

# ‘NON-EQUILIBRIUM’ TURBULENCE AND TWO-EQUATION MODELING

**Robert Rubinstein**  
Computational AeroSciences Branch  
NASA Langley Research Center  
Hampton, Virginia 23681 USA  
r.rubinstein@nasa.gov

## ABSTRACT

Two-equation turbulence models are analyzed from the perspective of spectral closure theories. Kolmogorov theory provides useful information for models, but it is limited to ‘equilibrium’ conditions in which the energy spectrum has relaxed to a steady state consistent with the forcing at large scales; it does not describe transient evolution between such states. Transient evolution is necessarily through ‘non-equilibrium’ states, which can only be found from a theory of turbulence evolution, such as one provided by a spectral closure. When the departure from equilibrium is small, perturbation theory can be used to approximate the evolution by a two-equation model. The perturbation theory also gives explicit conditions under which this model can be valid, and when it will fail. Implications of the non-equilibrium corrections for the classic Tennekes-Lumley balance in the dissipation rate equation are drawn: it is possible to establish both the cancellation of the leading order  $Re^{1/2}$  divergent contributions to vortex stretching and enstrophy destruction, and the existence of a nonzero difference which is finite in the limit of infinite Reynolds number.

## INTRODUCTION

The two-equation model is solved as part of many Reynolds averaged models, even in models that include Reynolds stresses and other more refined descriptions of the large-scale statistics. By itself, a two-equation model treats turbulent fluctuations as homogeneous isotropic turbulence. This simplification can be motivated by appealing to Kolmogorov theory, which supports the translation and rotation invariance of the statistics of fluctuations at small scales, regardless of the inhomogeneity and anisotropy of the large scales (Frisch, 1995). This paper analyzes the two-equation model as a description of the evolution of statistically non-stationary homogeneous isotropic turbulence. It poses the question of when two-equation modeling is possible, and when is it impossible. The answer is closely linked to the classic Tennekes and Lumley (1972) balance between vortex stretching and enstrophy destruction.

## NON-STATIONARY TURBULENCE

Kolmogorov theory implies that statistically stationary homogeneous isotropic turbulence is characterized by two parameters, equivalent to a length scale  $\ell$  and the dissipation rate  $\varepsilon$  (this choice of variables will be justified later). This idea is formulated in terms of the energy spectrum  $E(\kappa)$  as

$$E(\kappa) = E(\kappa; \ell, \varepsilon) \quad (1)$$

Such a state is sometimes called an ‘equilibrium,’ but it is more correct to call it simply a steady state, since thermodynamically, it is far from equilibrium. Eq. (1) includes a non-trivial idea from Kolmogorov’s theory, that because the dissipation rate equals the energy flux through the inertial range, it links the large and small scales; this justifies the characterization of the *entire* energy spectrum by a flux,  $\varepsilon$ , and a largest dynamically relevant scale,  $\ell$ .

But these considerations are purely ‘static;’ modeling attempts to treat temporal evolution under unsteady conditions. For the purposes of modeling, a non-stationary, or time-dependent energy spectrum cannot vary arbitrarily in time; for a finite dimensional model to be possible at all, it can only depend on time through finitely many parameters, which can reasonably be taken to be  $\ell$  and  $\varepsilon$  themselves. Then Eq. (1) is replaced by the time-dependent *ansatz*

$$E(\kappa, t) = E(\kappa; \ell(t), \varepsilon(t)) \quad (2)$$

This might be called a (temporally) *local* ‘equilibrium’ or steady state. This special solution structure, in which the time dependence occurs only through the time dependence of two parameters, can be compared to the *normal solution* of kinetic theory (Chapman and Cowling, 1970).

But how can evolution equations for  $\ell$  and  $\varepsilon$  be found? Kolmogorov theory is silent on this point. It states only that that starting from any state, under steady forcing, turbulence will evolve to an equilibrium steady state with  $\ell$  and  $\varepsilon$  determined by the forcing. Thus, Kolmogorov theory merely asserts the existence and analytical form of an *attractor* for the statistics of small scales of motion; it provides no information about the dynamics of reaching that state.

