ON THE STATIONARITY OF LINEARLY FORCED TURBULENCE IN FINITE DOMAINS

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ABSTRACT

A simple scheme of forcing turbulence away from decay was introduced by Lundgren some time ago, the 'linear forcing', which amounts to a force term linear in the velocity field with a constant coefficient. The evolution of linearly forced turbulence towards a stationary final state, as indicated by direct numerical simulations (DNS), is examined from a theoretical point of view based on symmetry arguments. In order to follow closely the DNS the flow is assumed to live in a cubic domain with periodic boundary conditions. The simplicity of the linear forcing scheme allows one to re-write the problem as one of decaying turbulence with a decreasing viscosity. Scaling symmetry considerations suggest that the system evolves to a stationary state, evolution that may be understood as the gradual breaking of a larger approximate symmetry to a smaller exact symmetry. The same arguments show that the finiteness of the domain is intimately related to the evolution of the system to a stationary state at late times, as well as the consistency of this state with a high degree of isotropy imposed by the symmetries of the domain itself. The fluctuations observed in the DNS for all quantities in the stationary state can be associated with deviations from isotropy. Indeed, self-preserving isotropic turbulence models are used to study evolution from a direct dynamical point of view, emphasizing the naturalness of the Taylor microscale as a self-similarity scale in this system. In this context the stationary state emerges as a stable fixed point. Self-preservation seems to be the reason behind a noted similarity of the third order structure function between the linearly forced and freely decaying turbulence, where again the finiteness of the domain plays an significant role.

INTRODUCTION

Maintaining a turbulent flow in a more or less stationary state, for better statistics in experiment or convenience in theoretical considerations, requires forcing the flow, that is feeding it energy which balances dissipation happening at the smallest scales. Numerical simulations of incompressible isotropic turbulent flows one usually solves the Navier-Stokes equations in a cubic box with periodic boundary conditions. There exists a variety of deterministic forcing schemes, see e.g. [1–8], in the sense that that there is no additional randomness introduced in the problem, as well as stochastic schemes, see e.g. [9–11], in which the details of the force term are determined by additional random variables following prescribed stochastic processes.

Lundgren proposed in [12] that we may simplify the deterministic models to the bare minimum, in some sense, assuming a force (per unit mass) $\mathbf{f} = A\mathbf{u}$, where A is plainly a constant. The 'linear forcing' scheme was further studied in [13] and [14]. Its simple force term $A\mathbf{u}$ has the same form in both the spectral and physical space. Thus, unlike other forcing schemes, it may be used equally easy in cases that need to be solved directly in the physical space with boundary conditions different than periodic [13]. Additionally, although in the linear forcing the injection of energy into the flow is not restricted to the larger scales, this scheme performs decently, and in fact possibly better, in the region between the inertial range and the integral scale than other forcing schemes in [12].

The performance of the linear forcing scheme with respect to its convergence properties was studied in considerable detail in [13] and useful remarks have been made in [15]. The clear conclusion is that linear forcing results in relatively large fluctuations in the stationary phase. From the practical point of view this is a disadvantage as good statistics require longer simulations. Also, the stationary state is reached after a relatively long transient period [13][15] requiring even more computational time.

On the other hand, linear forcing leads to quite controllable situations in the stationary state: Given the scales of the problem i.e. the rate A, the cubic box size l and the viscosity ν, the facts of the stationary state are predictable. The balance between the energy production and dissipation, $2AK = \varepsilon$, is indeed observed on the (time-) average validating the very concept of a stationary state; the dissipation length $L_\varepsilon = (2K)^{\frac{3}{2}}/\varepsilon$ turns out to be equal to the box size l within few percent error in all cases [13]; the Reynolds number $\mathrm{Re}_L = K^2/(\varepsilon\nu)$ may be re-written as $\frac{1}{4}AL_\varepsilon^2/\nu$ at the stationary state, should then be roughly equal to $\frac{1}{4}$ of the natural order of Re_L in this problem, Al^2/ν, in all cases, as it is indeed observed [14].

