ABSTRACT
The interaction between a turbulent supersonic boundary layer and an impinging shock wave is investigated numerically and analytically. The reflected-shock low-frequency motions are well captured even when using a narrow simulation domain, supporting the argument that one underlying key mechanism for the low-frequency shock motions is two-dimensional. Based on a two-dimensional approach, a stochastic ordinary differential equation for the low-frequency coupling between the reflected shock and the boundary layer is obtained. The system is closed and applied to a wide range of input parameters. It is argued that the low-frequency shock motions are not necessarily a property of the forcing, either from upstream or downstream of the shock, but are simply an intrinsic property of the coupled dynamical system.

INTRODUCTION

The physical mechanisms at the origin of the observed low-frequency shock motions in shock wave/turbulent boundary layer interactions (SBLI) are not fully understood. A number of tentative explanations have been proposed, usually falling into one of two categories: the first relates the low-frequency motions to specific events or flow structures from the upstream turbulent boundary layer, whereas the second looks for causal mechanisms within the interaction itself (i.e. downstream of the shock). In both cases, the difficulty resides in identifying a mechanism that can span timescales of the order of $10^3 \delta_0/\bar{u}_1$ to $10^5 \delta_0/\bar{u}_1$, where $\delta_0$ is the upstream boundary-layer 99% thickness and $\bar{u}_1$ the upstream freestream velocity.

The variety of the mechanisms proposed in the literature, together with the subsequent debate about the merits of one approach relative to another is symptomatic of the difficulty one has in identifying and then separating individual events from a (supposedly) non-linear (chaotic) system, where actual causal events may well be impossible to detect. Instead of reasoning about the relevance of one assumed mechanism against numerical/experimental data, an attempt to characterise in a useful way the properties of the dynamical system arising from the coupling between the shock and the boundary layer is sought.

The paper is organised as follows. The next section highlights the main steps for the derivation of a low-order model for the shock-foot low-frequency motions. Next, some implications of the model and its sensitivity to modelling errors are discussed.

A STOCHASTIC LOW-ORDER MODEL FOR THE SHOCK-FOOT MOTIONS

The momentum integral equation

Starting from the Navier–Stokes equations, and upon integrating the streamwise component of the momentum equation in the wall-normal direction (denoted $y$), one can derive a general form of the Momentum Integral Equation (MIE) where none of the classical assumptions (e.g. constant pressure in the wall-normal direction, steady state ...) are used. The resulting MIE is then expressed in the following moving coordinate system:

$$\xi \equiv \frac{x + l_0 - \varepsilon}{l_0 - \varepsilon + s}$$

where all notations are described in figure 1. Hence, in what follows, $\xi = 0$ is the instantaneous shock-foot position, $\varepsilon$ the shock-foot displacement with respect to its mean position and $\xi = 1$ the instantaneous location of the shock crossing. Note that due to the presence of the boundary layer, the shock does not reach the wall and the foot is defined as the linear extension of the shock to the wall.

The following assumptions are made to simplify the
where \( \rho \) is the fluid density, \( u \) the streamwise velocity component and \( p \) the pressure. Finally, \( C_f \) is the skin friction (i.e. \( 2\mu_w/(\rho u^2) \)) describing the streamwise evolution of the various boundary-layer thicknesses independent of the time variable, i.e.:

\[
F(\xi) = \frac{\delta_1(\xi) - \delta_1(\xi = 0)}{\Delta_i(t)} = \frac{\delta_i(\xi = 1) - \delta_i(\xi = 0)}{\Delta_i(t)}
\]  

where the subscript \( i \) is any of the following: 1, 2, \( \rho \), \( \rho \).

Mathematically, hypothesis 1 corresponds to the supposed existence of a separation of variables, an assumption which is
reasonably well supported by simulation results (see Touber and Sandham, 2011). The MIE becomes:

\[
\frac{1}{u_1 t_0} \left[ (1 - \xi) \frac{d \xi}{dt} + \frac{\xi}{l_0} \frac{d l}{d \xi} \right] (F' \Delta \rho - F' \Delta l) + \frac{1}{l_0} F' \Delta_2 + \frac{p_1}{\rho_1 \sqrt{2} \sqrt{l_0}} F' \Delta_\rho = \frac{1}{2} \left( \frac{1 - e}{l_0} + \frac{\theta}{l_0} \right) C_f \tag{8}
\]

with \( F' \equiv \frac{d F}{d \xi} \).

