{pp}(\mathbf{k})$ are given by

$$E^e_{pp}(k_1) = \frac{3}{14} K_p \langle\varepsilon\rangle^{4/3} k_1^{-7/3}, \tag{33}$$

and $\Delta E_{pp}(k_1) = 0$ for $S_{mn} = \delta_{m1}\delta_{m2}$, the latter of which is because of the anti-symmetry with respect to k_2 of $\Delta Q_{pp}(\mathbf{k})$. The scaling in (33) is consistent with studies so far reported (readers may refer to, e.g., Tsuji *et al.* (2007) Tsuji and Kaneda (2012) and references cited therein).

Recently, the spectra (27) - (33) were confirmed to be consistent with experiments of the pressure and velocity in a turbulent mixing layer generated by a jet, where the pressure was measured by newly developed pressure probe with resolution sufficiently high to resolve the inertial subrange Tsuji and Kaneda (2012). According to the measurements, $a_1 \approx -5.2$, $b_1 \approx -0.066$, $b_2 \approx -0.128$, $b_3 \approx 0.194$.

Table 1. h, the channel width; L_x, L_z, fundamental periodic lengths in the directions of streamwise ($x = x_1$), and spanwise ($z = x_3$) directions, respectively; N_x, N_y, N_z, the number of grid points in the x,y,z-directions; $\Delta x^+, \Delta y_c^+, \Delta z^+$, grid width in the x,y,z-directions; Δy_c is at the center of the channel.

	Re_τ	L_x/h	L_z/h	$N_x \times N_y \times N_z$	Δx^+	Δy_c^+	Δz^+
Case 1	320	π	$\pi/2$	$128 \times 192 \times 128$	7.9	7.9	3.9
Case 2	640	π	$\pi/2$	$256 \times 384 \times 256$	7.9	7.9	3.9
Case 3	1280	π	$\pi/2$	$512 \times 768 \times 512$	7.9	7.9	3.9
Case 4	2560	π	$\pi/2$	$1024 \times 1536 \times 1024$	7.9	7.9	3.9
Case 5	5120	π	$\pi/2$	$2048 \times 1536 \times 2048$	7.9	15.9	3.9

TURBULENT CHANNEL FLOW

In the above discussions, we have assumed that the flow boundary, if it exists, is sufficiently far from the local domain $\mathscr{D}$ under consideration, so that its direct influence on the statistics in $\mathscr{D}$ is negligible. One may then ask whether the idea discussed above is applicable to turbulent channel flows. In order to get some idea on this question, we have examined consequences of the theory by comparison with the data of DNS of turbulent channel flow up to $Re_\tau = 5120$. In the DNS we used the Fourier spectral method in the stream($x = x_1$)- and span($z = x_3$)-wise directions, and the Tchebycheff expansion method in the wall normal ($y = x_2$) direction. Some parameter values charactering the DNS are listed in Table 1. (Readers may refer to Morishita *et al.* (2011) for some details of the DNS methods and statistics up to $Re_\tau = 2560$.)

Figure 1 shows the mean streamwise velocity profile U as a function of y^+. Here the superscript $^+$ denotes the normalization by the wall units. It is seen that U fits well to the log-law

$$U^+(y^+) = \frac{1}{\kappa}\log y^+ + C, \tag{34}$$

in a certain range depending on Re_τ, for example, in the range $50 < y^+ < 1000$ at $Re_\tau = 5120$, where $\kappa \approx 0.4, C \approx 5$.

Figure 2 shows the mean rate of energy dissipation as a function of y^+. It is seen that $\langle\varepsilon\rangle^+$ fits well to the well-known relation (see, e.g., Tennekes and Lumley, 1972)

$$\langle\varepsilon\rangle^+ = \frac{1}{\kappa y^+}, \tag{35}$$

in the log-law region.

According to (34), the mean shear rate $S^+_{ij} = \partial U^+_i/\partial x^+_j$ is given by $\delta_{i1}\delta_{j2}/(\kappa y^+)$, so that the ratio defined by (5) is given by

$$\delta(\ell) = [\ell^+]^{2/3}/(\kappa y^+[\langle\varepsilon\rangle^+]^{1/3}) = (\ell/\kappa y)^{2/3}, \tag{36}$$

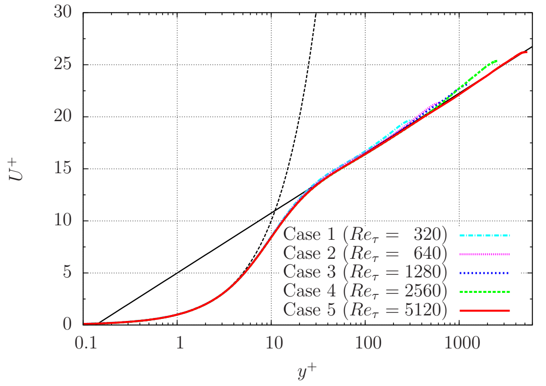

Figure 1. Mean velocity profile. Solid line shows the log-law $U^+(y^+) = (1/\kappa)\log y^+ + C$, with $\kappa = 0.4, C = 5.0$.

