ON HELICAL PROPERTIES OF HOMOGENEOUS TURBULENCE

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ABSTRACT

The helical properties of five prototypical turbulent flows are investigated: statistically steady forced isotropic turbulence, growing sheared turbulence, growing rotating sheared turbulence with rotation ratio $f/S = +0.5$, decaying rotating sheared turbulence with $f/S = +5$, and decaying rotating turbulence. These five turbulent flows were originally studied using direct numerical simulations. It was found that flows with growing turbulent kinetic energy and turbulent motion at large scales show a maximum in the velocity helicity probability distribution functions (PDFs) at $h_u = 0$, corresponding to a trend to two-dimensionalization of the flow with vorticity and velocity being perpendicular. Flows with decaying turbulent kinetic energy and turbulent motion at small scales, however, show a maximum in the velocity helicity PDFs at $h_u = \pm 1$, indicating a preference for helical motion with alignment or anti-alignment of vorticity and velocity. The PDFs of vorticity helicity h_ω always assume a maximum at $h_\omega = \pm 1$ for all cases. Joint PDFs of relative velocity helicity and relative vorticity helicity show that h_u and h_ω tend to have the same sign for all flows considered here, indicating that vorticity helicity diminishes velocity helicity.

INTRODUCTION

Helicity $H_u = \boldsymbol{u} \cdot \boldsymbol{\omega}$ is defined as the scalar product of velocity $\boldsymbol{u}$ and vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and it is an important topological quantity to characterize turbulence. Relative helicity $h_u = H_u/(|\boldsymbol{u}||\boldsymbol{\omega}|)$ describes the cosine of the angle between velocity and vorticity vectors. It allows to distinguish between helical structures (swirling motion) and non-helical structures. For helical structures, h_u has values of ± 1, which correspond to alignment or anti-alignment of vorticity and velocity, respectively. For non-helical structures, two-dimensionalization of the flow occurs. When vorticity is perpendicular to velocity, the velocity helicity h_u assumes a value $h_u = 0$.

Helicity is observed in atmospheric flows and it is known to play an important role in the evolution of tornadoes (Moffatt and Tsinober, 1992). Helicity is also of importance in the magneto-hydrodynamics of conducting fluids, in particular for the dynamo effect (Pouquet et al., 1976). Helicity furthermore plays a crucial role in the problem of relaxation to magnetostatic equilibrium. This is a problem of central importance in the context of thermonuclear fusion plasmas (Moffatt and Tsinober, 1992). Historically, helicity was first introduced by Betchov (1961), Moreau (1961), and Moffatt

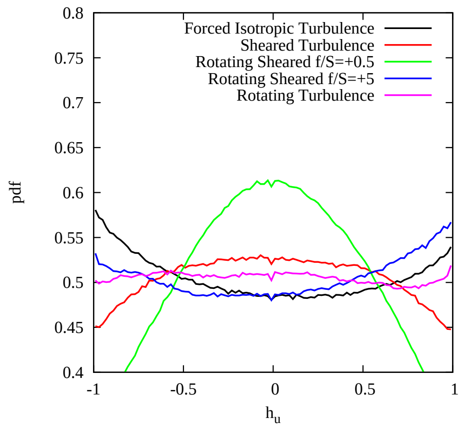

Figure 1. PDFs of relative velocity helicity h_u for the different turbulent flows.

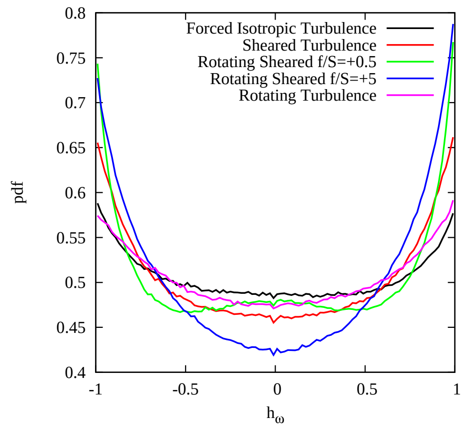

Figure 2. PDFs of relative vorticity helicity h_ω for the different turbulent flows.

