A SYMMETRY-PRESERVING DISCRETIZATION AND REGULARIZATION MODEL FOR COMPRESSIBLE FLOW

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ABSTRACT

Symmetry-preserving discretization and symmetry-preserving regularization are promising simulation strategies for incompressible flow. In this paper, it is shown that the symmetry-preserving discretization and regularization can be generalized to compressible flow straightforwardly, if the compressible Navier-Stokes equations are rewritten to a new form. The proposed symmetry-preserving methods for compressible flow are validated in a simulation of channel flow. The symmetry-preserving discretization for compressible flow is stable without artificial dissipation and produces accurate results on sufficiently fine grids, which makes the discretization very suitable for direct numerical simulation of compressible flow.

INTRODUCTION

The starting point of this research is the symmetry-preserving discretization for incompressible flow (Verstappen and Veldman, 2003). This discretization of the incompressible Navier-Stokes equations preserves important conservation properties on the discrete level; the discrete linear momentum is conserved and the discrete kinetic energy is bounded from above. Discrete kinetic energy is a norm of the numerical solution, and therefore the symmetry-preserving discretization attains numerical stability by preserving conservation laws. A convenient supplementary property of the symmetry-preserving discretization is that it captures channel flow accurately on very coarse grids without an explicit subgrid-scale model.

A subgrid-scale model that fits in naturally with the symmetry-preserving discretization is the symmetry-preserving regularization for incompressible flow (Verstappen, 2008; Trias et al., 2013). Symmetry-preserving regularization applies explicit filtering to the Navier-Stokes equations, but retains the important conservation properties of the unfiltered equations. The aim of the filtering is to suppress the creation of subgrid scales, and therefore regularization can be used as a subgrid-scale model for large-eddy simulation.

In this paper, the symmetry-preserving discretization and regularization for incompressible flow are generalized to compressible flow. As it turns out, this is straightforward if the compressible Navier-Stokes equations are rewritten to a form that expresses conservation properties in the language of functional analysis.

THE COMPRESSIBLE NAVIER-STOKES EQUATIONS AND FUNCTIONAL ANALYSIS

The Navier-Stokes equations for compressible flow can be expressed in different forms. Although the forms are mathematically equivalent, each form emphasizes different properties of compressible flow, and each form yields a different numerical discretization. In this section, a new form of the compressible Navier-Stokes equations is derived. This form straightforwardly expresses the skew-symmetric nature of convection, and reveals an energy bound for compressible flow. In the sequel, the skew-symmetry of convection and the energy bound will be preserved on the discrete level by the symmetry-preserving discretization and regularization.

The compressible Navier-Stokes equations are typically expressed in the conservative form

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \\ \partial_t \rho \vec{u} + \nabla \cdot (\rho \vec{u}\vec{u}) + \nabla p &= \nabla \cdot \sigma \\ \partial_t \rho E + \nabla \cdot (\rho \vec{u} E) + \nabla \cdot (p \vec{u}) &= \nabla \cdot (\sigma \cdot \vec{u}) + \nabla \cdot \vec{q} \end{aligned} \quad (1)$$

where ρ is the mass density, $\vec{u} = (u, v, w)$ the flow velocity, $E = \frac{1}{2}\vec{u} \cdot \vec{u} + e$ the total energy per unit mass, p the pressure, σ the stress tensor, and $\vec{q}$ the diffusive heat flux. In the conservative form, each term is either a divergence or gradient operator. Upon spatial integration these operators can be rewritten to surface integrals, and therefore the conservative form emphasizes conservation of mass, linear momentum, and energy in a compressible flow.