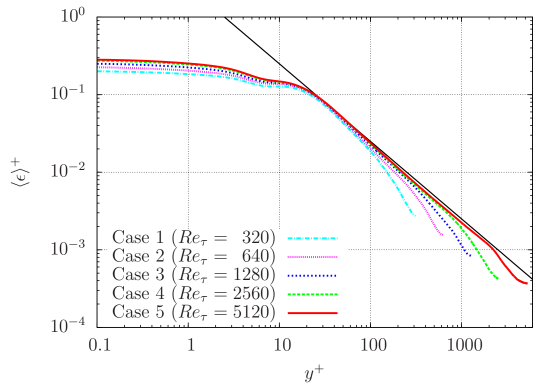

Figure 2. Mean dissipation rate $\langle\varepsilon\rangle^+$ vs. y^+. The solid line shows $\langle\varepsilon\rangle^+ = 1/(\kappa y^+)$ with $\kappa = 0.4$.

where we have used (35). For $\ell \sim \eta \equiv (\nu^3/\langle\varepsilon\rangle)^{1/4}$, (36) gives

$$\delta(\eta) \sim (\nu^3/\langle\varepsilon\rangle)^{1/6}/(\kappa y)^{2/3} \propto (\kappa y)^{-1/2}. \qquad (37)$$

In the DNS, the ratio $\delta(\eta)$ is small, typically less than 0.1, at the upper region of the log-law region at high Re_τ.

Equations (11) - (13) imply that the diagonal components of $Q_{ij}(\mathbf{k})$'s (i.e., the components with $i = j$) are dominated by the equilibrium spectrum $Q^e_{ij}(\mathbf{k})$, and therefore

$$E_{11}(k_1) \sim E_{33}(k_3) \sim C_K \langle\varepsilon\rangle^{2/3} k_1^{-5/3}, \qquad (38)$$

$$E_{22}(k_1) \sim E_{33}(k_1) \sim (4/3)C_K \langle\varepsilon\rangle^{2/3} k_1^{-5/3}, \qquad (39)$$

in the subrange $1/L_S \ll k_1 \ll 1/\eta$, where $C_K = (9/55)K_o$. The scaling is consistent with observations including those by McKeon and Morrison (2007), Smits *et al.* (2011) and Morishita *et al.* (2011). The DNS data fit fairy well with these relations with $C_1 \approx 0.5$ at $k_1\eta \sim 0.02$, and the fitting range increases with Re_τ.

Equations (11) - (13) also imply that $E_{12}(k_1)$ and E^{12}_{ii} are dominated by the second term of (11), i.e., $\Delta Q_{ij}(\mathbf{k})$, and they are given by (19) and (20). The DNS data are in fairy good agreement with (19) and (20) with the DNS value $A = -0.16, B = -0.40$ by noted above, as seen in Fig. 3. (Figures for E^{12}_{ii} are omitted.)

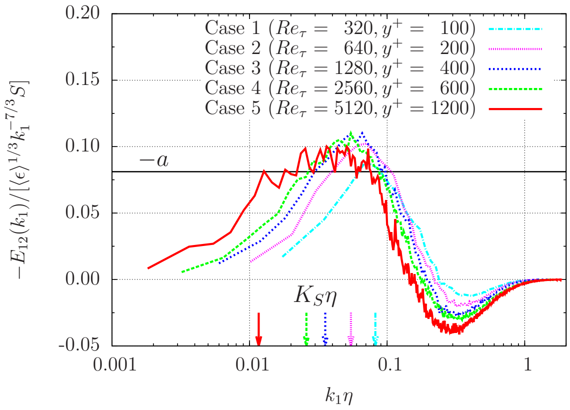

Figure 3. Spectrum $-E_{12}(k_1)/[\langle\varepsilon\rangle^{1/3}k_1^{-1/3}S]$ vs. $k_1\eta$ at y^+ in the upper region of the log-law region at each Re_τ. The value $K_S\eta$ is indicated by an arrow for each run, where $K_S \equiv 1/L_S$. The straight line shows the prediction (19) with (21) with $A = -0.16$ and $B = -0.40$.

So far we have considered the influence of the boundary wall to be only through the mean shear, and have ignored the inhomogeneity of the statistics. However, the statistics is in general not homogeneous in a strict sense because of the existence of the boundary wall. We assume here that the influences of the mean shear and the inhomogeneity on ρ or $Q_{ij}(\mathbf{k})$ are additive, so that $\Delta Q_{ij}(\mathbf{k})$ may be written as

$$\Delta Q_{ij}(\mathbf{k}) = \Delta^S Q_{ij}(\mathbf{k}) + \Delta^I Q_{ij}(\mathbf{k}), \qquad (40)$$

where $\Delta^S Q_{ij}$ is the change due to the mean shear given by (12), and $\Delta^I Q_{ij}$ the change due to the inhomogeneity.