Even if we take stationarity for granted, its characteristics i.e., the relatively large fluctuations and the 'predictability' of quantities describing the state of turbulence, certainly call for understanding. Simultaneously the very existence of a stationary state in this scheme is a fairly intriguing matter. The long-time effect of the energy production competing with dissipation is not a priori clear. From the dynamical point of view, it is clear that the dissipation term $\nu\nabla^2 u$ becomes stronger than the force term Au at scales smaller than $(\nu/A)^{\frac{1}{2}} \sim \mathrm{Re}_\lambda^{-1} l$, but it is not clear whether energy which is produced at all other scales up to l will be dissipated by an adequate rate at those smaller scales.

We will approach the problem as follows. The relative simplicity of linear forcing allows us to study its late-time evolution employing scaling symmetry arguments to an extent enjoyed possibly only in freely decaying turbulence.The predictability, as we called it above, of the stationary state, is enlightened through those symmetry arguments, essentially on the basis that there is no intrinsic large length scale in the dynamical equations apart from that introduced by the boundary conditions i.e. the finite size l of the domain. Then remains the question why the fluctuations observed in the stationary stateare so large. We shall argue, as analytically as we can, that the fluctuations can be associated with the deviations from isotropy accumulated by this forcing at all scales between the scale $(\nu/A)^{\frac{1}{2}}$ and the domain size l (unlike the limited bandwidth forcing schemes which feed anisotropy only at the domain size scale where isotropy is already broken). The method we shall use is to reduce the dynamical problem to a two-equation model. As a cross check of our previous conclusions, the stationary state re-emerges as a stable fixed point of the evolution, a byproduct of which is that fluctuations tend to be suppressed as long as turbulence is isotropic.

UNFORCED TURBULENCE WITH DECAYING VISCOSITY

Exact scaling symmetry at late times

We shall proceed as follows. Linearly forced turbulence is described the the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + A\mathbf{u}\,, \tag{1}$$

together with the incompressibility condition $\nabla\cdot\mathbf{u} = 0$. On the faces of the cubic domain we impose periodic boundary conditions: $\mathbf{u}(x,y,z) = \mathbf{u}(x,y+l,z) = \mathbf{u}(x,y,z+l) = \mathbf{u}(x+l,y,z)$.

Mathematically, we may re-write the problem as an equation for a new field $\mathbf{u}'$ w.r.t. a new time t' in the form:

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}'\cdot\nabla)\mathbf{u}' = -\frac{1}{\rho}\nabla p' + \frac{\nu}{At'+1}\nabla^2\mathbf{u}'\,, \tag{2}$$

where $\mathbf{u}' = e^{-At}\,\mathbf{u}$ and $At'+1 = e^{At}$. $\nabla\cdot\mathbf{u}' = 0$ still holds. We fix one integration constant by setting $t' = 0$ corresponding to $t = 0$, though the origin of t can be still be shifted arbitrarily. The problem has been changed to one with the form of decaying turbulence with decreasing viscosity $\nu' = \nu(At'+1)^{-1}$.

Equation (1) is invariant under time-translations. This translates to an exact scaling symmetry of equation (2). Even by inspection one may verify that the transformation

$$t' \to e^a t'\,, \qquad \mathbf{u}' \to e^{-a}\mathbf{u}'\,, \tag{3}$$

for any constant a is an exact symmetry of the previous equation (necessarily, $p' \to e^{-2a}p'$) for times $t' \gg A^{-1}$. Then the integration constants in the relation between t and t' are irrelevant. We may now forget equation (1) for a little while and focus on the unforced turbulence described by (2).