The usual approach to modeling postulates two equations: one is an exact energy balance, but the second is simply a fabrication. Briefly, it states that in a self-similar flow, since all quantities with the same units are proportional, the frequencies  $\dot{\varepsilon}/\varepsilon$ ,  $\varepsilon/k$ , and  $P/k$  ( $k$  is the turbulent kinetic energy and  $P$  is the rate of energy input) are constant multiples of each other, so that

$$\frac{\dot{\varepsilon}}{\varepsilon} = C_{\varepsilon 1} \frac{P}{k} - C_{\varepsilon 2} \frac{\varepsilon}{k} \quad (3)$$

The constants are determined by comparison with numerical or experimental data in self-similar flows, usually self-similar decaying turbulence and homogeneous shear flow, in which, to repeat, a model of this form is necessarily correct because of self-similarity (Clark, 1999). The extension of this equation to general flows assumes a ‘continuum mechanics’ paradigm, in which turbulence is viewed as a ‘fluid’ with the same ‘properties’ in every flow. However, this interesting paradigm is introduced without fundamental justification.

It is unnecessary to insist that this approach is scientifically unsatisfactory, regardless of any practical value it might be found to have, and indeed, this fact is well recognized in the modeling literature. Improvement requires equations of motion for  $E(\kappa, t)$ . It follows from the Navier-Stokes equations and the kinematic assumptions of homogeneity and isotropy that

$$\dot{E}(\kappa, t) = \Pi(\kappa, t) - \frac{\partial}{\partial \kappa} \mathcal{F}(\kappa, t) - 2\nu \kappa^2 E(\kappa, t) \quad (4)$$

where  $\Pi$  is a known, possibly time-dependent forcing (or ‘production’) spectrum, and where  $\mathcal{F}$  is the energy flux, a third order moment of the velocity field (Batchelor, 1953).

An energy balance is obtained by integrating Eq. (4) over all wavenumbers  $\kappa$ ,

$$\dot{k}(t) = P(t) - \varepsilon(t) \quad (5)$$

where total kinetic energy  $k(t) = \int_0^\infty d\kappa E(\kappa, t)$ , total production rate  $P(t) = \int_0^\infty d\kappa \Pi(\kappa, t)$ , and dissipation rate  $\varepsilon(t) = \int_0^\infty d\kappa 2\nu \kappa^2 E(\kappa, t)$ . The flux term integrates to zero because it is a gradient; recall that the gradient form reflects the conservative nature of energy transfer by nonlinear interaction. While this equation is attractive because it is exact, it must be emphasized that the nonlinearity embodied in the flux term  $\mathcal{F}$ , the heart of the turbulence problem, plays no role in this equation.

Similarly, a dissipation rate equation is obtained by multiplying Eq. (4) by  $2\nu \kappa^2$  and integrating,

$$\dot{\varepsilon} = S(t) - G(t) \quad (6)$$

where *vortex stretching*  $S$  is defined by

$$S(t) = - \int_0^\infty d\kappa 2\nu \kappa^2 \frac{\partial}{\partial \kappa} \mathcal{F}(\kappa, t)$$

$$= \int_0^\infty d\kappa 4\nu \kappa \mathcal{F}(\kappa, t) \quad (7)$$

and *enstrophy destruction* (or *palinstrophy*)  $G$  is defined by

$$G(t) = \int_0^\infty d\kappa 4\nu^2 \kappa^4 E(\kappa, t). \quad (8)$$

The integration by parts in Eq. (7) is justified by the plausible assumption of exponential decay of turbulence statistics when  $\kappa \rightarrow \infty$ , so that  $\lim_{\kappa \rightarrow \infty} 2\nu \kappa^2 \mathcal{F}(\kappa, t) = 0$ . Since production is assumed concentrated at large scales, the integral  $\int_0^\infty d\kappa 2\nu \kappa^2 \Pi(\kappa, t)$  is of order  $Re^{-1}$  and is ignored in Eq. (6) in the high Reynolds number limit. These results can also be obtained from the Kármán-Howarth equation (Batchelor, 1953).