(1969). Helicity allows a topological interpretation of the linkages of vortex lines present in the flow. The mean helicity $\langle H_u \rangle = \int H_u d^3x$ vanishes if the flow possesses a mirror symmetry property. Large helicity leads to the reduction of nonlinearity and consequently to the reduction of dissipation. The helical properties of isotropic turbulence and their relevance to vortical structures have been examined in the past. A comprehensive review on helicity can be found in Moffatt and Tsinober (1992) as well as in Sagaut and Cambon (2008).

Just as energy, the mean velocity helicity $\langle H_u \rangle$ satisfies a balance equation:

$$\frac{d}{dt}\langle H_u \rangle = -2\nu \langle H_\omega \rangle + \langle F \rangle \tag{1}$$

Here, $H_\omega = \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{\omega})$ is the vorticity helicity, $F = 2\boldsymbol{f} \cdot \boldsymbol{\omega}$ accounts for the forcing term $\boldsymbol{f}$ in the momentum equation, and ν is the kinematic viscosity of the fluid. Contrary to energy, neither velocity helicity H_u nor vorticity helicity H_ω are positive definite quantities. Therefore, the term involving vorticity helicity in equation (1) can only be interpreted as a velocity helicity dissipation term, if $\langle H_u \rangle$ and $\langle H_\omega \rangle$ have the same sign. Considering isotropic turbulence, Sanada (1993) conjectured that $\langle H_u \rangle$ and $\langle H_\omega \rangle$ indeed have the same sign and thus the vorticity helicity term acts as a dissipative mechanism for velocity helicity. Further evidence supporting Sanada's conjecture was given more recently by Galanti and Tsinober (2006) for isotropic turbulence with helical or non-helical forcing.

In the present study, the helicity properties of five flows are investigated: statistically steady forced isotropic turbulence (Vincent and Meneguzzi, 1991), growing sheared turbulence, growing rotating sheared turbulence with rotation ratio $f/S = +0.5$, decaying rotating sheared turbulence with $f/S = +5$ (Jacobitz et al., 2008; Jacobitz et al., 2010), and decaying rotating turbulence (Liechtenstein et al., 2005). All these flows were studied using direct numerical simulations and details can be found in the respective publications. These flows initially do not contain mean helicity and they remain free from it. However, this does not concern the local helicity and regions with strong helicity can exist in a flow free from mean helicity.

The purpose of this study is to answer three questions: First, what are the helical properties of the five turbulent flows and are the helical properties related to the fate, growth or decay, of the turbulence? Second, do helical properties vary with the scale of the turbulent motion? This question is addressed using a wavelet-based decomposition of the turbulent motion into different scales as proposed in Yoshimatsu et al. (2009). Third, does vorticity helicity h_ω act to diminish velocity helicity h_u?

RESULTS

In this section, the helical properties of statistically steady forced isotropic turbulence, growing sheared turbulence, growing rotating sheared turbulence with rotation ratio $f/S = +0.5$, decaying rotating sheared turbulence with $f/S = +5$, and decaying rotating turbulence are presented first. An overview of the different flows is given in table 1. All flows were studied using direct numerical simulations based on a Fourier-pseudospectral method at a resolution of 256^3 grid points. Then, a wavelet-based scale-dependent analysis considers helicity at different scales of turbulent motion. Finally, the role of vorticity helicity as a dissipative mechanism for velocity helicity is investigated.

Helical Properties of the Flows

Figure 1 shows the probability distribution functions (PDFs; estimated from a histogram with 100 equidistant bins) of relative velocity helicity h_u for the five flows. The two cases with growing turbulent kinetic energy, sheared turbulence and rotating sheared turbulence with $f/S = +0.5$, exhibit a maximum at $h_u = 0$, corresponding to a preference for velocity and vorticity to be perpendicular, i.e., two-dimensionalization of the flows. The two cases of statistically steady forced isotropic turbulence and decaying rotating sheared turbulence with $f/S = +5$ are characterized by maxima at $h_u = \pm 1$, corresponding to a preference for alignment or anti-alignment of velocity and vorticity, i.e., helical motion. The decaying rotat-