In order to get an estimate of $\Delta^I Q_{ij}$, it is worthwhile to note that (34) and (35) give $(dS/dy)/S \sim (d\langle\varepsilon\rangle/dy)/\langle\varepsilon\rangle \propto 1/y$, where $S = dU/dy$, i.e., the degree of inhomogeneity in S and $\langle\varepsilon\rangle$ decreases with the distance y from the wall in proportion to $1/y$, and the change in the inhomogeneity is in the direction normal to the wall. This suggests us to assume that $\Delta^I Q_{ij}$ is linear in $(1/y)\mathbf{n}$, i.e.,

$$\Delta^I Q_{ij}(\mathbf{k}) = C_{ijm}(\mathbf{k}) I_m, \quad I_m \equiv n_m/(n_k x_k), \qquad (41)$$

where $\mathbf{n}$ is the unit vector normal to the wall, and $\mathbf{x}$ is the position vector with $n_k x_k = 0$ on the wall.

Equation (41) can be derived also by generalizing the idea of K41, In order to take into account of the inhomogeneity, we assume that ρ may depend on $\nabla\langle\varepsilon\rangle$ in addition to $\langle\varepsilon\rangle$ and $\mathbf{r}$. Then retaining only terms up to the first order term in the expansion of ρ for small $\nabla\langle\varepsilon\rangle/\langle\varepsilon\rangle$, we have (41).

If the equilibrium state is characterized in accordance with K41, then a simple dimensional analysis gives for $n_m = \delta_{m2}$

$$\Delta^I Q_{ij} = \chi'(1/x_2)\langle\varepsilon\rangle^{2/3}\frac{k_2}{k}P_{ij}(\mathbf{k})k^{-14/3}, \qquad (42)$$

so that (11) with (40) and (42) yields

$$E^{(2)}_{\alpha\alpha}(k_1) = \chi(1/x_2)\langle\varepsilon\rangle^{2/3}k_1^{-5/3}, \qquad (43)$$

where χ' and χ are non-dimensional constants, $E^{(2)}_{\alpha\alpha}(k_1)$ (no summation over α) is defined as the Fourier transform with respect to r of $(\partial/\partial x_2)\langle u_\alpha(x_1+r,x_2,x_3)u_\alpha(x_1,x_2,x_3)\rangle$. Equation (43) could be also derived by assuming that $Q_{ij}(\mathbf{k})$ is given by (11)-(16) and the influence of the inhomogeneity is only through the x_2-dependence of $\langle\varepsilon\rangle$. This assumption gives $\chi = -(2/3)K_o$. Figure 4 shows the ratio $-[E^{(2)}_{\alpha\alpha}(k_1)x_2]/E_{\alpha\alpha}(k_1)$ vs. $k_1\eta$ for $\alpha = 1$. According to (43) and (11)-(16), this ratio must be $2/3$ in the inertial subrange. It is seen that the ratio is not far from $2/3$ in the range. The similar is also true for $\alpha = 2$ and 3 (figures are omitted).

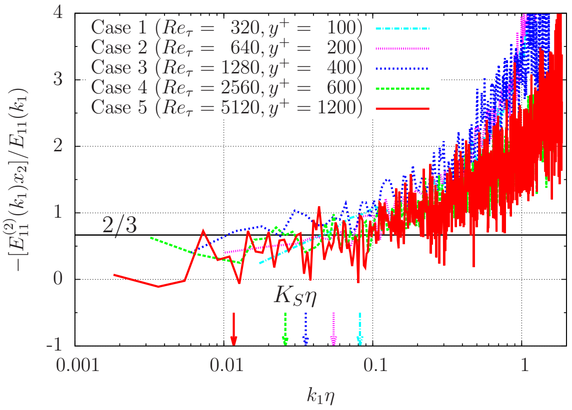

Figure 4. The ratio $r_{11} \equiv -[E^{(2)}_{11}(k_1)x_2]/E_{11}(k_1)$ vs. $k_1\eta$, where $x_2 = y$. The straight line shows $r_{11} = 2/3$.

CONCLUSION

A simple analysis suggests that in the absence of mean shear the statistics of sufficiently small eddies in turbulent flows at high Reynolds number (*Re*) is at a locally equilibrium state, and the influence of mean shear on the statistics may be regarded as a perturbation added to the equilibrium state. Although it is still not known how to accurately specify the statistics at the equilibrium state, one may apply an idea similar to the linear response theory familiar in the statistical mechanics for systems at or near thermal equilibrium state.

By the application, one can derive expressions for various multi-point correlation functions and spectra characteristics at small scales of turbulent shear flows at high *Re*. They imply that not only the slopes (scalings) of the spectra, but also some of the pre-factors are universal constants characterizing the equilibriums state. They have been confirmed to be consistent with DNS of homogeneous turbulent shear flow, and experiments/observations of turbulent boundary layers and a mixing layer. The idea is applicable also to turbulent channel flows. A comparison of the theoretical conjectures with the data of DNS of turbulent channel flows at Re_τ up to 5120 shows good agreements between them.

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