In this paper, the state of a compressible fluid is expressed as a real-valued vector function

$$\vec{h}(\vec{x}) = \left(\sqrt{\rho}, \sqrt{\rho}\vec{u}/\sqrt{2}, \sqrt{\rho e}\right)(\vec{x}) \quad (2)$$

where $\vec{x}$ is inside the domain Ω occupied by the fluid. A new form of the compressible Navier-Stokes equations can be obtained by deriving the evolution equation for the state vector $\vec{h}$

$$\partial_t \vec{h} = A(\vec{h})\vec{h} = \left(C(\vec{h}) + P(\vec{h}) + V(\vec{h}) + H(\vec{h})\right)\vec{h} \quad (3)$$

where the non-linear differential operator $A(\vec{h})$ represents the right-hand-side of the evolution equation, and the differential operators $C(\vec{h})$, $P(\vec{h})$, $V(\vec{h})$, and $H(\vec{h})$ are the terms related to respectively convection, pressure forces, viscous friction, and heat diffusion.

The form (3) also emphasizes conservation properties, but now in the language of functional analysis. To see this, consider the Hilbert spaces $L^2(\Omega)$ and $L^2(\Omega)^5$ of real-valued, square-integrable scalar and state vector functions on the domain Ω. The standard inner products are

$$(f,g) = \int_\Omega fg\,\mathrm{d}x \qquad \left\langle \vec{f},\vec{g}\right\rangle = \int_\Omega \vec{f}\cdot\vec{g}\,\mathrm{d}x \quad (4)$$

and the induced norms are denoted by $|h|$ and $||\vec{h}||$. To simplify the analysis, in this paper the domain Ω is assumed to be periodic.

The mass, linear momentum, kinetic energy, and internal energy inside Ω can be expressed as $L^2(\Omega)$ inner products of the components of $\vec{h}$. For example,

$$\sqrt{2}(\sqrt{\rho}, \sqrt{\rho}v/\sqrt{2}), \quad (\sqrt{\rho e}, \sqrt{\rho e}) \quad (5)$$

are respectively the linear momentum in the y direction inside Ω and the internal energy inside Ω. The $L^2(\Omega)^5$ norm of the dimensionless state vector $\vec{h}$ is equal to the sum of the mass and the energy inside Ω

$$||\vec{h}||^2 = \left\langle \vec{h},\vec{h}\right\rangle = \int_\Omega \rho\,\mathrm{d}x + \int_\Omega \rho\left(\frac{1}{2}\vec{u}\cdot\vec{u} + e\right)\mathrm{d}x \quad (6)$$

which is constant because mass and energy are conserved in a compressible flow. Thus, desired solutions $\vec{h}$ of the compressible Navier-Stokes equations satisfy the bound

$$||\vec{h}||^2 = \text{constant} < \infty \quad (7)$$

and therefore $\vec{h}$ is in $L^2(\Omega)^5$. The bound induces a property of the right-hand-side of (3)

$$\left\langle \vec{h}, A(\vec{h})\vec{h}\right\rangle = \left\langle \vec{h}, \partial_t\vec{h}\right\rangle = \frac{1}{2}\partial_t||\vec{h}||^2 = 0 \quad (8)$$

which again expresses conservation of mass and energy. In the sequel, the bound (7) and the equality (8) will provide numerical stability to the symmetry-preserving discretization.

By (5), the mass, linear momentum, kinetic energy, and internal energy inside Ω can be expressed as $L^2(\Omega)$ inner products of the components of $\vec{h}$. Therefore, the conservation properties of the compressible Navier-Stokes equations depend on the interaction of the differential operators from (3) with $L^2(\Omega)$ inner products; the so-called symmetries of differential operators. In this article, special attention is paid to the symmetries of the convection operator $C(\vec{h})$. The convection operator can be expressed as the componentwise application of the operator $c(\vec{u})$

$$C(\vec{h})\vec{\phi} = (c(\vec{u})\phi_1, c(\vec{u})\phi_2, c(\vec{u})\phi_3, c(\vec{u})\phi_4, c(\vec{u})\phi_5) \quad (9)$$

where

$$c(\vec{u})\phi = -\frac{1}{2}\vec{u}\cdot\nabla\phi - \frac{1}{2}\nabla\cdot(\vec{u}\phi) \quad (10)$$

