RESPONSE OF THE LINEARISED NAVIER-STOKES EQUATIONS: GENERATION OF NEAR-WALL STREAKS

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ABSTRACT

The near-wall streaks of a turbulent boundary layer are investigated using a modified version of a Low-Order Model (LOM) (Lockerby *et al.*, 2005; Carpenter *et al.*, 2007), which is based on the linearised Navier-Stokes equations (LNSE). The sensitivity of the LOM to the form of the source/forcing term that generates the streaks is explored. The response of the LNSE is gauged by a spatial energy-related norm, and the results are summarized on *response maps* containing the maximum temporal values of the response for different spanwise modes, evaluated at different wall-normal positions. All results are qualitatively similar, regardless of the form of vorticity source employed, and agree well with existing literature on streak characteristics; however, close quantitative agreement between the responses of the different forms of source term is not found. Finally, an alternative and more convenient approach is proposed that simplifies the interpretation and use of the LOM.

INTRODUCTION

There is substantial evidence that the formation of near-wall turbulent streaks is governed by an essentially linear process, even in fully nonlinear turbulent flows (see for example, Kim & Lim (2000), Joshi *et al.* (1997), Cortelezzi *et al.* (1998)). This suggests that the generation and early evolution of the near-wall streaks might be predicted using linear Navier-Stokes equations (LNSE), by introducing an initial condition or nonlinear source term (assuming this can be chosen appropriately). Even though such an approach does not model full turbulence, it offers researchers the possibility of studying near-wall streaks in isolation, providing a cleaned-up view of the streak generation and growth processes. Another distinct advantage that such linear streak models offer over full direct numerical simulation (DNS) is their numerical efficiency; extremely high-Reynolds number simulations can be performed with relative ease.

Most of the linear streak models that have been proposed come from two main methodologies: the optimal perturbation (OP) approach (e.g. Butler & Farrell, 1993; Chernyshenko & Baig, 2005; Hwang & Cossu, 2010); and low-order models (LOM), sometimes referred to as reduced-order models (Lockerby *et al.*, 2005; Carpenter *et al.*, 2007; Togneri & Davies, 2011). Although the same LNSE are solved in both cases (with some small differences), they employ rather different approaches to find and impose the initial perturbations that generate the streaks. In OP, the initial condition is chosen (out of all possible initial conditions) that most effectively amplifies streak energy, or some other appropriate measure of streak intensity. In LOM, a parameterised forcing term (representing the nonlinear source) is applied to the LNSE to generate the initial perturbation; the forcing parameters generating greatest disturbance amplification are those adopted. This is low-order compared to OP in the sense that, instead of optimising over all possible initial conditions, only a limited family of functions parameterised with a reduced number of variables is considered. The advantage of OP is accuracy, the advantage of LOM is computational efficiency. It is the latter approach that is the primary focus of this paper.

The starting point of the present work, is the LOM employed by Lockerby *et al.* (2005) and Carpenter *et al.* (2007), whereby the generation of turbulent streaks is produced by introducing a Lorentz-type body force to the LNSE; a coarse-grained optimisation over a small set of forcing parameters is then performed to find a near-optimal streak. This approach to generating streaks is particularly convenient as it is relatively easy, and computationally efficient, to incorporate the streak disturbance into an otherwise conventional computational fluid dynamics simulation. Given this advantage, this approach has been successfully employed in a variety of flow-control applications, such as: ‘closed-loop’ streak control using microjet actuators, Lockerby *et al.* (2005); ‘openloop' streak control using a spanwise-oscillating wall, Togneri (2010); and passive streak control using compliant surfaces, Carpenter *et al.* (2007). Despite its convenience and efficiency over full DNS and other linear streak models, the method as it currently stands has a number of less satisfactory aspects. The functional form of the forcing required to generate the streaks has a number of parameters which are either set empirically, or in a seemingly arbitrary way. This can give the impression that tuning of the free parameters is required to generate streaks, and that the body force is intended to represent some specific physical distribution. In fact, it is not well known how these parameters and the particular forcing form affect the streaks generated from such models.