Scaling symmetries, asymptotic behavior and isotropy

Consider an arbitrarily chosen moment of time t'_0 and the velocity field $\mathbf{u}'_0$ at that moment, and another moment $t' = e^a t'_0$ when velocity is $\mathbf{u}'$. Invariance means: $t'\mathbf{u}' = t'_0\mathbf{u}'_0$. Equivalently we may write

$$\mathbf{u}' = \frac{1}{t'}\,t'_0\,\mathbf{u}'_0\,. \tag{4}$$

Now in general a symmetry transformation moves us around the space of solutions. That is, all the previous relation means is that if there is a solution with velocity $\mathbf{u}'_0$ at time t'_0 then there is another solution with velocity field $\mathbf{u}'$ at time t'. i.e. in general $\mathbf{u}'$ and $\mathbf{u}'_0$ need not necessarily correspond to the same initial conditions. On the other hand, the symmetry holds for *large* times t' and t'_0. Even if it did not, that would be a convenient choice because the behavior (4) might then be an exact *asymptotic* result for a large class of solutions, meaning irrespectively of their initial conditions. That implies that $t'\mathbf{u}' = t'_0\mathbf{u}'_0$ is an actual constant at each point $\mathbf{r}$ in space depending only on the parameters of the equation and the boundary conditions.

Of course this cannot be true unless that constant is simply zero. he best context to apply this reasoning is given by the correlators of the velocity field, which of course makes much sense in the problem of turbulence. By correlator we mean an *ensemble average* $\overline{u'_{i_1}(\mathbf{r}_1,t'_1)u'_{i_2}(\mathbf{r}_2,t'_2)\cdots}$, and its derivatives. Denote such a correlator by a tensor $T'_{j_1j_2\cdots}$. Then the symmetry (3) might make more sense. If the correlator involves n times the velocity field the symmetry tells us that

$$T'_{j_1j_2\cdots} = \frac{1}{t'^n}\,t_0'^n\,T'_{0\,j_1j_2\cdots}\,. \tag{5}$$

Now $t_0'^n\,T'_{0\,j_1j_2\cdots}(\mathbf{r},t'_0)$ must be a constant at each point $\mathbf{r}$ in space. These quantities, if they are meaningful, are also independent of the initial conditions by their very definition. I.e. they represent some final steady state where all solution evolve to. Then the solenoidal nature of the velocity field and the symmetries of the cubic domain force many correlators to vanish. For example, $t_0'^2\,\overline{u'_{0i}u'_{0k}}$ must be equal to $\delta_{ik}\,t_0'^2\,\overline{u'_{0i}u'_{0i}}$(no sum) $= \frac{1}{3}\delta_{ik}\,t_0'^2\,\overline{u'_{0j}u'_{0j}}$, i.e. essentially a scalar. Moreover, by the incompressibility condition $\nabla\cdot\mathbf{u}' = 0$ we see that the scalar itself must be constant in space. The same reasoning applies to any correlator with free solenoidal indices.

One should note that the situation resembles very much that of isotropic i.e. also homogeneous turbulence. There is of course an amount of anisotropy allowed by the problem but it is much less than what would call anisotropy in general. Thus we will proceed by assuming isotropy and analyze what that implies; then, as isotropy cannot hold at scales comparable to the cubic box size l, the effects of the finiteness of the domain eventually play a key role.

We define now a few important scalars for the description of turbulence, their symmetry and transformation properties and their expected late time behavior according to our arguments.

The r.m.s. value q of the velocity and the dissipation rate ε are defined by $q^2 = \overline{\mathbf{u}\cdot\mathbf{u}}$ and $\varepsilon = \nu\,\overline{\partial_j u_i \partial_j u_i}$. Also by $K = \frac{1}{2}q^2$ we shall denote the total kinetic energy per unit mass. Similar expressions hold for the primed quantities.