Next, (8) is evaluated at the shock foot (\( \xi = 0 \)). Functions \( s \), \( \Delta \), and the slow-varying part of \( C_f(\xi = 0) \) are expressed in terms of linear functions of \( \varepsilon \). By retaining the leading-order terms only, a first-order stochastic ordinary differential equation (ODE) for the reflected-shock-foot motions is obtained (please see Touber and Sandham, 2011, for details), resembling the equation proposed by Plotkin (1975), who postulated that the shock displacement was obeying a first-order stochastic ODE with an associated characteristic timescale, which needs to be determined \textit{a posteriori} from existing data. In the present work, an expression for the characteristic timescale is readily available.

Closed form of the system

After some modelling efforts, combined with series expansions of the shock-jump relations, the system is closed so that the following dynamical equation for the shock-foot motions may be written (see Touber and Sandham, 2011):

\[
\frac{1}{u_1} \frac{d \xi}{dt} + \phi \frac{e}{L} = \Pi C_f'(t) \tag{9}
\]

with:

\[
\Pi = \frac{\tan \beta}{2 F'(t) (\tan \alpha + \tan \beta)} \tag{10}
\]

\[
\phi = \frac{2 \gamma + \gamma (\gamma - 1) M_f^2}{(\gamma + 1) [1 + (1 - r) P_2 - r P_3]} \left\{ \Pi \left[ \frac{1}{\tan \alpha} + \frac{1}{\tan \beta} \right] \right\}
\]

\[
\left( C_{f_0} - \Lambda \right) + \frac{C_{f_0} \tan \alpha}{\tan \beta} + \left[ 1 - \tan \alpha \tan \beta \right] \left( \frac{r \gamma M_f^2 C_{f_0}}{P_2 - 1} - r' D - \frac{P_3}{\gamma + 1} \left( \frac{M_2}{M_1} \right)^2 \right) \tag{11}
\]

\[
\kappa = \frac{\tan \alpha + \tan \beta}{\tan \beta (1 - \frac{1}{\tan \alpha})} \sin (2 \alpha) \sin [2 (\alpha + \theta)] \tag{12}
\]

\[
D = \frac{M_3}{M_1} \left\{ \left( \frac{1}{\sqrt{R_3 P_2}} \frac{M_3}{M_1} \right) + \frac{1}{\sqrt{R_2 P_3}} A \right\} + \frac{M_1}{M_3} \sqrt{R_3 P_2 - 2 P_3} C \tag{13}
\]

\[
A = \frac{\gamma M_f^2}{1 + \gamma M_f^2} P_2 \tag{14}
\]

\[
B = k R_3 \left\{ \frac{1}{2 \sin^2 (\alpha + \theta)} \right\} \tag{15}
\]

\[
C = \frac{M_3}{M_1} \left\{ \kappa \left[ \frac{(\gamma - 1) M_2^2}{8 + 4 (\gamma - 1) M_2^2 \sin^2 (\alpha + \theta)} - \frac{\gamma M_2^2}{2 (1 - \gamma) + 4 \gamma M_2^2 \sin^2 (\alpha + \theta)} \right] - \left( \tan \alpha + \tan \beta \right) \cos^2 \alpha \right\} \frac{1}{\tan \beta (1 - \frac{1}{\tan \alpha}) - 1} \tag{16}
\]

where \( \alpha, \beta, P_2 \equiv \frac{p_2}{p_1}, P_3 \equiv \frac{p_3}{p_1}, R_3 \equiv \frac{\rho_3}{\rho_1}, M_2 \) and \( \tilde{M}_3 \) are computed from the inviscid shock reflection problem for a given pair of wedge angle \( \theta \) and upstream Mach number \( M_1 \). Factors \( F'(0), r, r' \) and \( r'' \) are assumed to take the values of 0.12, 0.2, -0.14 and 0.2, respectively. Term \( \tilde{C}_{f_0} \) is an input parameter, together with the upstream Mach number \( M_1 \) and wedge angle \( \theta \). The coefficient \( \Lambda \), although of the same order as \( \tilde{C}_{f_0} \), is not an input parameter and is not generally known. In this work, it is taken to be \( 3 \times 10^{-3} \) (based on simulation data). The term \( C_{f_0} \) corresponds to the skin-friction turbulence-related variations at the reflected-shock foot and therefore constitutes the dynamical-system input signal.