is the scalar convection operator. The scalar convection operator is skew-symmetric with respect to the $L^2(\Omega)$ inner product; $(\psi, c(\vec{u})\phi) + (c(\vec{u})\psi, \phi) = 0$. The evolution equation of each component of $\vec{h}$ is of the form $\partial_t h_i = c(\vec{u})h_i + \ldots$, so that for each pair h_i and h_j

$$\partial_t(h_i, h_j) = (h_i, c(\vec{u})h_j) + (c(\vec{u})h_i, h_j) + \ldots \quad (11)$$

which vanishes by skew-symmetry of the scalar convection operator. Thus, products of the components of $\vec{h}$ are conserved by convection. This implies that convection conserves mass, linear momentum, kinetic energy, and internal energy. In the sequel, the skew-symmetry of the convection operator will be preserved on the discrete level, so that the discrete mass, linear momentum, kinetic energy, and internal energy are conserved by discrete convection.

This section demonstrates that by rewriting the compressible Navier-Stokes equations to an evolution equation for $\vec{h}$, conservation properties can be expressed in the language of functional analysis. This line of thought was also pursued by Vabishevich (2007). He found the same scalar convection operator, but applied the operator to the energy variable $\sqrt{\rho}e$. In this paper, the bound (7) is important for numerical stability, and therefore $\sqrt{\rho e}$ should be chosen as the energy variable.

SYMMETRY-PRESERVING DISCRETIZATION

The aim of a symmetry-preserving discretization is to obtain a stable numerical discretization by preserving conservation properties on the discrete level (Verstappen and Veldman, 2003). In the previous section it was shown that if the state vector is $\vec{h}$, then conservation properties can be expressed in terms of inner products. Therefore, to transfer conservation properties to the discrete level in a natural way, the numerical solution is stored as a state vector $\vec{h}$, and discrete inner products are defined as

$$(f,g) = \sum_j \Omega_j f_j g_j \qquad \left\langle \vec{f},\vec{g}\right\rangle = \sum_j \Omega_j \vec{f}_j\cdot\vec{g}_j \quad (12)$$

where j is a grid cell number, Ω_j is the size of grid cell j, and f, g, $\vec{f}$, and $\vec{g}$ are collocated grid functions. By the

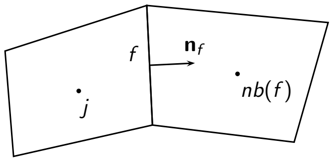

Figure 1. The grid cell j, its outward-pointing unit normal $\vec{n}_f$ at face f, and its neighbour at face f denoted by $nb(f)$.

remarks preceding (5), the definition of discrete inner products induces natural discretizations of the discrete mass, linear momentum, kinetic energy, and internal energy inside Ω.

The norms induced by the discrete inner products (12) are again denoted by $|h|$ and $||\vec{h}||$. If the discrete norm $||\vec{h}||$ of a real-valued numerical solution $\vec{h}$ is bounded from above, then the numerical solution is locally bounded. Thus, if the continuous bound (7) is preserved on the discrete level, numerical stability is attained. The aim of a symmetry-preserving discretization is to apply the bound (7) to numerical solutions $\vec{h}$ by preserving conservation properties on the discrete level.

Spatial discretization

In this paper, a spatial discretization is called symmetry-preserving if the equality (8) is preserved on the discrete level, and if the conservation properties of mass, linear momentum, kinetic energy, and internal energy of each operator in the right-hand-side of (3) are preserved on the discrete level. Discretizations with these conservation properties already exist (Kok, 2009; Morinishi, 2010) and are reported to have good stability properties. The existing discretizations are not derived from the form (3), do not identify the energy bound (7), and do not explain all the conservation properties of the convection operator from a skew-symmetry.