The primary purpose of this paper is to investigate the sensitivity of LOM streak generation to a broad range of different forcing forms and with different parameters. Are the streak responses quantitatively and/or qualitatively similar? How much does the result of such models depend on the artificial force that creates the perturbation? This is critical to understand in order to gauge how much we can rely on their results. The secondary purpose of this paper is to propose an alternative approach, which is not only more generic and more numerically convenient, but does not require adjustable or empirically-fitted parameters at all.

The paper is presented as follows: first we overview the numerical methodology and discuss the approach for measuring/gauging the streak strength; secondly we look at streak generation using a variety of forcing forms and parameters; in the third section the results of the alternative forcing forms are presented and discussed; the results of the previous section motivates a fourth part in which we propose a spatio-temporal impulse perturbation as a less arbitrary and more convenient alternative to the vorticity forcing in LOM.

NUMERICAL METHODOLOGY

A velocity-vorticity formulation of the Navier-Stokes equations, particularly suitable for modelling wall-bounded incompressible viscous flows, has been adopted. Originally developed and presented by Davies & Carpenter (2001) and expanded by Davies (2005), this formulation is easily adapted to a low-order streak model, as in Lockerby *et al.* (2005). Only a very brief summary is given here; for a detailed exposition the reader is referred to Davies (2005) and Lockerby *et al.* (2005).

In this formulation, the flow field is decomposed into a mean base flow $\mathbf{U}_b = (U_b, V_b, W_b)$, and a disturbance field $\mathbf{u} = (u, v, w)$, the total velocity is expressed as simply: $\mathbf{U} = \mathbf{U}_b + \mathbf{u}$. In the same manner the total vorticity field can be expressed as $\Omega = \Omega_b + \omega$, with $\Omega_b = (\Omega_{b,x}, \Omega_{b,y}, \Omega_{b,z})$ and $\omega = (\omega_x, \omega_y, \omega_z)$. The components for the perturbation fields are defined in the x-streamwise, y-spanwise and z-wall-normal directions, respectively. In the present work we assume a parallel flow assumption, and a homogeneous turbulent mean velocity profile is selected as the underlying mean-base flow: $\{\mathbf{U}_b = U(z), \Omega_b = \Omega_{b,y}(z)\}$. Linearisation of the vorticity transport equations is then performed by omitting products of perturbations that arise in the terms for convection.

Numerically, a hybrid discretization is used: finite difference in the streamwise direction; spectral in the spanwise direction, (Fourier modes); and spectral in the wall-normal direction, (Chebyshev polynomials). The code has been validated extensively over a range of problems in linear stability; again the reader is referred to Davies (2005) for a full discussion.

The results presented in this paper have been obtained using a computational grid with 128 collocation points in the wall-normal direction, 2000 discrete points in the streamwise direction, and a total of 24 spanwise Fourier modes.

Measure and Quantification of Growth

In order to quantify the temporal and spatial response of the LNS equations we use a 'measure' based on amplification of streamwise kinetic energy, with respect to an initial state. In a similar fashion to that proposed by Chernyshenko & Baig (2005), kinetic energy is evaluated at particular wall-normal locations (herein referred to as *visualisation planes*). The time-dependent measure μ, is as follows:

$$\mu(t) = \int_{\Omega_S} \mathbf{u}_{z_v}^2(t)\, d\Omega_S \tag{1}$$

with Ω_S representing the 2D-domain at a visualization plane z_v. This particular measure allows us to investigate the response as a function of the wall-normal position and, since streak characteristics are known to vary with wall-normal position, is more appropriate than a measure evaluated over the whole domain.

MECHANISMS OF STREAKS GENERATION - LOW ORDER MODEL

The low-order model approach of Lockerby *et al.* (2005) generates streaks using an artificial body forcing term, F_x, which is introduced into the streamwise vorticity transport equation[1]. Note that in this approach the wall-normal vorticity is also indirectly forced in order to maintain solenoidality of the total vorticity field.

To ensure a finite excitation, the forcing used must have a finite duration. In previous uses of LOM this duration was set empirically (based on the length of time taken to create streaks artificially in experiment). This is reminiscent of the empirical mechanism employed by Butler & Farrell (1993) in their optimal perturbation study, who specified a restricted growth period in order to obtain reasonable streak scale predictions. In the present work we aim to remove the arbitrariness and empiric nature of such approaches. We have therefore made the force effectively instantaneous, noting that in fact an extended duration is not required to generate streaks at all. In all the simulations of this paper the forcing term is applied for no more than two time steps, as an approximation to a delta function in time.