## EQUILIBRIUM EVOLUTION

Important properties of an equilibrium state in which  $\mathcal{F}$  is constant and  $E \sim \kappa^{-5/3}$  in the inertial range follow because the integrals defining  $S$  and  $G$ , Eqs. (7) and (8), are dominated by contributions near the Kolmogorov scale  $\kappa_d \propto (\varepsilon/\nu^3)^{1/4}$  (Smith and Reynolds, 1991). Then

$$S \approx \nu \kappa_d^2 \propto \nu^{-1/2} \sim Re^{1/2} \quad (9)$$

and

$$G \approx \nu^2 \kappa_d^{10/3} \propto \nu^{-1/2} \sim Re^{1/2}, \quad (10)$$

the famous  $Re^{1/2}$  divergences identified by Tennekes and Lumley (1972). Since in an equilibrium state,  $\dot{\varepsilon} = S - G = 0$ , it follows that  $S = G$  also in a local equilibrium defined by Eq. (2). Therefore, *evolution through exactly local equilibrium states is inconsistent with evolution of  $\varepsilon$* : any attempt to derive an  $\varepsilon$  equation based on the assumption of a local equilibrium must be inconclusive.

Further consequences of equilibrium evolution follow from the Kolmogorov similarity form of the energy spectrum for steady state turbulence

$$E(\kappa; \ell, \varepsilon) = \varepsilon^{2/3} \ell^{5/3} \phi(\ell \kappa) \quad (11)$$

The function  $\phi$  is ‘universal’ insofar as it does not depend on  $\ell$  or  $\varepsilon$ , however, the particular analytical form of  $\phi$  only applies in one class of flows: forced steady state homogeneous isotropic turbulence. From

$$k = \varepsilon^{2/3} \ell^{2/3} \int_0^\infty dx \phi(x) \quad (12)$$

the Kolmogorov relation

$$\varepsilon = C_\varepsilon \frac{k^{3/2}}{\ell} \quad (13)$$

with

$$C_\varepsilon = \left( \int_0^\infty dx \phi(x) \right)^{-3/2} \quad (14)$$

follows. The constant  $C_\varepsilon$  is ‘universal’ in the restricted sense noted above: whereas Eq. (13) can apply more generally, for example to decaying turbulence, the dimensionless spectrum  $\phi$  can be flow-dependent, so that the constant  $C_\varepsilon$  can take different values in different flows (Bos *et al.*, 2007).

Similar considerations apply to the spectral eddy viscosity, which will be discussed further in the next section,

$$v_t = \int_0^\infty d\kappa \sqrt{\frac{E(\kappa)}{\kappa^3}} \quad (15)$$

Inserting the self-similar form Eq. (11),

$$v_t = \frac{\int_0^\infty dx \sqrt{\phi(x)/x^3} k^2}{\left( \int_0^\infty dx \phi(x) \right)^2} \frac{k^2}{\varepsilon} \quad (16)$$

The spectral eddy viscosity is therefore a constant multiple of the two-equation model viscosity  $k^2/\varepsilon$ ; the proportionality constant has a theoretical value that does not depend on  $\varepsilon$  and  $\ell$ , but that could depend on the flow class through the function  $\phi$  (Besnard *et al.*, 1996).

Because they refer to equilibrium conditions alone, these results do not depend on any particular expression for the flux  $\mathcal{F}$  in Eq. (4). This fact can account for their ubiquity in turbulence modeling.