In order to see what this result means back in the variables of the system (1), we use the transformation of velocity and the time-dependent viscosity ν' to obtain the transformations of K and ε:

$$K' = (At'+1)^{-2}K\,, \qquad \varepsilon' = (At'+1)^{-3}\varepsilon\,. \tag{6}$$

The dissipation length scale L_ε and the Reynolds number Re_L defined by $L_\varepsilon = q^3/\varepsilon$ and $\mathrm{Re}_L = K^2/(\varepsilon\nu)$, and transforming as

$$\mathrm{Re}'_L = \mathrm{Re}_L\,, \qquad L'_\varepsilon = L_\varepsilon\,. \tag{7}$$

Linearly forced isotropic turbulence

The 'Karman-Howarth equation' [16][17] derived from (2) under the conditions of homogeneity and isotropy reads

$$\frac{\partial}{\partial t'}(q_1'^2 f') = \frac{1}{r^4}\frac{\partial}{\partial r}\left\{r^4\left(q_1'^3 h' + \frac{\nu}{At'+1}2q_1'^2\frac{\partial f'}{\partial r}\right)\right\}. \tag{8}$$

The one-direction r.m.s. value of the velocity, q_1, does not depend on the direction, i.e. $q^2 = 3q_1^2$. The functions f and h are defined by $\overline{u_l(0)u_l(\mathbf{r})} = q_1^2\,f(r)$ and $h(r)$ by $\overline{u_l(0)u_l(0)u_l(\mathbf{r})} = q_1^3\,h(r)$, where u_l is the velocity component in the direction of the separation $\mathbf{r}$.

The following scaling arguments are borrowed from [18] where they are applied to the freely decaying turbulence. Define the group of transformations

$$t' \to e^a t'\,,\ r \to e^{\gamma a} r\,,\ q' \to e^{\gamma a - a} q'\,,\ f \to f\,,\ h \to h\,. \tag{9}$$

Each value of the γ a subgroup which is basically time evolution.

At late times $t' \gg A^{-1}$ equation (8) transforms to

$$\frac{\partial}{\partial t'}(q_1'^2 f') = \frac{1}{r^4}\frac{\partial}{\partial r}\left\{r^4\left(q_1'^3 h' + e^{-2\gamma a}\frac{\nu}{At'}2q_1'^2\frac{\partial f'}{\partial r}\right)\right\}.$$

We observe that for very high Reynolds, essentially for inviscid flow, the group (9) is a symmetry of the Karman-Howarth equation. It is an approximate symmetry for high Reynolds. On the other hand the $\gamma = 0$ subgroup is an *exact symmetry* of (8) for any Reynolds number. In fact it is nothing but the exact symmetry (3) of the Navier-Stokes.

Observe now that the group (9) leaves the quantities $t'^{-\gamma}L'$ and $t'^{2-2\gamma}K'$ invariant, which means constant. We can then deduce the time-evolution of all quantities of interest.

$$L' = \text{constant } t'^{\gamma}\,,\ K' = \text{constant } t'^{2\gamma-2}\,, \qquad \varepsilon' = \text{constant}\,t'^{2\gamma-3}\,,\ \mathrm{Re}'_L = \text{constant}\,t'^{2\gamma}\,. \tag{10}$$

We may now study a case where we may 'watch' the flow evolving to the stationary state. Consider a flow that starts off with velocities of order u_0 and a box of size l such that $u_0 \gg Al$. Equivalently the turn-over time is much smaller than forcing time scale A^{-1}, that is, $l/u_0 \ll A^{-1}$.

Given A, l and ν there is a naturally defined Reynolds number in the problem:

$$\mathrm{Re}_A = \frac{Al^2}{\nu}\,. \tag{11}$$

That is, the condition $l/u_0 \ll A^{-1}$ can be rephrased as that the flow starts off with a very high Reynolds number, $\mathrm{Re}_L \gg \mathrm{Re}_A$.

Consider then times t' such that $l/u_0 \ll t' \ll A^{-1}$. Looking at the previous equation we understand that for those times the turbulent flow is merely freely decaying with constant viscosity ν. If all previous inequalities hold strongly enough, then there will be time for the flow to evolve adequately towards its developed stage. That means that the quantities describing turbulence will evolve according to the power laws (10).