In this paper a curvilinear, structured, collocated computational grid is used. The scalar convection operator (10) is rewritten to $c(\vec{u})\phi = \frac{1}{2}(\nabla\cdot\vec{u})\phi - \nabla\cdot(\vec{u}\phi)$, which can be discretized to second-order accuracy as

$$\begin{aligned}(c(\vec{u})\phi)_j &= \frac{1}{2}\phi_j \frac{1}{\Omega_j}\sum_{f\in F_j} A_f \vec{n}_f\cdot\vec{u}_f \\ &\quad - \frac{1}{\Omega_j}\sum_{f\in F_j} A_f\vec{n}_f\cdot\vec{u}_f \frac{1}{2}\left(\phi_j+\phi_{nb(f)}\right) \\ &= -\frac{1}{\Omega_j}\sum_{f\in F_j}\frac{1}{2}A_f\vec{n}_f\cdot\vec{u}_f\phi_{nb(f)} \qquad (13)\end{aligned}$$

where F_j are the faces of cell j, A_f the area of face f, and $\vec{u}_f$ some interpolation of $\vec{u}$ to face f (see figure 1). This discretization is skew-symmetric with respect to the discrete $L^2(\Omega)$ inner product, and locally conserves discrete mass, linear momentum, kinetic energy, and internal energy. The discretization from Kok (2009) is obtained for the interpolation $\vec{u}_f = \frac{1}{2}((\rho\vec{u})_j + (\rho\vec{u})_{nb(f)})/\sqrt{\rho_j\rho_{nb(f)}}$. In this paper the interpolation $\vec{u}_f = \frac{1}{2}(\vec{u}_j + \vec{u}_{nb(f)})$ is preferred, because this leads to a more natural implementation of regularization models.

The pressure force operator $P(\vec{h})$ is discretized as in Kok (2009). Richardson extrapolation is applied to the discretizations of the convection operator and the pressure force operator as in Kok (2009). The result is an optimized, fourth-order accurate, symmetry-preserving discretization of the compressible Euler equations.

The viscous friction operator $V(\vec{h})$ conserves mass, linear momentum, and total energy. The heat diffusion operator $H(\vec{h})$ conserves internal energy. These conservation properties are preserved already by a standard finite-volume discretization. Therefore, the discretizations of the viscous friction operator and the heat diffusion operator are derived from a standard second-order accurate finite-volume discretization. Richardson extrapolation is applied with two control volumes, yielding a fourth-order accurate discretization.

The symmetry-preserving spatial discretization differs from the standard finite-volume discretization because the numerical solution is stored in the state vector $\vec{h}$, and because the convection is discretized as a skew-symmetric discrete operator. Skew-symmetric discretization of the convection operator is observed to considerably enhance the stability of a finite-volume method (Kok, 2009). On coarse grids, a general finite-volume discretization may transfer internal energy to kinetic energy by convection. This unphysically alters the energy balance in the flow and causes numerical instability. To suppress or postpone the instability, often artificial dissipation is added to the discretization. This unphysically changes the energy balance again, and there is no reason to believe that the sum of the two counteracting energy transfers produces a sensible numerical solution. In fact, flow phenomena that are very sensitive to the energy balance, such as turbulence and acoustic waves, are known to be suppressed heavily by artificial dissipation.

Skew-symmetric discretization of the convection operator eliminates the unphysical convective transfer of internal energy to kinetic energy. Therefore, no artificial dissipation is needed to suppress the corresponding numerical instability. The low level of artificial dissipation makes the symmetry-preserving discretization for compressible flow a very suitable method for the simulation of turbulence and acoustic waves.

Temporal discretization

One of the new ideas in this paper is that transferring conservation properties to the discrete level is easy if the state vector $\vec{h}$ is used. To demonstrate this once more, consider symmetry-preserving time integration.