## NON-EQUILIBRIUM EVOLUTION

While considerations based on Kolmogorov theory are very useful, they only carry us so far. Woodruff and Rubinstein (2006) advocated a new viewpoint on two-equation modeling, that the two-equation model should determine an approximate solution of the spectral evolution equation Eq. (4) under some closure assumption for the energy flux  $\mathcal{F}$ . For analytical simplicity, Woodruff and Rubinstein (2006) used the classical Heisenberg model (Batchelor, 1953), in which the energy flux  $\mathcal{F}$  is the functional of the energy spectrum,

$$\mathcal{F}[E(\kappa)] = C_h \int_0^\kappa d\mu \mu^2 E(\mu) \int_\kappa^\infty dp E(p) \theta(p) \quad (17)$$

with the ‘algebraic’ time-scale

$$\theta(p) = \frac{1}{\sqrt{p^3 E(p)}} \quad (18)$$

The Heisenberg model is written in the form of two equations, Eqs. (17) and (18) in order to emphasize the connection of this

model to more complete closure theories based on triad interactions (Kraichnan, 1987; Rubinstein and Clark, 2004). The Heisenberg model treats the energy flux at wavenumber  $\kappa$  as the product of a scale-dependent eddy viscosity at wavenumber  $\kappa$ , the second integral on the right side of Eq. (17), and the square of the effective strain at wavenumber  $\kappa$ , given by the first integral on the right side of Eq. (17).

The time-independent form of Eq. (4),

$$0 = P(\kappa) - \frac{\partial}{\partial \kappa} \mathcal{F}(\kappa) - 2\nu \kappa^2 E(\kappa) \quad (19)$$

has a steady solution consistent with the Kolmogorov  $\kappa^{-5/3}$  inertial range; then the constant  $C_h$  can be chosen to match the familiar Kolmogorov constant, treated as an empirical input. The existence of a Kolmogorov steady state solution establishes the *consistency* of the Heisenberg model with Kolmogorov theory, but the crucial fact is that the Heisenberg model is more general, because it states equations for the evolution of the energy spectrum without imposing Kolmogorov scaling in advance. From this viewpoint, spectral closures like the Heisenberg model stand in the same relation to Kolmogorov theory as the Boltzmann equation to equilibrium thermodynamics.

The forcing spectrum  $\Pi(\kappa, t)$  is assumed to be completely specified by a single length-scale  $L(t)$  such that  $\Pi(\kappa, t)$  is nonzero only near the wavenumber  $\pi/L$ , and by the total energy input rate  $P(t) = \int_0^\infty d\kappa \Pi(\kappa, t)$ . If  $P$  and  $L$  do not depend on time, then the corresponding solution of Eq. (19) is

$$E_{\text{eq}}(\kappa) = E(\kappa; \ell, \varepsilon) \quad \ell = L \quad \varepsilon = P \quad (20)$$

This connection between the energy spectrum and the forcing spectrum justifies the somewhat unconventional choice of  $\ell$  and  $\varepsilon$  as basic variables. It is obvious that if  $L = L(t)$  and  $P = P(t)$  depend on time, then the local equilibrium

$$E_{\text{loc}}(\kappa, t) = E(\kappa; \ell(t), \varepsilon(t)) \quad (21)$$

with

$$\ell(t) = L(t) \quad \varepsilon(t) = P(t) \quad (22)$$

satisfies

$$0 = P(\kappa, t) - \frac{\partial}{\partial \kappa} \mathcal{F}(\kappa, t) - 2\nu \kappa^2 E(\kappa, t) \quad (23)$$

because the flux function  $\mathcal{F}$  depends on time only through  $E$ . But the complete spectral evolution model Eq. (4) is not satisfied by  $E_{\text{loc}}(\kappa, t)$  because of the time derivatives on the left. Thus, *the spectral evolution equation cannot be satisfied by evolution through local equilibrium states*. If the time derivatives are simply ignored, then there is no need for a two-equation model, because the equalities in Eq. (22) state that