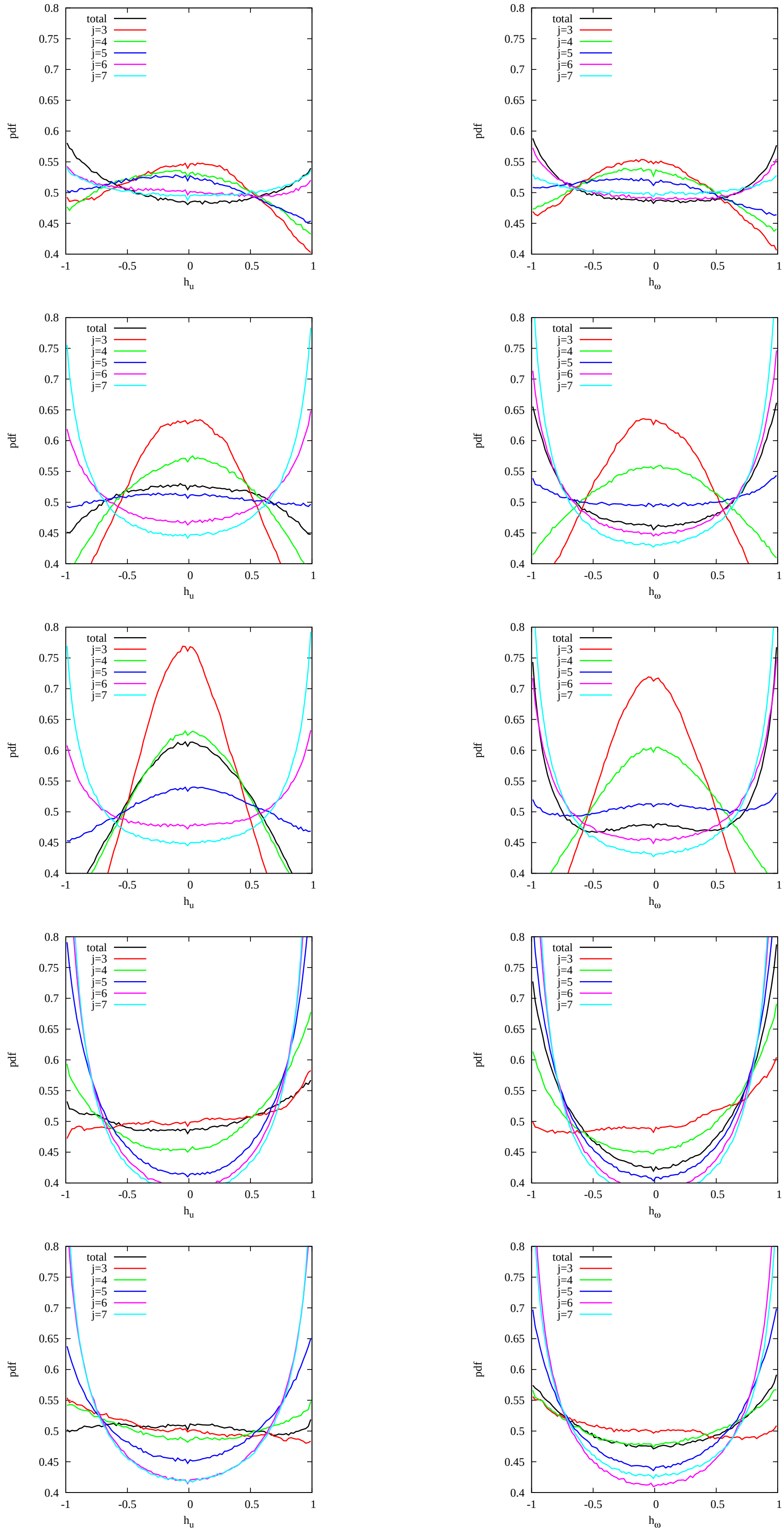

Figure 3. Scale-dependent PDFs of velocity helicity h_{u^j} (left column) and PDFs of vorticity helicity h_{ω^j} (right column) for forced isotropic turbulence (1st row), sheared turbulence (2nd row), growing sheared rotating turbulence with $f/S = +0.5$ (3rd row), decaying sheared rotating turbulence with $f/S = +5$ (4th row), and rotating turbulence (5th row).

Table 1. Properties of the turbulent flows considered in this study.