When t' becomes of order of A^{-1} 'linear forcing' kicks in. Now one should recall that Reynolds number is always decreasing, therefore some time before or after that moment it will drop enough so that the viscosity term cannot be neglected. That means that from that moment on the group (9) is not much of a symmetry anymore: the only symmetry remaining is its subgroup (3) corresponding to $\gamma = 0$, which is exact and therefore holds at all times. Viscosity now decreases with time therefore energy will be dissipated with an ever decreasing rate. We may then picture, very roughly, the flow evolving by going through stages of smaller exponents γ, following simple of less simple laws parameterized by it, eventually reaching the specific value for which a subgroup of (9) is an exact symmetry of the system: $\gamma = 0$. Now, the power laws (10) imply that L' and Re'_L are constant, $\varepsilon' = \text{constant}\,t'^{-3}$ and $K' = \text{constant}\,t'^{-2}$. The transformations (6) of K' and ε' back to the original variables of equation (1) show that everything, that is K, ε, L_ε and Re_L, is constant. We have reached the stationary state.

Effects of the finite domain

Recall now that the system (1), or (2), is solved in a domain of some finite size l. An infinite size l is meaningless: In the absence of another large length scale, this means that the total energy production rate in the domain depends on it and diverges, $2AK\rho l^3 \to \infty$. Also large l essentially means a large Al compared with any specific initial condition u_0: an infinitely large

l is equivalent to initial conditions u_0 infinitely close to zero.

There is a major implication following the presence of a finite size domain. *Its fixed size l breaks the symmetry (9), as the presence of a fixed length says that it must be $b = 0$.* That is, the domain size breaks the larger symmetry (9) down to its subgroup $\gamma = 0$, the exact symmetry.

We may now think of the evolution of the flow from another point of view, that of the integral scale. As long as the integral scale L' is small compared to l the group (9) is a fairly good approximate symmetry. Then L' increases with time as $\sim t'^{\gamma}$. As L' grows larger, (9) is a less and less good approximate symmetry. As before, we may then roughly picture the flow as going through stages of smaller γ reaching the stage with $\gamma = 0$ which is consistent with the exact symmetry. This means that L' will become constant. The natural order for that constant L' (as well as L, recalling that $L = L'$ by (7)) is the domain size l. As mentioned in the Introduction, DNS have shown that specifically $L_\varepsilon = l$ within a few percent error [13]. Quite similarly the Reynolds number Re'_L becomes constant when the system evolves to the state with $\gamma = 0$. We see that the peculiar 'decaying' turbulent flow (2) reaches a peculiar kind of stationarity: its Reynolds number becomes constant i.e. *turbulence as such is not decaying at all.* Also by $\mathrm{Re}'_L = \mathrm{Re}_L$, relation (7), this is also the Reynolds number of the linearly forced turbulence described by (1). The order for that constant Reynolds number is of course set by Re_A. This is indeed what follows from the DNS [14][13].

SELF-PRESERVING TURBULENCE AND STABILITY OF STATIONARITY

The periodic boundary conditions on the cubic domain imply that the flow evolves in an essentially boundaryless but homogeneous space of finite size. The latter means that the condition of isotropy cannot hold at scales comparable to the domain size. Therefore evolution to stationarity is compromised. Deviations from isotropy at the larger scales may then generate deviations from stationarity. Now consider small enough scales where turbulence is adequately isotropic. Deviations from stationarity appear at those scales through the cascade as well as forcing. Although perfect stationarity is shaken by the domain size scales, if those smaller scale deviations decay and vanish fast enough, then we could still have an imperfect stationarity, than a completely different state.

The evolution laws derived previously can be deduced from models relating the kinetic energy K and dissipation rate ε. These models may be derived by self-similarity arguments, which date back to the work of von Karman and Howarth [19] and Batchelor [20], see also [21][22]. One looks for self-similar solutions of the equations w.r.t. a single length scale $L(t)$, 'self-preserving' turbulent flows. Assuming that the larger scales of the flow are evolving in such a self-preserving manner, one chooses $L(t)$. The simple model so obtained can also be regarded as describing self-preserving turbulence of all scales for infinitely high Reynolds numbers, essentially for inviscid flow.