the spectrum adjusts instantaneously to the forcing. But if the time derivatives are small (this assumption can be made precise by perturbation theory: compare (Woodruff and Rubinstein, 2006)), then it is reasonable to construct an approximate solution of Eq. (4) by treating the local equilibrium as a leading order solution, say

$$E_0(\kappa; \ell(t), \varepsilon(t)) = E_{\text{loc}}(\kappa, t) \quad (24)$$

where analytical form in Eq. (21) is assumed, but Eq. (22) is not imposed. A perturbative solution of Eq. (4) is then sought in the form

$$E(\kappa, t) = E_0(\kappa; \ell(t), \varepsilon(t)) + E_1(\kappa, t) \quad (25)$$

where  $E_1 \ll E_0$  (again, this statement can be made precise in perturbation theory). This approach to solving the spectral evolution equation is motivated by the Chapman-Enskog and Hilbert expansions of kinetic theory (Chapman and Cowling, 1970).

The correction  $E_1$  satisfies the linear integral equation

$$\frac{\partial}{\partial \kappa} \mathcal{L}[E_1(\kappa)] = -\frac{\partial}{\partial t} E_0(\kappa; \ell(t), \varepsilon(t)) \quad (26)$$

where  $\mathcal{L}$  is the flux linearized about the zero order solution; for the Heisenberg model Eq. (17),

$$\begin{aligned} \mathcal{L}[E_1] = & C_h \int_0^\kappa d\mu \mu^2 E_1(\mu) \int_\kappa^\infty dp \sqrt{\frac{E_0(p)}{p^3}} \\ & + C_h \frac{1}{2} \int_0^\kappa d\mu \mu^2 E_0(\mu) \int_\kappa^\infty dp \frac{E_1(p)}{\sqrt{p^3 E_0(p)}} \end{aligned} \quad (27)$$

Thus,  $E_1$  cancels the error in Eq. (4) due to the time dependence of  $E_0$ .

The crux of the analysis of Woodruff and Rubinstein (2006) is that the possibility of solving Eq. (26) requires that certain compatibility conditions be satisfied, and these conditions provide the two-equation model for the unknown functions  $\ell(t)$  and  $\varepsilon(t)$ .

Assuming that the solution exists, it is clear from Eqs. (26) and (27) that the correction  $E_1$  has the structure

$$E_1(\kappa, t) = A(\kappa) \frac{\dot{\varepsilon}(t)}{\varepsilon(t)} + B(\kappa) \frac{\dot{\ell}(t)}{\ell(t)} \quad (28)$$

and Eq. (26) shows that in the inertial range where  $E_0 \propto \kappa^{-5/3}$ ,

$$E_1 \propto \kappa^{-7/3} \quad (29)$$

The existence of such a ‘non-equilibrium’ correction to the energy spectrum was first proposed by Yoshizawa (1994). Although it is very difficult to extract this contribution even in

closure calculations, by using a novel method to analyze the data, Horiuti and Ozawa (2011) have recently given conclusive evidence for its existence in direct numerical simulations.

This analysis demonstrates that in a regime described by a two-equation model, spectral evolution cannot be through exactly equilibrium states, and that non-equilibrium corrections for the time dependence necessarily arise. These corrections have a crucial role in answering the famous and fundamental question posed by Tennekes and Lumley (1972): given that  $S$  and  $G$  in Eq. (6) are shown in Eqs. (7) and (8) to diverge as  $\nu^{-1/2} \sim Re^{1/2}$ , how is it possible for  $\dot{\varepsilon} = S - G$  to be finite and nonzero as  $Re \rightarrow \infty$ ?

The statement that the difference between vortex stretching and enstrophy destruction is independent of Reynolds number expresses the crucial property that the evolution of the small scales depends entirely on the large scales; or, equivalently, that  $\varepsilon$  does indeed link the large and small scales, *even in a time-dependent turbulent flow*. This conclusion is absolutely critical to the possibility of modeling: it bears repeating that this conclusion does not follow from Kolmogorov theory.