Assume that a Runge-Kutta method is used to integrate the semi-discrete system $\partial_t\vec{h}_j = (A(\vec{h})\vec{h})_j$. The energy bound (7) and the discrete mass, linear momentum, kinetic energy, and internal energy inside Ω can all be expressed as discrete inner products of the components of $\vec{h}$. Therefore, a Runge-Kutta method that preserves discrete inner products of the integrated quantities is symmetry-preserving. These Runge-Kutta methods are called symplectic. All the symplectic Runge-Kutta methods are implicit. For an extensive investigation of symplectic methods as symmetry-preserving time-integration methods for incompressible flow, see Sanderse (2013).

In this work, symmetry-preserving time-integration methods will not be used. Instead a sufficiently accurate four-stage, low-storage, explicit Runge-Kutta method is used for time integration.

REGULARIZATION MODELLING

The computational costs of a direct numerical simulation are high, and therefore in practice often a large-eddy simulation is performed. In a large-eddy simulation, the large eddies in a flow are resolved, and the effect of the small unresolved eddies is modelled by a subgrid-scale model. A promising subgrid-scale model for incompressible flow is regularization, which applies explicit filtering to the Navier-Stokes equations. The aim of the filtering is to suppress the creation of subgrid scales, and therefore regularization can be used for large-eddy simulation.

A well-known regularization model for incompressible flow is the Leray regularization. Leray regularization applies explicit filtering to the convecting velocity field

$$\begin{aligned} \partial_t \vec{u} + \overline{\vec{u}} \cdot \nabla \vec{u} + \nabla p &= \nu \Delta \vec{u} \\ \nabla \cdot \overline{\vec{u}} &= 0 \end{aligned} \tag{14}$$

where the bar denotes filtering. The Leray regularization model was originally proposed as a tool for the mathematical analysis of the Navier-Stokes equations, but recently it was recognized that Leray regularization can also be used as a subgrid-scale model for large-eddy simulation (Geurts and Holm, 2003).

Another regularization model for incompressible flow is the symmetry-preserving regularization by Verstappen (2008). The simplest symmetry-preserving regularization is obtained by filtering the convection operator three times

$$\begin{aligned} \partial_t \vec{u} + \overline{\overline{\vec{u}} \cdot \nabla \overline{\vec{u}}} + \nabla p &= \nu \Delta \vec{u} \\ \nabla \cdot \overline{\vec{u}} = \nabla \cdot \vec{u} &= 0 \end{aligned} \tag{15}$$

where the filter operation is self-adjoint $(\overline{u}, v) = (u, \overline{v})$ with respect to the $L^2(\Omega)$ inner product and commutes with differentiation. The symmetry-preserving regularization preserves the skew-symmetry of the convection operator, and therefore preserves the important conservation properties of convection.

Regularization modelling is justified by an exhaustive body of mathematical analysis. However, upon discretization this mathematical analysis loses its rigour, and it becomes hard to justify the use of regularization as a practical subgrid-scale model. Nevertheless, regularization models produce promising time-averaged results for incompressible flow (Geurts and Holm, 2003; Verstappen, 2008; Trias et al., 2013), which makes them an interesting research topic.

Compressible symmetry-preserving regularization

In this section, a symmetry-preserving regularization for compressible flow is proposed. This regularization applies explicit filtering to the scalar convection operator in order to suppress the creation of subgrid scales, but preserves the skew-symmetry of this operator, so that the important conservation properties of convection are preserved.

A simple symmetry-preserving regularization is obtained by filtering the convecting velocity field

$$c_\alpha(\vec{u})\phi = c(\overline{\vec{u}})\phi \tag{16}$$

which yields the compressible Leray regularization. The operator $c_\alpha(\vec{u})$ is skew-symmetric with respect the $L^2(\Omega)$ inner product, and therefore conserves the mass, linear momentum, kinetic energy, and internal energy inside Ω.