Case	Source	Re_λ	K	Z	fate
Forced Isotropic Turbulence	Vincent and Meneguzzi (1991)	150	1.35813224	151.630432	steady
Sheared Turbulence	Jacobitz et al. (2008)	72	1.15730989	175.995407	growth
Rotating Sheared $f/S = +0.5$	Jacobitz et al. (2008)	100	1.78920579	217.145401	growth
Rotating Sheared $f/S = +5$	Jacobitz et al. (2008)	35	0.228543386	28.3713894	decay
Rotating Turbulence	Liechtenstein et al. (2005)	50	0.0620917268	2.37317228	decay

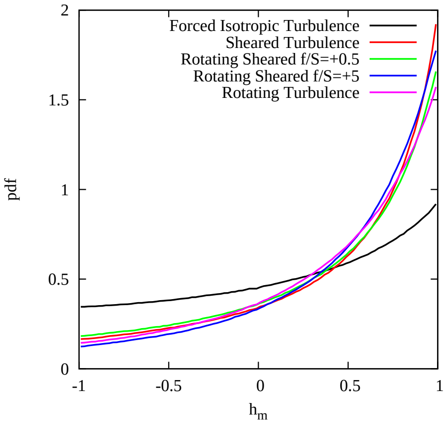

Figure 4. PDFs of the product of velocity $\boldsymbol{u}$ and bi-vorticity $\nabla \times \nabla \times \boldsymbol{u}$ for the different turbulent flows.

ing turbulence case shows an approximately even distribution of relative velocity helicity. It appears that growing turbulence has a tendency to two-dimensionalization of the flow, while decaying turbulence is characterized by helical motion.

The corresponding PDFs of relative vorticity helicity h_ω are given in figure 2. In contrast to the relative velocity helicity h_u, the relative vorticity helicity h_ω shows pronounced maxima at $h_\omega = \pm 1$ for all five flows and does not aid with a further classification of the fate of the flows.

Scale-Dependent Analysis

The scale-dependent velocity helicity, proposed in Yoshimatsu et al. (2009), is defined as $H_{u^j} = \mathbf{u}^j \cdot \boldsymbol{\omega}^j$, where $\mathbf{u}^j$ and $\boldsymbol{\omega}^j$ are velocity and vorticity at scale 2^{-j}, respectively. For $j \neq 0$, H_{u^j} is a property of the flow which is invariant to Galilean transformations, though H_u itself is not. The scale contributions of velocity $\mathbf{u}^j$ (and similarly for vorticity and its curl) are obtained by decomposing $\mathbf{u} = (u^1, u^2, u^3)$, given at resolution $N = 2^{3J}$ with $J = 8$, into an orthogonal wavelet series

$$\mathbf{u}(\mathbf{x}) = \sum_\lambda \widetilde{\mathbf{u}}_\lambda \psi_\lambda(\mathbf{x}) \tag{2}$$

where the multi-index $\lambda = (j, \mathbf{i}, \mu)$ denotes scale j (with $0 \leq j \leq J-1$), spatial position $\mathbf{i}$ (with 2^{3j} values for each j and μ) and seven spatial directions $\mu = 1, ..., 7$ of each wavelet ψ_λ (Farge, 1992). Orthogonality implies that the wavelet coefficients are given by $\widetilde{\mathbf{u}}_\lambda = \langle \mathbf{u}, \psi_\lambda \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product. The coefficients measure fluctuations of $\mathbf{u}$ at scale 2^{-j} and around position $\mathbf{i}/2^j$ for each of the 7 possible directions. Fixing j and summing only over $\mathbf{i}$ and μ in eq.(2) the contribution of $\mathbf{u}$ at scale j is obtained and by construction we have $\mathbf{u} = \sum_j \mathbf{u}_j$.

The scale-dependent relative velocity helicity is defined by $h_{u^j} = H_{u^j}/(|u^j||\omega^j|)$. Analogously, the scale-dependent vorticity helicity $H_{\omega^j} = \boldsymbol{\omega}^j \cdot (\nabla \times \boldsymbol{\omega})^j$ and the corresponding relative quantity h_{ω^j} can be obtained. The above scale-dependent quantities will help to gain a better understanding of geometrical statistics at different scales of motion and in the following we analyze the different turbulent flows.

PDFs of scale-dependent relative velocity helicity h_{u^j} (left column) and relative vorticity helicity h_{ω^j} (right column) are presented in figure 3 for the different turbulent flows studied here. Note that scales $j = 1$ and $j = 2$ are not shown due to the small number of wavelet modes at those scales.