In our case these analysis leads to the equation

$$\frac{d\varepsilon}{dt} = -C_\varepsilon^A \frac{\varepsilon^2}{K} + c_1 A\varepsilon\,, \tag{12}$$

where $c_1 = 3$ and C_ε^A is a dimensionless constant. From (1) one may easily derive

$$\frac{dK}{dt} = -\varepsilon + 2AK\,. \tag{13}$$

The system of equations (13) and (12) is consistent with a static solution only for $2C_\varepsilon^A = c_1$. The special case $C_\varepsilon^A = c_1/2 = 3/2$, predicted by large scale self-preservation, implies that $L_\varepsilon =$ constant at all times the model holds. This is consistent with the general idea about it. The model can be easily solved exactly and indeed predicts that the flow approaches stationarity exponentially fast for all $C_\varepsilon^A > 1$ (the case $C_\varepsilon^A = 1$ is trivially consistent with stationarity).

A more elaborate analysis of the evolution of isotropic turbulence has been presented in the past in the Refs. [23–26]. In those works the self-similarity hypothesis is applied at the viscous equations of the flow i.e. self-preservation is required to be true for all scales of turbulence for finite Reynolds. In the terminology of Ref. [25], self-preservation is complete. An implication of this requirement is that the self-similarity scale is the Taylor microscale λ_g.

From the point of view of the linearly forced turbulence all that sound very relevant and interesting for the following reason. There is a natural scale where energy production balances dissipation: $\lambda_A = (\nu/A)^{\frac{1}{2}}$. This can be understood both from the physical space equation (1), or in another form from the spectral energy balance equation $\partial_t E(k) = -\partial_k T(k) - 2\nu k^2 E(k) + 2AE(k)$, where $E(k)$ is the spectral density of the kinetic energy K and $T(k)$ is the spectral energy flux. Now the Taylor microscale $\lambda_g = (10\nu K/\varepsilon)^{\frac{1}{2}}$ becomes equal to $\sqrt{5}\lambda_A$ in the stationary state (upon applying $2AK = \varepsilon$, which follows e.g. from (13)). That is the problem possesses a specific scale for the Taylor microscale, the scale λ_A.

We may then proceed as follows. There is another exact equation which we may use along with (13). It reads

$$\frac{d\varepsilon}{dt} = \frac{7|S|}{3\sqrt{15\nu}}\varepsilon^{3/2} - \frac{7G}{15}\frac{\varepsilon^2}{K} + 2A\varepsilon\,, \tag{14}$$

where S (the velocity gradient distribution skewness) and G are defined by $S = \lambda_g^3 \partial_r^3 h|_{r=0}$ and $G = \lambda_g^4 \partial_r^4 f|_{r=0}$. Also, λ_g is essentially defined by $\lambda_g^2 \partial_r^2 f|_{r=0} = -1$. Equation (14) can be derived from the Karman-Howarth equation associated with (1),

$$\frac{\partial}{\partial t}(q_1^2 f) = \frac{1}{r^4}\frac{\partial}{\partial r}\Big\{r^4\Big(q_1^3\, h + 2\nu\, q_1^2\,\frac{\partial f}{\partial r}\Big)\Big\} + 2A\, q_1^2 f\,, \tag{15}$$

and using the definitions of S and G.

The system of equations (13) and (14) is not closed. Assume now that at time t_0 the flow becomes self-similar with a self-similarity scale λ_0. Then f and g are functions of r/λ_0 alone, modulo a possible dependence on the initial conditions at t_0. By the definition of the Taylor microscale, we have that λ_0/λ_g must be a *constant*. Then S and G are constant and equal to the values they have at t_0: $S = S_0$ and $G = G_0$. Now the system (13) and (14) is closed and we may study it.