Equation (9) is a first-order linear stochastic differential equation resembling the Langevin equation for Brownian motion. It is possible to show that the autocorrelation function of the shock-foot motions in response to a white-noise forcing with amplitude \( 2 \phi \) is an exponential if computed after the initial transients from starting up the flow (see Touber and Sandham, 2011, for details). Therefore, the Power Spectral Density function (PSD), which is the Fourier transform of the autocorrelation function, may be explicitly written:

\[
\mathcal{S}(S_t) = \frac{A_0}{1 + (S_t/\phi_{\text{max}})^2} \tag{17}
\]

where \( A_0 \equiv \frac{q (L/(u_1 \phi))}{\phi_{\text{max}}^2} \), \( \phi_{\text{max}} \equiv \theta / (2 \pi) \) and \( S_t \) is the Strouhal number \( (S_t = f L/u_1) \). The PSD being more easily given for the wall pressure, the above expression which is valid for the shock-foot motions may be converted to wall pressure at the shock foot using:

\[
\mathcal{S}_p(S_t) \approx \frac{A_0 (\frac{d p_w}{d x}|_{x_0})^2}{1 + (S_t/\phi_{\text{max}})^2} \tag{18}
\]

where \( d p_w/d x |_{x_0} \) is the mean wall-pressure gradient at the mean shock-foot position.

**IMPLICATIONS AND SENSITIVITY OF THE MODEL**

First, the model is tested against both the experimental and simulation results for the 8-degree shock-reflection configuration of Dupont et al. (2006), as shown in figure 2. Both
A constant upstream Mach number, the fundamental property of the shock/boundary-layer system indicated by (18) is its low-pass filter behaviour. As shock foot location are reasonably well predicted.

The broadband nature and the frequency of the most energetic low-frequency pressure fluctuations in the vicinity of the shock foot location are reasonably well predicted.

The fundamental property of the shock/boundary-layer system indicated by (18) is its low-pass filter behaviour. As such, a transfer of energy from higher (turbulence-related) to lower frequencies is not required to explain the spectra in figure 2. Instead, the shock/boundary-layer system simply damps fluctuations greater than the cutoff frequency $\phi_{\text{max}}$, while any existing fluctuations smaller than this cutoff frequency are amplified.

The use of white noise to force (9) may not seem ideal, as this is not, a priori, representative of turbulence fluctuations. However, it is argued that at low frequencies, skin-friction fluctuations solely associated with the contribution from the turbulence resembles that of a white noise, i.e. the spectrum of $\phi_{\text{max}}$ is “flat” at low frequencies, where (9) is to be applied.

One great advantage of the model is the possibility to use it for any given values of $M_1$ and $\theta$. For a constant wedge angle, $\phi_{\text{max}}$ increases with increasing Mach number and for a constant upstream Mach number, $\phi_{\text{max}}$ decreases with increasing wedge angle. The latter trend can be tested against the experimental results of Dupont et al. (2006), as shown in figure 3. The agreement is well within the model and measurement uncertainties. Figure 4a shows the map of $\phi_{\text{max}}$ for $M_1$ ranging from 1 to 6 and $\theta$ from $2^\circ$ to $30^\circ$, whenever a regular reflection exists. Most values are within the range $10^{-2}$ to $10^{-1}$, which is consistent with the experimental observations of SBLI (see Dussauge et al., 2006).

Although the final model is described by a linear equation, it does not mean that none of the non-linearities of the coupled shock/boundary-layer system are accounted for. Significant non-linear effects are mechanically embedded in the timescale $\phi^{-1}$. Looking at the constituents of $\phi$, one can see that even if the model is expressed in the form of point-particle dynamics (i.e. the shock-foot position), it does not convey a direct relation between a given velocity fluctuation and the shock response to it, as linearised Euler would do (the resulting spectrum would then be similar to the forcing), but instead it accounts for integrated effects by means of the different thicknesses which are non-linear functions of the velocity perturbations.

As discussed earlier, the model describes the coupled shock/boundary-layer system as a low-pass filter with characteristic timescale $\tau_s \sim \phi^{-1}$. One remarkable result is that this timescale is significantly larger than any characteristic timescales of the incoming boundary layer $(\phi/(2\tau_s))$ is in the $10^{-2}$ to $10^{-1}$ range giving $\tau_s \sim 10$ to $100L/\bar{u}_1$, to compare with $\bar{\delta}_0/\bar{u}_1 \sim L/\bar{u}_1$, assuming that the interaction length scales with $\bar{\delta}_0$). This conforms to experimental observations (e.g. Dupont et al., 2006), and the known issue in numerical simulations that such flows have long initial transients, even for laminar cases (indeed, in the absence of forcing, the convergence to the steady solution would be as exp $(-t/\tau_s)$).

The low-pass filtering property of the system indicates that, strictly speaking no transfer of energy from the higher to the lower frequencies is occurring. Instead, any high frequency is damped and any low frequency is amplified, with the frontier between high and low being determined by $\phi$. Therefore, the system itself is simply amplifying existing low-frequency fluctuations, even if energetically insignificant, while it filters out any high-frequency content. Moreover,
the resulting broadband spectrum about a particular Strouhal number is not a property of the forcing but a characteristic of the shock/boundary-layer system itself (figure 2).