For growing sheared turbulence, the scale-dependent relative velocity helicity PDFs of the larger scales with $j = 3, 4$, and 5 show a maximum at $h_{u^j} = 0$, corresponding to a trend to two-dimensionalization of the flow at large scales. The smaller scales with $j = 6$ and 7 have maxima at $h_{u^j} = \pm 1$, corresponding to a trend to helical motion at small scales. For growing rotating sheared turbulence with $f/S = +0.5$ and for statistically steady isotropic turbulence, an identical result is obtained, but the preference for two-dimensionalization at large scales and helical motion at small scales is even more pronounced for strongly growing rotating sheared turbulence and less pronounced for statistically steady forced isotropic turbulence. The scale-dependent relative vorticity helicity PDFs of these three cases also yield a maximum at $h_{\omega^j} = 0$ for the larger scales with $j = 3, 4$ and 5 and maxima at $h_{\omega^j} = \pm 1$ at the smaller scales $j = 6$ and 7. For decaying rotating turbulence and decaying rotating sheared turbulence with $f/S = +5$, the scale-dependent velocity helicity PDFs show maxima for $h_{u^j} = \pm 1$ at all scales $j > 3$. Similarly, the scale-dependent relative vorticity helicity PDFs yield maxima for $h_{\omega^j} = \pm 1$ for all scales considered.

Dissipation of Velocity Helicity

For isotropic turbulence, Sanada (1993) discussed the balance equation for total helicity (1) and conjectured that the dissipation of total velocity helicity is determined by total vorticity helicity. As neither total velocity helicity nor total vorticity helicity are positive definite, it is required that both quantities assume the same sign for this to be true. Thus this sign correlation strongly impacts the dynamics of helicity.

As shown in figure 4, the PDF of the cosine of the angle between velocity $\boldsymbol{u}$ and $-\nabla^2\boldsymbol{u} = \nabla \times \nabla \times \boldsymbol{u}$ indicates a much larger probability that the two vectors are aligned. This can also be directly seen from the fact that the mean value is positive $-\langle \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \rangle = \langle \boldsymbol{\omega} \cdot \boldsymbol{\omega} \rangle > 0$. This implies the tendency that $H_u = \boldsymbol{u} \cdot \boldsymbol{\omega}$ and $H_\omega = -\nabla^2 \boldsymbol{u} \cdot \boldsymbol{\omega}$ have the same sign (Sanada, 1993). This tendency also holds for the relative helicities (Galanti and Tsinober, 2006).

To verify this conjecture for the five different flows considered here, the joint PDFs of relative velocity helicity h_u with relative vorticity helicity h_ω are shown in figure 5. For all cases a strong correlation of the signs of the two helicities is indeed observed. This sign correlation thus supports that vorticity helicity diminishes velocity helicity. In addition, the joint PDFs are approximately symmetric along their diagonal axis for all cases with decaying turbulent kinetic energy. This symmetry is broken for the cases with growing turbulent kinetic energy.

CONCLUSIONS

To summarize, helical properties of five prototypical turbulent flows were investigated. For the PDFs of velocity helicity h_u, a maximum at $h_u = 0$ was observed for cases with growing turbulent kinetic energy, while a maximum at $h_u = \pm 1$ was found for decaying cases. Thus, for growing cases, the PDFs of velocity helicity h_u indicate a larger probability that velocity and vorticity are perpendicular, corresponding to two-dimensionalization of the flow. For decaying cases, the PDFs of velocity helicity h_u show a preference for the alignment or anti-alignment of velocity and vorticity, corresponding to helical flow. For all cases, however, the PDF of vorticity helicity h_ω always assumes a maximum at $h_\omega = \pm 1$.

Scale-dependent PDFs of velocity helicity show that large scales tend to have a maximum at $h_{u^j} = 0$, corresponding to two-dimensionalization of the flows, while small scales tend to show a maximum at $h_{u^j} = \pm 1$, corresponding to helical motion. These observations hold for all types of turbulent flows considered in this study.

Joint PDFs of relative velocity helicity and relative vorticity helicity show a high probability that h_u and h_ω have the same sign even locally. Thus, vorticity helicity tends to diminish velocity helicity for the different homogeneous turbulent flows studied here.