Aiming to further extend the work in Lockerby *et al.* (2005), Carpenter *et al.* (2007) and Togneri & Davies (2011), and to investigate the sensitivity of the streak response, we have numerically experimented with a broad set of forcing terms covering particular domains in space with different topological features. Some of the forces that have been

[1] even though this is referred to as a body forcing, strictly speaking this is a spatially-distributed streamwise vorticity source.

considered are, namely: i) a concentrated-Gaussian vorticity source; ii) a constant vorticity patch; iii) a wall-normal-linear and streamwise-Gaussian vorticity source; iv) a wall-normal-constant, streamwise-quadratic vorticity source; and v) a wall-normal-sinusoidal, streamwise-Gaussian vorticity source These forces are formulated mathematically in equations (2) to (6), in the order mentioned, respectively:

$$F_{1,x} = G\,\mathbf{e}^{\left(-a\,(x-x_f)^2 - b\,(z-z_f)^2\right)}\mathbf{e}^{\mathbf{i}\beta y}\;;\{x>0; z>0\} \qquad (2)$$

$$F_{2,x} = G\,\mathbf{e}^{\mathbf{i}\beta y}\;;\{x_{\min} \le x \le x_{\max}; z_{\min} \le z \le z_{\max}\} \qquad (3)$$

$$F_{3,x} = G\,\frac{z}{z_{\max}}\,\mathbf{e}^{\left(-a\,(x-x_f)^2\right)}\mathbf{e}^{\mathbf{i}\beta y}\;;\{z \le z_{\max}\} \qquad (4)$$

$$F_{4,x} = G\,\frac{x^2}{x_{\max}^2}\,\mathbf{e}^{\mathbf{i}\beta y}\;;\{x \le x_{max}\;; z_{min} \le z \le z_{max}\} \qquad (5)$$

$$F_{5,x} = G\,\sin\left(\frac{\pi z}{z_f}\right)\mathbf{e}^{\left(-a\,(x-x_f)^2\right)}\mathbf{e}^{\mathbf{i}\beta y}\;;\{z \le z_{\max}\} \qquad (6)$$

where G is the intensity or strength of the distribution, a and b are parameters defining the spread of the corresponding Gaussian function, x_f and z_f are the coordinates of the spatial centroid for each distribution in the streamwise and wall-normal directions, respectively, and $\{x_{\min}, x_{\max}\}$ and $\{z_{\min}, z_{\max}\}$ define, where appropriate, the extents of the domain for each distribution used. The results reported here have been obtained for values of x_{min} and x_{max} adjusted to ensure the centroid of each forcing distribution was located at a streamwise location in inner units, $x_f^+ = 178.53$ from the origin of the domain (this value has no particular significance, except that it is a comfortable distance away from the upstream and downstream boundaries). All the results presented here correspond to forcing functions located at $z_f^+ = 10$, except where stated.

NUMERICAL EXPERIMENTS AND RESULTS

A series of simulations have been performed to explore the sensitivity of the streak response to changes in the form of the body forcing. The results presented in this section represent a small set of all the forcing types and flow conditions considered in this study, but for which results have been broadly similar to those presented; space precludes a full account of these simulations. Here we report results for $Re_{\delta^*} = 4500$ (note, variation in Reynolds number does not significantly affect the streak spanwise spacing, when non-dimensionalised with inner units).

For each of the six forces considered (equation (2) to (6)) over a range of spanwise modes, *all* responses have exhibited development of streak disturbances in u velocity perturbation (the signature being transient growth: algebraic growth followed by viscous decay). A representative example of this behaviour is shown in Figure 1 for source type $F_{1,x}$ and measure μ evaluated at five different wall-normal planes. We conclude from this general behaviour, that the generation of streaks is largely insensitive to the form of force adopted; in respect to the validity of LOM it suggests that the exact form of forcing chosen is not particularly important for the generation of streaks.

