# TURBULENT SCALING LAWS AND WHAT WE CAN LEARN FROM THE MULTI-POINT CORRELATION EQUATIONS

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## ABSTRACT

We presently show that the infinite set of multi-point correlation equations, which are direct statistical consequences of the Navier-Stokes equations, admit a rather large set of Lie symmetry groups. This set is considerable extended compared to the set of groups which are implied from the original set of equations of fluid mechanics. Specifically a new scaling group and translational groups of the correlation vectors and all independent variables have been discovered. These new statistical groups have important consequences on our understanding of turbulent scaling laws to be exemplarily revealed by two examples. Firstly, one of the key foundations of statistical turbulence theory is the universal law of the wall with its essential ingredient is the logarithmic law. We demonstrate that the log-law fundamentally relies on one of the new translational groups. Furthermore, we consider a rotating channel flow, whose scaling behavior can only be described using the new statistical symmetries. It can be seen that the direction of rotation axes plays an important role, because different axes result in very different scaling laws.

## 1 Introduction

The special importance of turbulence is determined by its ubiquity in innumerable natural and technical systems. Examples for natural turbulent flows are atmospheric flow and oceanic current which to calculate is a crucial point in climate research. Only with the advent of super computers it became apparent that the Navier-Stokes equations provide a very good continuum mechanical model for turbulent flows. Still, the exclusive and direct application of the Navier-Stokes equations to practical flow problems at high Reynolds numbers without invoking any additional assumptions is still several decades away.

However, in most applications it is not at all necessary to know all the detailed fluctuations of velocity and pressure present in turbulent flows but for the most part statistical measures are sufficient.

This was in fact the key idea of O. Reynolds who was the first to suggest a statistical description of turbulence. The Navier-Stokes equations, however, constitute a non-linear and, due to the pressure Poisson equation, a non-local set of equations. As an immediate consequence of this the equations for the mean or expectation values for velocity and pressure leads to an infinite set of statistical equations, or, if truncated at some level of statistics, an un-closed system is generated.

In order to obtain a much deeper insight into the statistical behavior of turbulence we presently apply Lie symmetry group theory to the full infinite set of statistical equations investigating two canonical turbulent flow situations.

This work presents the most important results from Oberlack & Rosteck (2010), extended by new developments concerning the extension of the set of symmetries in Rosteck & Oberlack (2011) and applications to various rotating channel flows.

## 2 Equations of statistical turbulence theory

### 2.1 Navier-Stokes equations

The initial point of the entire analysis to follow is based on the three dimensional Navier-Stokes equations for an incompressible fluid under the assumption of a Newtonian material with constant density and viscosity. In Cartesian tensor notation we have the continuity equation

$$\frac{\partial U_k}{\partial x_k} = 0 \tag{1}$$

and the momentum equation writes

$$\frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} \quad , \quad i = 1,2,3 \,. \tag{2}$$

$t \in \mathbb{R}^+$, $x \in \mathbb{R}^3$, $U = U(x,t)$ and $P = P(x,t)$ represent time, position vector, instantaneous velocity vector and pressure. The density $\rho$ and the viscosity $\nu$ are positive constants. Furthermore pressure can be normalized with the constant density. The new pressure term reduces to $P^* = \frac{P}{\rho}$ which, inserted into (2), leads to a modified momentum equation and the asterisk is omitted from here on

$$\mathscr{M}_i(x) = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} = 0 , \quad (3)$$

$i = 1,2,3$, where all terms have been collected on one side.

### 2.2 Reynolds averaged transport equations

After $U$ and $P$ are decomposed according to the Reynolds decomposition, i.e. $U = \bar{U} + u$ and $P = \bar{P} + p$, we gain an averaged versions of the continuity equation

$$\frac{\partial \bar{U}_k}{\partial x_k} = 0 \ , \quad (4)$$

and the momentum equations

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k} , \ i = 1,2,3. \quad (5)$$

At this point we observe the well-known closure problem of turbulence since, compared to the original set of equations, the unknown Reynolds stress tensor $\overline{u_i u_k}$ appeared. However, rather different from the classical approach we will not proceed with deriving the Reynolds stress tensor transport equation which contains additional four unclosed tensors. Instead the multi-point correlation approach is put forward the reason being twofold.

First, if the infinite set of correlation equations is considered the closure problem is somewhat bypassed. Second, the multi-point correlation delivers additional information on the turbulence statistics such as length scale information which may not be gained from the Reynolds stress tensor, which is a single-point approach.