$S$  and  $G$  can be written in terms of contributions from  $E_0$  and  $E_1$  as

$$S = S_0 + S_1 \quad G = G_0 + G_1 \quad (30)$$

but since in local equilibrium,  $S_0 = G_0$ ,

$$\dot{\varepsilon} = S - G = S_1 - G_1 \quad (31)$$

To evaluate  $S_1$ , note that the contribution to the flux from the correction  $E_1$  is

$$\mathcal{F}_1 = \mathcal{L}[E_1] \sim \kappa^{-2/3} \quad (32)$$

in the inertial range, where  $\mathcal{L}[E_1]$  is defined in Eq. (27); consequently,

$$S_1 \sim \int_0^\infty d\kappa \nu \kappa \mathcal{F}_1 \sim \nu \kappa_d^{4/3} \sim \varepsilon \sim Re^0 \quad (33)$$

Similarly,

$$G_1 \sim \int_0^\infty d\kappa \nu^2 \kappa^4 E_1 \sim \nu^2 \kappa_d^{8/3} \sim \varepsilon^2 \sim Re^0 \quad (34)$$

Thus,  $S_1 - G_1$  can be nonzero and finite in the high Reynolds number limit. This argument establishes the Tennekes-Lumley balance in a perturbation theory of slow spectral evolution.

The Tennekes-Lumley balance has a crucial role in the formulation of a dissipation rate equation: it states that the quantity that must be modeled is the unknown  $O(Re^0)$  difference between two correlations of order  $O(Re^{1/2})$ ; the difficulty is that this difference does not correspond to any definite correlation of flow quantities. Consequently, the program of ‘correlation by correlation’ modeling is not possible. The

result is the usual heuristic approach to the dissipation rate equation. The present analysis provides a possible explanation for this impasse: the dissipation rate equation arises as a first order effect in perturbation theory. Since it is therefore tightly linked to non-equilibrium effects, it is not accessible to derivations based on equilibrium considerations alone.

The time dependence also induces corrections to the equilibrium eddy viscosity and length scale relations in Eqs. (13) and (15). The correction to eddy viscosity has the form

$$v_{t1} = \frac{1}{2} \int_0^\infty d\kappa \frac{E_1(\kappa)}{\sqrt{\kappa^3 E_0(\kappa)}} = \frac{1}{2} \frac{\dot{\varepsilon}(t)}{\varepsilon(t)} \int_0^\infty d\kappa \frac{A(\kappa)}{\sqrt{\kappa^3 E_0(\kappa)}} + \frac{1}{2} \frac{\dot{\ell}(t)}{\ell(t)} \int_0^\infty d\kappa \frac{B(\kappa)}{\sqrt{\kappa^3 E_0(\kappa)}} \quad (35)$$

where Eq. (28) has been substituted for  $E_1$ . Non-equilibrium corrections to the eddy viscosity have been introduced by Yoshizawa and Nisizima (1993). However, their goal was to find a correction to the eddy viscosity in steady state turbulence that makes it appropriate for homogeneous shear flow as well. The present perspective is that the eddy viscosity coefficient in homogeneous shear flow will be different from that in steady state turbulence because the function  $\phi$  in Eq. (16) will not be the same. The non-equilibrium corrections in Eq. (35) are *small* corrections to the equilibrium eddy viscosity in weakly time-dependent homogeneous isotropic turbulence; they do not apply to any other class of turbulent flows.

The analogous corrections to Eq. (13) are non-equilibrium corrections to the Kolmogorov law  $\varepsilon \sim k^{3/2}/\ell$ . Attempts to derive a dissipation rate equation by manipulation of this equilibrium law are futile because the dissipation rate equation depends on non-equilibrium effects.