However, the same conclusion cannot be drawn on the streaks' structural properties (e.g. spanwise spacing). There exists a range of streak structure scales, which is why the LOM must perform a coarse-grained optimisation, so that the 'most likely' or 'most representative' streak can be selected. The question that naturally follows is how does the choice of forcing type affect the structural properties of the 'near-optimal' streak? Is this, too, largely independent on the particular forcing type?

For each of the six forcing types we consider a range of spanwise modes. The spanwise wavelength ($\lambda^+ = 2\pi/\beta$) can thus be thought of as the free optimisation parameter for each of the forces, which are otherwise topographically quite different. Figures 2(a) to 2(f) show results as contour plots, herein named *response maps*, for the six forcing configurations. These response maps summarise the general signature of the system in terms of the maximum value in time for the selected measure μ, for different spanwise wavelengths (the free parameter) with μ evaluated at different wall-normal locations. These maps, then, show the position of the near-optimal response of the LNSE (if it exists) in terms of the forcing spanwise wavelength λ^+, and the wall-normal visualization plane z_v^+.

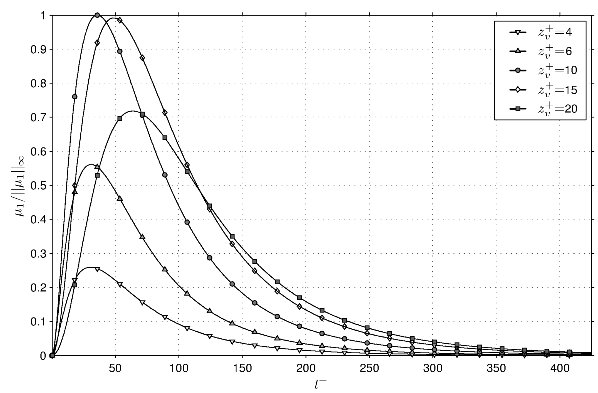
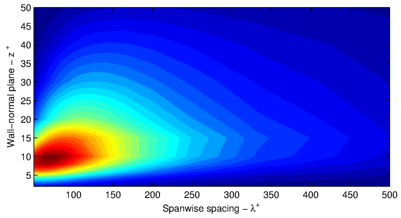
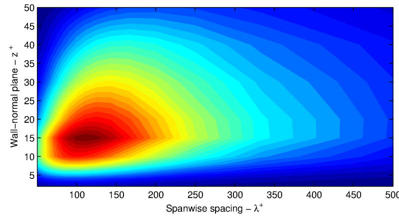


Figure 1. Transient growth of streamwise kinetic energy at different wall-normal locations z_v^+ for body force at z_f^+ =10 and $\lambda^+ = 100$. Plots have been normalised using the global maximum.

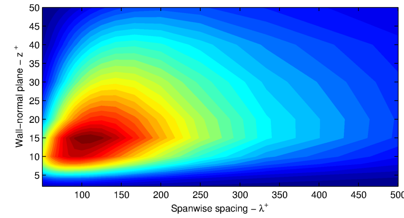
What is immediately striking from Figures 2(a) to 2(f) is the qualitative similarity between all of the response maps. Considering the diversity of forcing form, this is perhaps surprising. Looking more closely, we can see that the optimum values correspond to spanwise wavelengths of $\lambda^+ = 75-120$, which is in consonance with the experimentally observed values of spanwise streak spacing. Furthermore, the location in the wall-normal direction at which this maximum response occurs lies at $z_f^+ \approx 10-15$ (which is affected, but not completely determined, by the wall-normal forcing location). This is also in broad agreement with the experimental literature on streaks (e.g. Smith & Metzler (1983)). Finally, we also see in every response map, the optimal spanwise wavelength for a given wall-normal visualization plane increases with distance away from the wall. This is also a well-known streak behaviour: streaks farther from the wall are typically more separated (Tomkins & Adrian, 2005).