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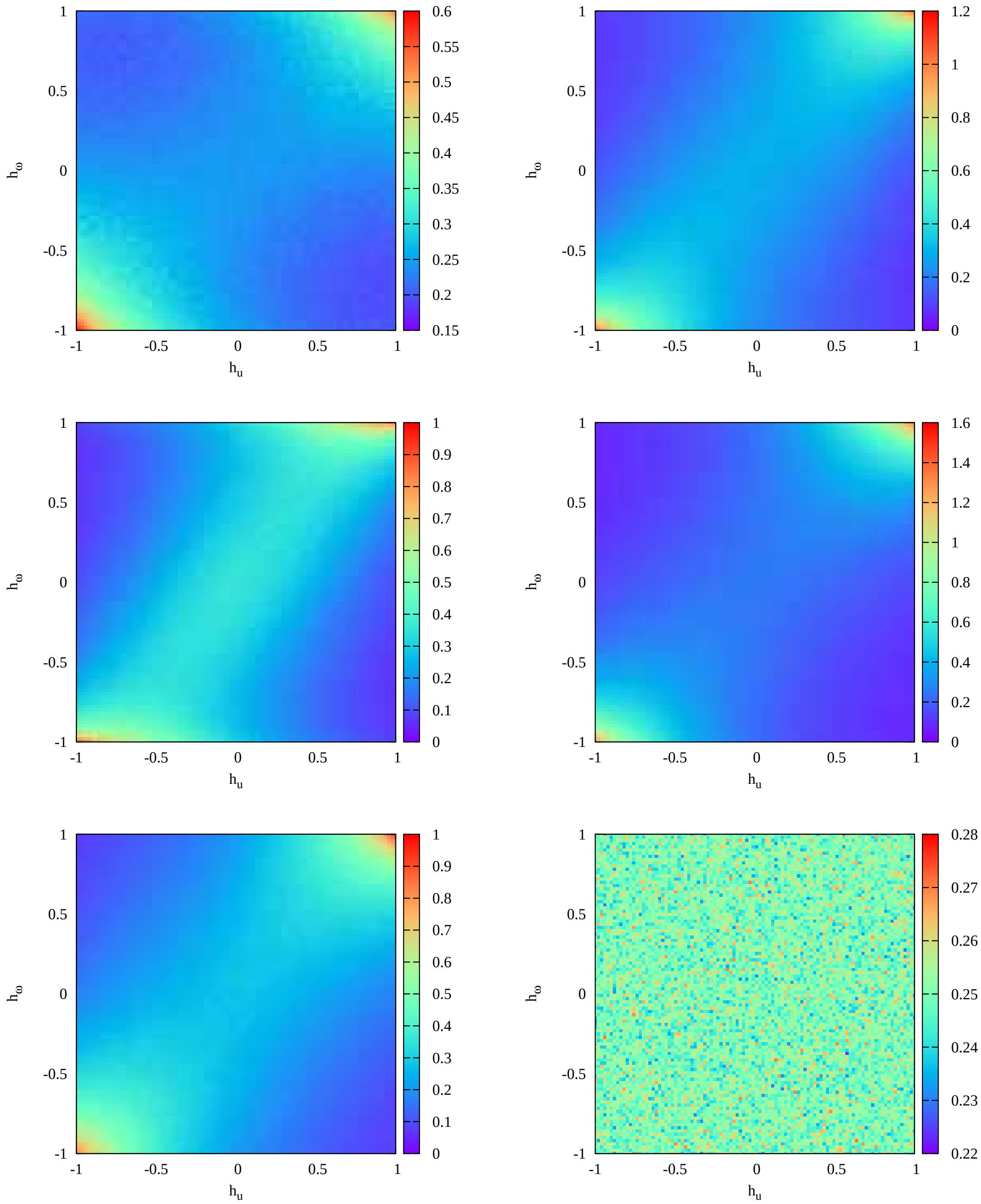

Figure 5. Joint PDFs of velocity helicity h_u and vorticity helicity h_ω for forced isotropic turbulence (top left), sheared turbulence (top right), growing rotating sheared turbulence with $f/S = +0.5$ (center left), decaying rotating sheared turbulence with $f/S = +5$ (center right), rotating turbulence (bottom left), and random fields (bottom right).