For this we need the equations for the fluctuating quantities $u$ and $p$ which are derived by taking the differences between the averaged and the non-averaged equations, i.e. (1)/(4) and (3)/(5). The resulting fluctuation equations read

$$\frac{\partial u_k}{\partial x_k} = 0 \ , \quad (6)$$

and

$$\mathscr{N}_i(x) = \frac{\bar{\mathrm{D}} u_i}{\bar{\mathrm{D}} t} + u_k \frac{\partial \bar{U}_i}{\partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k} + \frac{\partial u_i u_k}{\partial x_k} + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = 0 \quad (7)$$

with $i = 1,2,3$.

### 2.3 Multi-point correlation equations

The idea of two- and multi-point correlation equations in turbulence was presumably first established by Keller & Friedmann (1924). At the time it was assumed that all correlation equations of orders higher than two may be neglected. Theoretical considerations led to the result that all higher correlations have to be taken into account. Consequently, all multi-point correlation equations have to be considered in the symmetry analysis to follow.

Two different sets of multi-point correlation (MPC) equations will be derived below. The first is based on the instantaneous values of $U$ and $P$ while the second one is in accordance with the classical notation based on the fluctuating quantities $u$ and $p$.

#### 2.3.1 MPC equations: instantaneous approach

In order to write the MPC equations in a very compact form, we introduce the following notation. The multi-point velocity correlation tensor of order $n+1$ is defined as follows:

$$H_{i_{\{n+1\}}} = H_{i_{(0)} i_{(1)} \ldots i_{(n)}} = \overline{U_{i_{(0)}}(x_{(0)}) \cdot \ldots \cdot U_{i_{(n)}}(x_{(n)})} \ , \quad (8)$$

where the first index of the **H** tensor defines the tensor character of the term and the second index in braces denotes the order of the tensor. The curly brackets point out that not an index of a tensor but an enumeration is meant. It is important to mention that the indices start with 0 which is an advantage when introducing a new coordinate system based on the Euclidean distance of two space points. The value in curly brackets is the actual order of the tensor and takes into account that counting starts at zero. Apparently we have the connection to the mean velocity according to $H_{i_{\{1\}}} = H_{i_{(0)}} = \bar{U}_i$.

In some cases the list of indices is interrupted by one or more other indices which is pointed out by attaching the replaced value in brackets to the index

$$H_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]} = \overline{U_{i_{(0)}}(x_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(x_{(l-1)}) \cdot} \overline{\cdot U_{k_{(l)}}(x_{(l)}) U_{i_{(l+1)}}(x_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(x_{(n)})} \ . \quad (9)$$

This is further extended by

$$H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[x_{(n+1)} \mapsto x_{(l)}] = \overline{U_{i_{(0)}}(x_{(0)}) \cdot \ldots \cdot U_{i_{(n)}}(x_{(n)}) U_{k_{(l)}}(x_{(l)})} \ , \quad (10)$$

where not only that index $i_{(n+1)}$ is replaced by $k_{(l)}$, but also that the independent variable $x_{(n+1)}$ is replaced by $x_{(l)}$. If indices are missing e.g. between $i_{(l-1)}$ and $i_{(l+1)}$ we define

$$H_{i_{\{n\}}[i_{(l)} \mapsto \emptyset]} = \overline{U_{i_{(0)}}(x_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(x_{(l-1)}) U_{i_{(l+1)}}(x_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(x_{(n)})} . \quad (11)$$

Finally, if the pressure is involved we write

$$I_{i_{\{n\}}[l]} = \overline{U_{i_{(0)}}(x_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(x_{(l-1)}) P(x_{(l)}) U_{i_{(l+1)}}(x_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(x_{(n)})} , \quad (12)$$

which is, considering all the above definitions, sufficient to derive the MPC equations from the equations of instantaneous velocity and pressure i.e. equation (1) and (3).

Applying the Reynolds averaging operator according to the sum below

$$
\begin{aligned}
&\mathscr{S}_{i_{\{n+1\}}}(x_{(0)},\dots,x_{(n)}) = \\
&\quad \overline{\mathscr{M}_{i_{(0)}}(x_{(0)})U_{i_{(1)}}(x_{(1)})\cdot\ldots\cdot U_{i_{(n)}}(x_{(n)})} \\
&\quad +\overline{U_{i_{(0)}}(x_{(0)})\mathscr{M}_{i_{(1)}}(x_{(1)})U_{i_{(2)}}(x_{(2)})\cdot\ldots\cdot U_{i_{(n)}}(x_{(n)})} \\
&\quad +\ldots \\
&\quad +\overline{U_{i_{(0)}}(x_{(0)})\cdot\ldots\cdot U_{i_{(n-2)}}(x_{(n-2)})\mathscr{M}_{i_{(n-1)}}(x_{(n-1)})U_{i_{(n)}}(x_{(n)})} \\
&\quad +\overline{U_{i_{(0)}}(x_{(0)})\cdot\ldots\cdot U_{i_{(n-1)}}(x_{(n-1)})\mathscr{M}_{i_{(n)}}(x_{(n)})}\ ,
\end{aligned} \tag{13}
$$