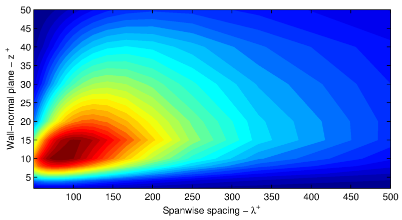
(a) Distribution 1 centred at $z_f^+ = 5$



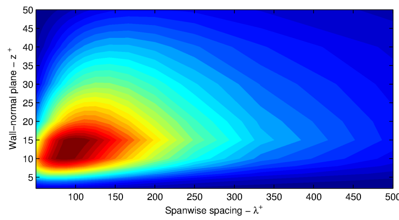
(b) Distribution 1 centred at $z_f^+ = 15$



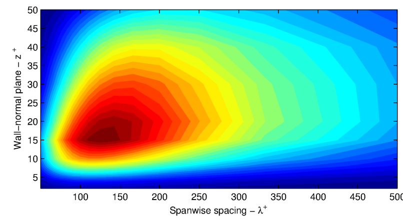
(c) Distribution 2 centred at $z_f^+ = 10$



(d) Distribution 3 centred at $z_f^+ = 10$



(e) Distribution 4 centred at $z_f^+ = 10$



(f) Distribution 5 centred at $z_f^+ = 10$

Figure 2. Response maps for $\max_t(\mu)$ for fields (a) $F_{1,x}$ at $z_f^+ = 5$, (b) $F_{1,x}$ at $z_f^+ = 15$, (c) $F_{2,x}$ at $z_f^+ = 10$, (d) $F_{3,x}$ at $z_f^+ = 10$, (e) $F_{4,x}$ at $z_f^+ = 10$, and (f) $F_{5,x}$ at $z_f^+ = 10$

It appears, then, that it is not at all necessary for the free parameters of an LOM to be finely tuned in order to achieve good qualitative comparison with the experimental literature on streaks; in fact it would appear that almost any near-wall streamwise-vorticity forcing will generate a reasonable response. For example, the response maps in Figures 2(c) and 2(e), corresponding to elongated patches of vorticity in the x-streamwise direction, show no significant difference from the other response maps, for which the distributions are very concentrated in x-streamwise direction.

Having said that, exact quantitative agreement of the streak optimum is not shown between forcing types; in our wider search, not reported here, we were unable to find a measure that provided close quantitative agreement between all forcing forms. In fact, the magnitudes of the measure for the different forcing types were in most cases very different (for this reason we have deliberately omitted a numerical scale in presentation of the response maps). It is also clear from Figures 2(a) and 2(f) that results of these particular forcing distributions deviate slightly from the common pattern exhibited by other cases. All together, this suggests that if precise quantitative information is required from such models, an empirical calibration for a particular forcing type and measure would be needed.

Figures 3(a) to 3(c) show response maps from $F_{1,x}$ centred at various normal distances from the wall (z_f^+=5,10,15). Again, the responses are similar, though clearly the wall-normal location of the forcing does have an effect on the exact scale of the optimum streak. In order to understand the fundamental difference between the forcing in each case, spectral distributions along the wall-normal direction (at x_f) were obtained. As our intention is to examine the domain corresponding only to the boundary layer, then a DST-I discrete Fourier transform was selected, in order to account for the differences in boundary conditions at the wall and at the boundary layer edge. DST-I spectra using 512 points are shown in Figures 4(a) to 4(c) for $F_{1,x}$ with centroid at $z_f^+ = 5, 10, 15$. What is shown by the spectra, and what is perhaps obvious on reflection, is that a structurally identical forcing function (a double delta function), applied at different wall-normal positions, is responsible for weighting certain modes more than others along the wall-normal direction. The forcing farthest from the wall (z_f^+=15) forces more of the lower value (spatially larger) modes, and is thus probably why this response has an optimal at a larger spanwise spacing.

This raises some interesting questions regarding the LOM approach. For example, is it necessary to perturb the 'correct' scales in the wall-normal direction in order to obtain a correct prediction of the relevant characteristics of streaks? For quantitative prediction the answer is probably yes; for qualitative prediction, possibly not. Irrespective of the use of such a model (for qualitative or quantitative prediction) a standardised and non-arbitrary approach is far preferable, and this is what is currently missing in the LOM method.