An advantage of using the state vector $\vec{h}$ is that the skew-symmetric scalar convection operator $c(\vec{u})$ for compressible flow is similar to the convection operator for incompressible flow. Therefore, the symmetry-preserving regularization for incompressible flow (Verstappen, 2008) can be applied directly to the compressible convection operator

$$\begin{aligned} c_2(\vec{u}) &= \overline{c(\widetilde{\overline{\vec{u}}})\overline{\phi}} \\ c_4(\vec{u}) &= c(\widetilde{\overline{\vec{u}}})\overline{\phi} + \overline{c(\widetilde{\overline{\vec{u}}})\phi'} + \overline{c(\vec{u}'')\overline{\phi}} \\ c_6(\vec{u}) &= c(\widetilde{\overline{\vec{u}}})\overline{\phi} + c(\widetilde{\overline{\vec{u}}})\phi' + c(\vec{u}'')\overline{\phi} + \overline{c(\vec{u}'')\phi'} \end{aligned} \tag{17}$$

where the bar denotes a self-adjoint filter with residual $\phi' = \phi - \overline{\phi}$, and the tilde denotes some filter with residual $\phi'' = \phi - \widetilde{\phi}$. In this paper one filter is used; $\widetilde{\phi} = \overline{\phi}$. By the self-adjointness of the filter, the regularizations (17) are skew-symmetric with respect to the $L^2(\Omega)$ inner product, so that global conservation of mass, linear momentum, kinetic energy, and internal energy inside Ω is preserved. The regularizations are respectively second-order, fourth-order, and sixth-order accurate approximations of $c(\vec{u})$ with respect to the filter length.

Selection of the filter

In order to preserve the symmetries of the Navier-Stokes equations on both the continuous and the discrete level, the bar filter from (17) should be self-adjoint with respect to the continuous $L^2(\Omega)$ inner product, and its discretization should be self-adjoint with respect to the discrete $L^2(\Omega)$ inner product. In this paper, a conservative and self-adjoint differential filter is used

$$\overline{u} = u + \sum_{i=1}^{3} \partial_i \left(\frac{\Delta_i^2}{24} \partial_i u \right) \tag{18}$$

where Δ_i is the filter length in the direction i. As a first try, the filter length is chosen proportional to the local filter length $\Delta_i = r\Delta x_i$. Note that modern symmetry-preserving regularization models for incompressible flow use advanced algorithms that locally determine the filter length (Trias et al., 2013). Possibly, compressible symmetry-preserving regularization models can also benefit from these algorithms.

The differential filter (18) with filter length $\Delta_i = r\Delta x_i$ is discretized as (see figure 1)

$$\overline{u}_j = u_j + \frac{1}{\Omega_j} \sum_f \frac{r^2}{24} A_f \vec{n}_f \cdot (\vec{x}_{nb(f)} - \vec{x}_j)(u_{nb(f)} - u_j) \tag{19}$$

where $\vec{x}_j$ denotes the location of the centre of cell j. This discretization is self-adjoint with respect to the discrete $L^2(\Omega)$ inner product, and therefore preserves the skew-symmetry of the regularizations (17) on the discrete level.

RESULTS AND DISCUSSION

To assess the proposed symmetry-preserving discretization and regularization model, the methods are vali-

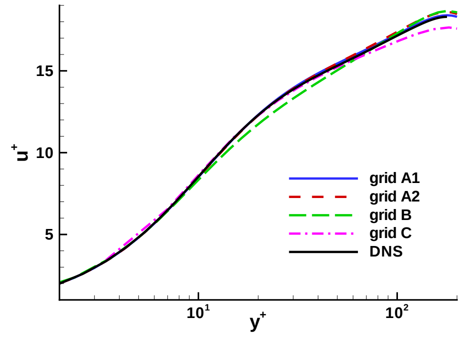

Figure 2. Mean streamwise velocity profiles without regularization.