We wish to study the stability of the constant solution $\varepsilon = \varepsilon_s$ and $K = K_s$. ε_s and K_s can be thought of as the stationary state values of ε and K. They are related by $\varepsilon_s = 2AK_s$. It will be convenient to define the quantity $g \equiv \frac{7G_0}{15}$. It is easy to check that $g > 1$. The constant g is related to the stationary state value of the Taylor-scale Reynolds number $\mathrm{Re}_\lambda = (\frac{20}{3}\mathrm{Re}_L)^{\frac{1}{2}}$:

$$\mathrm{Re}_{\lambda s} = \frac{30}{7|S_0|}\,(g-1)\,. \tag{16}$$

Define small fluctuations ξ and ζ of ε and K around their stationary values $\varepsilon = \varepsilon_s\,(1+\xi)$ and $K = K_s\,(1+\zeta)$. Inserting these expressions into (13) and (14) and keeping only linear terms we obtain the following system of equations:

$$\frac{d\xi}{dt} = -A(1+g)\,\xi + 2Ag\,\zeta\,, \quad \frac{d\zeta}{dt} = -2A\,\xi + 2A\,\zeta\,. \tag{17}$$

Its eigenvalues Γ read

$$\Gamma = \frac{1}{2}A\left(-(g-1) \pm \sqrt{(g-1)(g-9)}\right)\,. \tag{18}$$

By $g > 1$ we see that the real part of both eigenvalues is always negative. Fluctuations around the stationary state die out exponentially fast. That is, *modulo finite domain effects, the stationary state is stable as a complete self-preserving isotropic solution.* In other words all solutions will evolve to this state.

For $g < 9$ there is an imaginary part in the rates Γ. The relative size and the phase difference such solutions $\zeta = \zeta_0 e^{\Gamma t}$and $\xi = \xi_0 e^{\Gamma t}$, are given the relation $\xi_0 = \sqrt{g}\,e^{-i\phi}\,\zeta_0$ with $\tan\phi = (g+3)^{-1}\sqrt{(g-1)(9-g)}$. These relations and (18) describe small fluctuations. Arbitrary fluctuations are solutions of the full non-linear system (13) and (14) which must be solved numerically. It turns that the previous small fluctuation relations describe fairly well also the general features of the large fluctuations. For example, when $\mathrm{Re}_{\lambda s}$ is roughly in the neighborhood of 70 and higher, the large fluctuations stop evolving as damped oscillations and vanish exponentially without flickering.

We may now return to the issue we started our discussion with in this section. As mentioned, the deviations from isotropy reasonably originate from scales of order of the domain size l. The same can be said about the fluctuations around the stationary state. That is, one may attribute the generation of fluctuations to the interaction of the larger eddies with the periodicity i.e. the restriction to their size. Then, through both forcing and cascade, fluctuations are generated at all scales from l down to a certain scale where isotropy becomes a good approximation. There things are different. We may define correlators as spatial averages $\langle X\rangle_V$ over volumes V smaller than that maximum isotropic scale i.e. within these volumes turbulence is isotropic (meaning homogeneity as well) to a good approximation. Then K and ε understood as spatial averages $\langle X\rangle_V$ obey similar equations to those studied above. The entire analysis given above goes through. That finally means that at adequately small scales fluctuations decay and vanish indeed, as we hoped they did, although at all higher scales are maintained through forcing and cascade. The maximum isotropic scale should reasonably be related to the characteristic Taylor microscale of linear forcing $\lambda_A = (\nu/A)^{\frac{1}{2}}$, as below that scale dissipation becomes stronger to energy production.

DISCUSSION

The importance of the finiteness of the domain and its effects cannot be over-emphasized in the linearly forced turbulence. In a limited bandwidth forcing scheme, deterministic or stochastic, the inverse wave numbers at which one forces the flow imitate, very roughly, the scale of a physical stirring of an incompressible fluid existing in slightly larger 'box'. In linear forcing there is no such intrinsic scale. This simplifies things in some sense because there is no interaction between the forcing and domain size scales. On the other hand it is left entirely to the domain to set the large scales, becoming an essential part of the forcing itself. Also the large scale is introduced geometrically as a matter of size and not dynamically as in the bandwidth schemes, and there is no actual control over the extent forcing is consistent with isotropy. Turbulence is expected to behave quite differently under linear forcing than under a bandwidth scheme. There some additional interesting properties of linear forcing we have not yet commented on.