## LIMITATIONS OF TWO-EQUATION MODELS

This analysis demonstrates that two-equation modeling requires an explicit treatment of the departure of a turbulent flow from equilibrium conditions. But it also restricts the validity of models to *weak* departures from equilibrium, in which the ratios  $\dot{\varepsilon}/\varepsilon$  and  $\dot{\ell}/\ell$ , which quantify the departure from equilibrium, are ‘small’ in a sense made precise in the theory. An extreme case of violation of this restriction occurs if either  $\dot{\varepsilon}/\varepsilon$  or  $\dot{\ell}/\ell$  is large enough to cause  $E_0 + E_1$  to become negative. This possibility can be compared to ‘continuum breakdown’ (Bird, 1994) in kinetic theory, in which the Navier-Stokes equations become invalid because the Chapman-Enskog distribution is non-realizable. The possibility of formulating this limitation depends on a key feature of the present analysis, that it constructs an approximate energy spectrum, as well as equations of evolution for the turbulence descriptors  $\varepsilon$  and  $\ell$ .

It should be stressed that the explicit formulation of conditions under which the two-equation model can be valid is a particular strength of the present perturbative approach; nothing comparable is possible if the model is simply postulated by the usual heuristic arguments.

Weak departure from equilibrium imposes relations between large and small scales: they are linked to leading order in perturbation theory through the dissipation rate  $\varepsilon$  as if

the flow were steady, and the evolution is therefore such that the small scale evolution is basically determined completely by the large scales. This relation is sometimes expressed by saying that the small scales are ‘slaved’ to the large scales; however, from the perspective of closure theories, this relation between the small and large scales is very special, not a general feature of all turbulent flows.

Of course, the two-equation model can always be solved, even when it is not valid. In such cases, the relation between small and large scales that it imposes can make the evolution unrealistic. An example occurs in the problem of homogeneous isotropic turbulence with linearly increasing production rate (Rubinstein *et al.*, 2004), in which turbulence evolves from a Kolmogorov steady state to a self-similarly growing state. A two-equation model consistent with both the initial steady state and with the state of self-similar growth misrepresents the evolution of the dissipation rate: whereas closure and direct numerical simulation data suggest a delay in the growth of  $\varepsilon$  followed by a very rapid rise, two-equation models predict a gradual growth of  $\varepsilon$  throughout, as if the true dynamics were subjected to a low-pass filter. This ‘filtering’ effect arises because the two-equation model links the evolution of the dissipation rate to large scales at all times.

During the initial transient, the large scales are evolving more rapidly than the small scales; but when the small scales finally start to respond to the large scales, they do so on their own fast time scales, not on the time scales of the large scale motion. In both cases, the link between large and small scales is broken, and the the energy spectrum simply does not admit the two-parameter *ansatz* of Eq. (2). A more familiar example of breaking the link between large and small scales is the rise in dissipation rate during the initial transient of decaying turbulence, before the Kolmogorov spectrum is formed. In this problem, there is no ‘production’ mechanism at large scales to cause the dissipation rate to increase; the small scales evolve independently of the large scales, and consequently, the two-equation model cannot predict the evolution. Compare also the analysis of front propagation in transient turbulent evolution by Connaughton and Nazarenko (2004).

The breakdown of the two-equation model in these problems does not imply that a statistical approach is impossible: as noted earlier, spectral closure theories do not make any special assumptions about Kolmogorov scaling, and are therefore candidates to analyze flows in which a Kolmogorov spectrum does not exist.

## EXTENSIONS OF THE THEORY

The term ‘equilibrium’ is used in turbulence modeling very generally, to describe self-similar states, even when they are time-dependent, as in decaying turbulence and in homogeneous shear. Another important feature of the present analysis is that perturbations of each such ‘equilibrium’ state requires a separate analysis: there is no assertion that a model developed for departures from steady-state turbulence will correctly describe turbulent decay. A possible topic for future research is the development along the present lines of a model for perturbations of self-similar turbulent boundary layers.