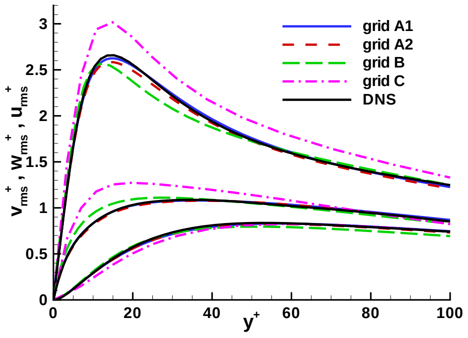

Figure 3. Velocity fluctuations without regularization.

dated in a simulation of compressible channel flow at Mach 0.2. The dimensions of the channel are $4\pi H \times 2H \times 2\pi H$, where H is the half-height of the channel. The channel flow is driven by a uniform body force per unit mass, and the bulk Reynolds number based on the channel half-height is fixed at 2800. The Prandtl number is set to 0.72. At the bulk Mach number of 0.2 the channel flow is approximately incompressible, and therefore the incompressible direct numerical simulation by Moser et al. (1999) at wall Reynolds number $\mathrm{Re}_\tau = 178$ can be used for validation.

The computational grid is uniform in the streamwise and spanwise directions. Following Verstappen and Veldman (2003), the grid is stretched in the wall-normal direction with a stretching parameter γ. The four grids used in this paper are listed in table 1. The time step size is chosen so that the Courant number based on the speed of sound is approximately 1. The finest grid used in this study has 128 grid cells in the wall-normal direction, just like the grid used by Moser et al. (1999). However, Moser et al. (1999) use a spectral method for incompressible flow, whereas here a finite-difference method for compressible flow is used.

Simulations without regularization have been performed on all the computational grids. All the simulations are numerically stable without artificial dissipation. Time-averaged results of the simulations without regularization are shown in figure 2 and 3. The obtained wall-Reynolds numbers are shown in table 1. The results obtained on the grids A1 and A2 coincide with the incompressible direct numerical simulation by Moser et al. (1999). The high accuracy on sufficiently fine grids and the absence of artificial dissipation make the symmetry-preserving discretization a very suitable method for direct numerical simulation of compressible turbulent flow. On the coarse grids B and C the obtained wall Reynolds number is acceptable, but the slope in the log layer is not captured correctly.

Table 1. The grids used in this study, and the wall Reynolds number obtained without regularization model.

Grid	Dimensions	γ	$\Delta y^+_{\mathrm{min}}$	Re_τ
A1	$256 \times 128 \times 128$	7.0	0.6	178.7
A2	$128 \times 128 \times 128$	7.0	0.6	177.8
B	$128 \times 64 \times 64$	7.0	1.2	179.3
C	$32 \times 32 \times 32$	6.0	3.4	182.3

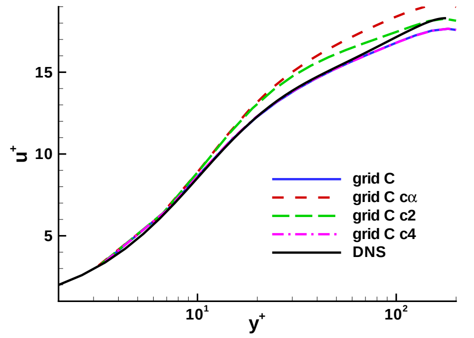

Figure 4. Mean streamwise velocity profiles with regularization.

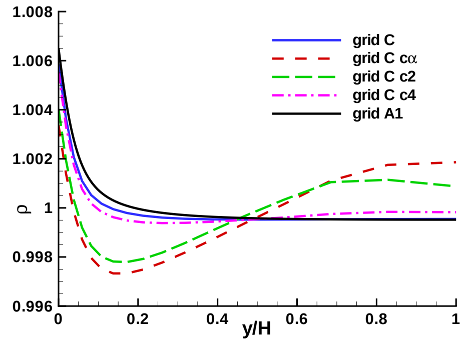

Figure 5. Time-averaged density profile with regularization.