Based on the preceding discussion, it is inferred that the origin of the low-frequency oscillations is not in the forcing but in the dynamics of the system formed by the shock/boundary-layer interaction. Of course, if one applies any specific forcing below the natural frequency of the system, such forcing will be picked up and magnified. A specific forcing could be any significantly-long upstream coherent structures (see Ganapathisubramani et al., 2007, and references therein) or particular flow features within the interaction itself (see Dussauge and Piponniau, 2008; Piponniau et al., 2009; Pirozzoli and Grasso, 2006, and references therein). However, we stress that, mathematically speaking, these are not necessary and the low-frequency motions can simply arise from a background (white) noise, as successfully demonstrated in figure 2.

The robustness of the model to modelling errors is partially investigated in figure 4. First, the sensitivity of the model to the mean boundary-layer properties is weak for $\hat{C}_{f1}$ and insignificant for $r''$, suggesting that the map in figure 4a is a good estimate for other mean boundary-layer properties (as long as the hypotheses used to derive the model hold). The mean boundary-layer properties thus play a major role in setting the interaction length (see the steady-state equation in Touber and Sandham, 2011) but their effect on the final dynamical equation is only weak. Second, the accuracy of the model for $\Delta_2$ and to a lesser extent for $\Delta_p$ is crucial. While $r$ can be easily determined to a relatively good accuracy, $r'$ is the most critical aspect of the present model and further improvements could be sought in the future. Nevertheless, the overall monotonicity of the map of $\phi_{\text{max}}$ and the order of magnitude of the predicted $\phi_{\text{max}}$ are maintained even for these sensitive cases. This demonstrates that the Strouhal-number value for the most energetic low-frequency shock motions is robust with values remaining below 0.1 for a wide range of configurations, as argued by Dussauge et al. (2006).

Finally, it is important to bear in mind that the model is based on an approximate form of the momentum integral
The wedge angle was found to increase with increasing $M_1$ in Dussauge et al. (2006). The most energetic Strouhal number $S_	heta$ of the Strouhal number with the shock crossing point located above the incoming (i.e. the spanwise wrinkling of the shock was not considered). The derivation assumes two-dimensional motions what was initially a partial differential equation into an ordinary one. The derivation assumes two-dimensional motions (i.e. the spanwise wrinkling of the shock was not considered) with the shock crossing point located above the incoming boundary-layer height $\delta_0$. Under such conditions, a governing equation for the shock-foot motions is obtained and linearised on the basis of sufficiently small shock displacements combined with the analysis of LES data. This final form of the governing equation is mathematically identical to the one postulated by Plotkin (1975), and capable of reproducing the wall-pressure low-frequency spectrum in the vicinity of the mean shock-foot position.

Upon modelling the constituents of the derived governing equation, the dynamical system can be closed and expressed in terms of its input parameters: the upstream Mach number $M_1$, the wedge angle $\theta$ and the upstream boundary-layer properties (i.e. skin friction and momentum thickness). Although the upstream boundary-layer properties are found to be important at setting up the interaction length, the dynamical system is mainly controlled by $M_1$ and $\theta$. A wide range of input ($M_1$, $\theta$) pairs was tested and the predicted most energetically significant low-frequency motions, expressed in the form of the Strouhal number $S_	heta$, were shown to remain in the range 0.01 to 0.1, confirming the experimental evidence collected in Dussauge et al. (2006). The most energetic Strouhal number was found to increase with increasing $M_1$ for a constant wedge angle $\theta$, whereas it decreased with increasing wedge angle for constant $M_1$.

Mathematically speaking, the derived governing equation corresponds to a first-order low-pass filter and the analytical spectrum derived from forcing the system with white noise is in excellent agreement with the available experimental and numerical spectra. This result is consistent with the findings of Plotkin (1975); Poggie and Smits (2001, 2005) and leads to the suggestion that the low-frequency motions observed in SBLI need not be a characteristic of the forcing but simply the result of the low-pass filtering property of the dynamical system formed by the coupling between the boundary layer and the reflected shock, as demonstrated by the white-noise forcing. This does not mean that specific forcing from upstream (see Ganapathisubramani et al., 2007; amongst others) or downstream (see Pirozzoli and Grasso, 2006; Piponniau et al., 2009; Robinet, 2007; Touber and Sandham, 2009) does not play a role, but that they are not necessary. Obviously, if present and acting below the system cutoff frequency, they will inevitably be picked up by the system.

Further improvements to the proposed model are clearly possible and could be considered in the future: include spanwise shock wrinkling, derive better models for $\Delta_\theta$, extend the derivations to compression ramps and/or hot/cold walls.

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