## CONCLUSIONS

Time-dependent evolution through equilibrium states is not possible; any description of statistically unsteady dynamics requires some theory to quantify the departures from equilibrium. When the time dependence is sufficiently slow, perturbation theory can be used to derive a two-equation model which provides an approximate description of the evolution. The perturbation theory also indicates the limitations of the two-equation model. Completely general transient evolution can only be predicted by solving a complete spectral evolution equation. The use in this analysis of the Heisenberg model was entirely for analytical convenience; a complete perturbative development of turbulence models must be based on more accurate closures that include the dynamics of triad interactions in turbulence (Kraichnan, 1959).

## REFERENCES

- Batchelor, G. K., 1953, *The theory of homogeneous turbulence*, Cambridge University Press, Cambridge UK.
- Besnard, D. C., Harlow, F. H., Rauen Zahn, R. M., and Zemach, C., 1996, "Spectral transport model for turbulence," *Theoretical and Computational Fluid Dynamics*, Vol. 8, p. 1.
- Bird, G. A., 1994, *Molecular gas dynamics and the direct simulation of gas flows*, Oxford University Press, Oxford.
- Bos, W. J. T., Shao, L., and Bertoglio, J.-P., 2007, "Spectral imbalance and the normalized dissipation rate of turbulence," *Physics of Fluids*, Vol. 19, paper 045101.
- Chapman, S., and Cowling, T. G., 1970, *The mathematical theory of non-uniform gases*, Cambridge University Press, Cambridge UK.
- Clark, T. T., 1999, "Two-point closures and statistical equilibrium," in *Modeling Complex Turbulent Flows*, Salas, M. D., Hefner, J. N., and Sakell, L. (editors), Kluwer Academic Publishers, Dordrecht.
- Connaughton, C., and Nazarenko, S., 2004, "Warm cascades and anomalous scaling in a diffusion model of turbulence," *Physical Review Letters*, Volume 92, paper 044501.
- Frisch, U., 1995, *Turbulence: The legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge UK.
- Horiuti, K., and Ozawa, T., 2011, "Multimode stretched spiral vortex and non-equilibrium energy spectrum in homogeneous shear flow turbulence," *Physics of Fluids*, Vol 23, paper 035107.
- Kraichnan, R. H., 1959, "The structure of isotropic turbulence at very high Reynolds numbers," *Journal of Fluid Mechanics*, Vol. 5, p. 497.
- Kraichnan, R. H., 1987, "An interpretation of the Yaglom-Orszag turbulence theory," *Physics of Fluids*, Vol. 30, p. 2400.
- Rubinstein, R., Clark, T. T., Livescu, D., and Luo, L. S., 2004, "Time-dependent isotropic turbulence," *Journal of Turbulence*, Vol. 5, N11.
- Rubinstein, R. and Clark, T. T., 2004, "A generalized Heisenberg model for turbulent spectral dynamics," *Theoretical and Computational Fluid Dynamics*, Vol. 17, p. 249.
- Smith, L. M., and Reynolds, W. C., 1991, "The dissipation-range spectrum and the velocity-derivative skewness in turbulent flows," *Physics of Fluids A*, Vol 3, p. 992.
- Tennekes, H., and Lumley, J. L., 1972, *A first course in turbulence*, MIT Press, Cambridge MA.
- Woodruff, S. L., and Rubinstein, R., 2006, 'Multiple-scale perturbation analysis of slowly evolving turbulence,' *Journal of Fluid Mechanics*, Vol. 565, p. 93.
- Yoshizawa, A., 1994, "Non-equilibrium effect of the turbulent energy production process on the inertial range energy spectrum," *Physical Review E*, Vol. 49, p. 4065.
- Yoshizawa, A., and Nisizima, S., 1993, "A non-equilibrium representation of the turbulent viscosity based on a two-scale turbulence theory," *Physics of Fluids A*, Vol. 5, p. 3302.