The question is now: in a standardised LOM, what distribution of wall-normal modes of the vorticity forcing should be chosen? Incidentally, the same should be asked of the spanwise modes, because the relative weighting of these will also affect the optimal response; LOM currently equally forces the spanwise modes. To be consistent with this, we propose that all wall-normal modes of the vorticity forcing should be equally weighted. This can be generated by a small-scale near-wall forcing: i.e. an approximation to an impulse at the wall. In the limit (as the forcing approaches a wall impulse) this modified LOM has *no* input parameters, except for the impulse's location in the streamwise and spanwise directions, reflecting that no specific scales are being forced more than others. Furthermore, the equal weighting of spanwise modes now has a clearer and more explicit interpretation; i.e., it can be thought of as arising from an impulsive forcing in the spanwise direction.

However, a remaining ambiguity in the LOM exists. This is because the form of the vorticity forcing does not correspond directly to the cross flow perturbations it is used to generate. It is therefore very difficult to normalise the results consistently. For example, as mentioned above, the spanwise modes of the vortical forcing are weighted evenly with respect to each other, when assessing the optimal streak. However, this does not imply that initially the spanwise modes of each of the velocity perturbations are also weighted equally. To remove this uncertainty, and in light of the fact that we have demonstrated that forcing need not be applied for an extended duration to generate streaks, we propose replacing the vortical forcing with a fixed initial condition on the velocity field. This impulse response approach is investigated in the following section.

SPATIO-TEMPORAL IMPULSE: AN ALTERNATIVE APPROACH TO LOM

An initial condition is prescribed to the LNSE to simulate an approximate impulse response. The initial crossflow velocity field is prescribed such that it is approximately impulsive (Gaussian) in the wall-normal direction for the wall-normal velocity, impulsive in the spanwise-direction for the spanwise velocity, and so that continuity is preserved:

$$u = 0 \quad (7)$$

$$v = 2Gb(z-z_f)\delta(y-y_f)\mathbf{e}^{\left(-a(x-x_f)^2-b(z-z_f)^2\right)} \quad (8)$$

$$w = G\delta'(y-y_f)\mathbf{e}^{\left(-a(x-x_f)^2-b(z-z_f)^2\right)} \quad (9)$$

where the parameter z_f is made very small (i.e. approaching the wall). The location parameters x_f and y_f have no material significance, as the response is translationally invariant in the x and y directions. To approximate the impulse the parameters a and b are made as large as the fineness of the discretisation will allow; above a certain value they do not affect the form of the response. Both the delta function δ and its spanwise derivative δ' can be approximated without a Gaussian function, using the spanwise fourier decomposition in the current numerical method. Note, here, unlike in the previous section, for presentation purposes spanwise modes are combined in order to get the total form and energy of the response. This does not mean, though, that an optimum spacing cannot be extracted from the impulse response; all that is required is that the spanwise modes are instead considered separately, as before.

The temporal energy evolution for this initial condition (at a wall-normal plane $z^+ = 10$)) is shown in Figure 5. The same transient growth is observed as was shown in Figure 1 for the standard LOM approach. However, a combination of modes is presented here (a full three-dimensional representation of the response) which, on average, have less persistence than the mode shown in Figure 1 (corresponding to λ^+=100). The solid line and dashed line correspond to initial conditions centred at $z_f^+ = 1$ and z_f^+=0.5, respectively. The response is nearly identical, demonstrating that given the initial condition is sufficiently close to the wall, the response is independent of the exact location.

Figure 6 shows a series of plan-view contours of streamwise velocity perturbation at progressive points in the evolution. The development of the long streaky nature of the disturbance is evident. Note that the colour contours are different in value; the energy of the disturbance in Figure 6(c) is much less than in figure 6(b). Although difficult to see from the figure,

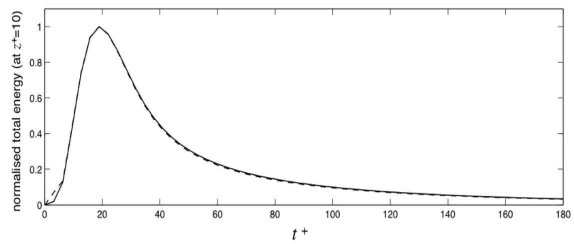


Figure 5. The impulse response: normalised time evolution of μ evaluated at $z^+ = 10$ for impulse initial conditions. Solid line: $z_f^+ = 1.0$; Dashed line: $z_f^+ = 0.5$.

the spanwise scale of this response changes in time, reflecting the difference in persistence of the spanwise modes (the larger scales last longer, as might be expected). An average over time of the dominant spanwise mode is approximately 100 wall units; again, as with the LOM results of the previous section, this is in general agreement with experimental findings. Note, in the figures, the distance between the positive and negative streak disturbances corresponds to half the spanwise spacing.