On all the grids small spurious wiggles are observed in the instantaneous internal energy near the wall. The wiggles gradually disappear upon grid refinement, which suggests that the internal energy is not yet completely resolved (Gresho and Lee, 1981). We believe that the wiggles are produced by viscous dissipation, which transfers kinetic energy to internal energy near the wall. The viscous source term of internal energy is $\sigma : \nabla\vec{u}$, which is quadratic in the velocity gradient. If the velocity is barely resolved, then viscous dissipation tries to create even smaller scales in the internal energy. These scales cannot be captured on the computational grid, and therefore appear as spurious wiggles.

The symmetry-preserving discretization for incompressible flow correctly captures channel flow already on very coarse grids without an explicit subgrid-scale model (Verstappen and Veldman, 2003). Apparently, the implicit subgrid-scale model of this discretization is very suitable for the large-eddy simulation of channel flow. This property is not inherited by the symmetry-preserving dis-

Table 2. The wall Reynolds numbers obtained on grid C.

	no model	c_α	c_2	c_4
Re_τ	182.3	168.8	175.2	182.1

cretization for compressible flow; on coarse grids the compressible symmetry-preserving discretization is stable and yields acceptable wall Reynolds numbers, but the law of the wall is not captured correctly. Thus, unlike the symmetry-preserving discretization for incompressible flow, on coarse grids the symmetry-preserving discretization for compressible flow needs an explicit subgrid-scale model.

Simulations with the proposed symmetry-preserving regularization subgrid-scale models have been performed on grid C. In these simulations, the filter length was set equal to the mesh spacing; $r = 1$ and $\Delta_i = \Delta x_i$. Time-averaged results of the simulations are shown in figure 4 and 5, and the obtained wall Reynolds numbers are listed in table 2. The wall Reynolds numbers obtained with the Leray and c_2 regularizations are considerably lower than the wall Reynolds number obtained without regularization. Both the Leray and the c_2 regularization do not correctly capture the law of the wall. The results obtained with c_4 regularization collapse on the results obtained without regularization; the obtained wall Reynolds number is acceptable, but the slope in the log layer is not captured correctly. Recall that c_4 regularization is a more accurate approximation of the convection operator than the Leray and c_2 regularizations. It seems that for channel flow the c_4 regularization resembles the unfiltered convection operator so closely, that the results obtained with c_4 regularization become practically identical to the results obtained without regularization.

An unexpected side-effect of regularization is observed in the time-averaged density profile (see figure 5); regularization of the continuity equation moves air to the centre of the channel. We currently do not completely understand this phenomenon.

CONCLUSION AND OUTLOOK

In this paper, it was shown that the symmetry-preserving discretization and regularization for incompressible flow can be generalized to compressible flow by rewriting the compressible Navier-Stokes equations to a new form. The proposed symmetry-preserving discretization and regularizations for compressible flow were validated in simulations of channel flow.

The simulations of compressible channel flow are stable without artificial dissipation. Results obtained on sufficiently fine grids accurately coincide with the results of a direct numerical simulation. The high accuracy and the absence of artificial dissipation make the symmetry-preserving discretization a very suitable method for direct numerical simulation of compressible flow. On coarse grids the simulations of compressible channel flow are stable without artificial dissipation and yield acceptable wall Reynolds numbers, but the slope in the log layer is not captured correctly. Apparently, unlike the symmetry-preserving discretization for incompressible flow (Verstappen and Veldman, 2003), the symmetry-preserving discretization for compressible flow needs an explicit subgrid-scale model to correctly capture channel flow.

Therefore, the proposed symmetry-preserving regularization subgrid-scale models have been applied in the compressible channel flow simulations. As a first test, the filter length was set equal to the local mesh spacing. The symmetry-preserving regularizations for compressible flow do not capture the law of the wall correctly. Possibly the proposed regularization models can be improved by local calculation of the filter length (Trias et al., 2013).

Our future research will consider subgrid-scale modelling for compressible flow. One of the goals of the research is to improve the proposed symmetry-preserving regularization model. Another goal of the research is the derivation of an eddy-viscosity model for compressible flow with minimal subgrid-scale dissipation in resolved regions (Verstappen, 2011).

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