Consider the structure functions $\overline{(\Delta u_l)^2} = 2q_1^2(1-f)$ and $\overline{(\Delta u_l)^3} = 6q_1^3 h$, where Δu_l is the longitudinal velocity difference. In the inertial range of scales $\overline{(\Delta u_l)^2} = C_2(\varepsilon r)^{2/3}$, where C_2 a constant. Consider first decaying turbulence. It obeys power laws similar to (10), at least for the dimensionful quantities. The law for ε can be deduced. It is the $K-\varepsilon$ model equation (12) for $C_\varepsilon = \frac{3-2\gamma}{2-2\gamma}$ and of course $A = 0$. By the the Karman-Howarth equation we may show to show [27][28] that for *very high but finite* Reynolds numbers, and within the inertial range (more specifically as long as r/λ_g is a number of $O(1)$), we obtain corrections to the four-fifths law of the third order structure function:

$$\overline{(\Delta u_l)^3} = -\frac{4}{5}\varepsilon r\times \tag{19}$$

$$\times\left(1 - \frac{5}{17}C_\varepsilon C_2 \mathrm{Re}_\lambda^{-\frac{2}{3}}\left(\frac{r}{\lambda_g}\right)^{\frac{2}{3}} - \left(\frac{25}{3}\right)^{\frac{1}{3}} C_2 \mathrm{Re}_\lambda^{-\frac{2}{3}}\left(\frac{r}{\lambda_g}\right)^{-\frac{4}{3}}\right)\,.$$

Consider the same question in the linearly forced turbulence, working with the Karman-Howarth equation (15). One finds a result entirely similar to (19) upon replacing

$$C_\varepsilon \to -\frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt} + \frac{3AK}{\varepsilon} \,. \tag{20}$$

Observe now that if we think of the r.h.s. of this substitution as a constant, then we re-discover the model equation (12); the constant is what we denoted there by C_ε^A. Equation (12) is derived assuming self-similarity (self-preservation) of the larger scales of turbulence with respect to the integral scale L for high Reynolds numbers, in both the linearly forced ($A \neq 0$) and freely decaying case ($A = 0$). In all, by self-preservation we obtain a similar result of the form (19) in both kinds of turbulence, differing only in the value of the constants C_ε^A and C_ε. On the linearly forced side, self-preservation requires $C_\varepsilon^A = 3/2$ and equation (12) and (13) require that $L =$ constant.

Now if we require $C_\varepsilon^A = C_\varepsilon$ then $\gamma = 0$. I.e. if the decaying turbulence evolves according to $L \sim$ constant (and $K \sim t^{-2}$) then its structure function expression (19) is exactly similar to that of the linearly forced turbulence. That is, the correspondence between the two flows can be exact.

The $K \sim t^{-2}$ evolution is too fast compared to the usually observed decay laws. Such power laws can be reproduced if choose the constant C_ε to be different than 3/2, a fact regarded as an imperfection of the correspondence in the Ref. [12], where it was first pointed out. On the other hand the origin and the nature of the correspondence seem to have been overlooked in [12].

The key role is played again by the finiteness of the domain. As emphasized in section a container is a necessary thing when turbulence is linearly forced. Lacking an intrinsic length scale, linear forcing essentially requires a large scale to be provided by the boundary conditions. It is therefore not much of a surprise that similarities between linearly forced and freely decaying turbulence are more detailed when the decaying side evolves in a way consistent with the existence of a container: For adequately high Reynolds numbers that means $L \sim$ constant. Then the mathematics of self-similarity of turbulence with respect to the scale L imply exactly the same formula (19) for both kinds of turbulence.

The next obvious question is, what kind of modifications does linear forcing need in order to reproduce aspects of a generic decaying turbulence and a more general evolution law $L \sim t^\gamma$? Two immediate guesses are to consider a time-dependent rate $A = A(t)$ or, to consider a time-dependent box whose size l evolves according to $l \sim t^\gamma$. The analysis of such possibilities is left for future work.

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