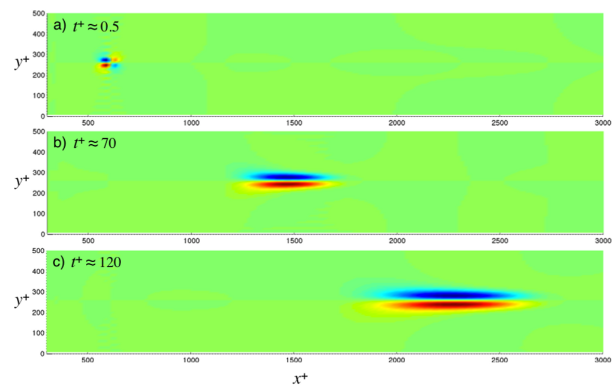
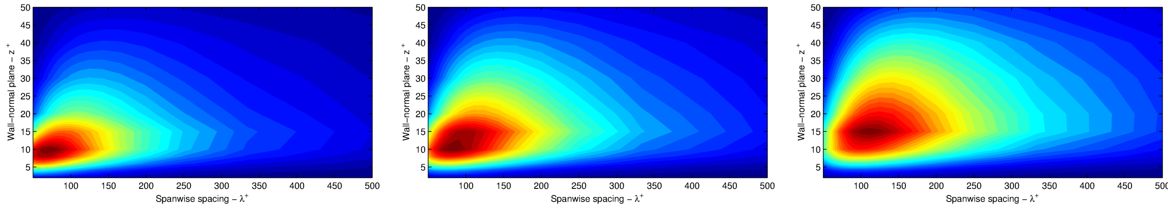


Figure 6. Plan view contours (taken at $z^+ = 10$) of streamwise velocity perturbation at times (a) $t^+ \approx 0.5$, (b) $t^+ \approx 70$ and (c) $t^+ \approx 120$.

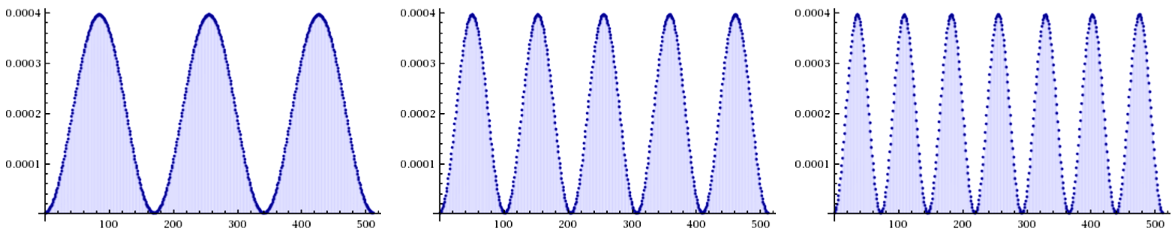
SUMMARY

For the characteristics investigated, streaks generated using a low-order model are generally insensitive to the form of the source or body force term in the LNSE: the optimum streak scales generated are all within a range of what might be expected experimentally. However, the exact quantitative features do depend on the form of that force, and thus there are limits to an uncalibrated application of such an approach. A more convenient alternative to LOM, and one which does not have the arbitrariness of a parameterised forcing, is offered by a spatio-temporal impulse perturbation at the wall of the boundary layer.



(a) (b) (c)

Figure 3. Response maps using measure $\max_t(\mu)$ for $F_{1,x}$ located at (a) $z_f^+ = 5$, (b) $z_f^+ = 10$ and (c) $z_f^+ = 15$



(a) (b) (c)

Figure 4. Wall-normal spectras using DST-I with 512 points for $F_{1,x}$ located at (a) $z_f^+ = 5$, (b) $z_f^+ = 10$ and (c) $z_f^+ = 15$

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