we obtain the $\mathscr{S}$-equation which writes

$$
\begin{aligned}
\mathscr{S}_{i_{\{n+1\}}} = &\frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[x_{(n+1)}\mapsto x_{(l)}]}{\partial x_{k_{(l)}}}\right. \\
&\left. + \frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right] = 0 \\
&\text{for}\quad n=1,\dots,\infty\ .
\end{aligned} \tag{14}
$$

Loosely speaking equation (14) implies the full multi-point statistical information of the Navier-Stokes equations at the expense to deal with an infinite dimensional chain of differential equations starting with order 2 i.e. $n = 1$. The rather remarkable consequence of the derivation is that (14) is a linear equation which considerably simplifies the finding of Lie symmetries to be pointed out below.

From equation (1) a continuity equation for $H_{i_{\{n+1\}}}$ and $I_{i_{\{n\}}[l]}$ can be derived. This leads to

$$
\frac{\partial H_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]}}{\partial x_{k_{(l)}}} = 0 \quad \text{for} \quad l=0,\dots,n \tag{15}
$$

and

$$
\frac{\partial I_{i_{\{n\}}[k][i_{(l)}\mapsto m_{(l)}]}}{\partial x_{m_{(l)}}} = 0 \quad \text{for} \quad k,l=0,\dots,n \ \text{and}\ k\neq l\ . \tag{16}
$$

At this point we adopt the classic notation of distance vectors. Accordingly the usual position vector $x$ is employed and the remaining independent spatial variables are expressed as the difference of two position vectors $x_{(l)}$ and $x_{(0)}$. The coordinate transformation are

$$
x = x_{(0)}\ ,\qquad r_{(l)} = x_{(l)} - x_{(0)} \quad \text{with} \quad l=1,\dots,n. \tag{17}
$$

#### 2.3.2 MPC equations: fluctuation approach

In the present subsection we adopt the classical approach i.e. all correlation functions are based on the fluctuating quantities $u$ and $p$ as introduced by Reynolds and not on the full instantaneous quantities as in the previous sub-section. Hence, similar to (8) we have the multi-point correlation for the fluctuation velocity

$$
R_{i_{\{n+1\}}} = R_{i_{(0)}i_{(1)}\dots i_{(n)}} = \overline{u_{i_{(0)}}(x_{(0)})\cdot\ldots\cdot u_{i_{(n)}}(x_{(n)})}\ . \tag{18}
$$

Further, all other correlations defined in sub-section 2.3.1 are defined accordingly i.e. equivalent to the definitions (9)-(12) we respectively define $R_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]}$, $R_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[x_{(n+1)}\mapsto x_{(l)}]$, $R_{i_{\{n\}}[i_{(l)}\mapsto\emptyset]}$ and $P_{i_{\{n\}}[l]}$.

Finally, we define the correlation equation in analogy to (13) where $\mathscr{M}_i$ is replaced by the equation for the fluctuations (7) denoted by $\mathscr{N}_i$ and $U_i$ and $P$ are substituted by $u_i$ and $p$. The resulting equation is denoted by $\mathscr{T}_{i_{\{n+1\}}}$

$$
\begin{aligned}
\mathscr{T}_{i_{\{n+1\}}} = &\frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n}\left[\bar{U}_{k_{(l)}}(x_{(l)})\frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}}\right. \\
&+ R_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]}\frac{\partial \bar{U}_{i_{(l)}}(x_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} \\
&- \nu\frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)}\mapsto\emptyset]}\frac{\partial\overline{u_{i_{(l)}}u_{k_{(l)}}}(x_{(l)})}{\partial x_{k_{(l)}}} \\
&\left. + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[x_{(n+1)}\mapsto x_{(l)}]}{\partial x_{k_{(l)}}}\right] = 0 \\
&\text{for}\quad n=1,\dots,\infty\ .
\end{aligned} \tag{19}
$$

The first tensor equation of this infinite chain propagates $R_{i_{\{2\}}}$ which has a close link to the Reynolds stress tensor, i.e.

$$
\lim_{x_{(k)}\to x_{(l)}} R_{i_{\{2\}}} = \lim_{x_{(k)}\to x_{(l)}} R_{i_{(0)}i_{(1)}} = \overline{u_{i_{(0)}}u_{i_{(1)}}}(x_{(l)}) \quad \text{mit} \quad k\neq l\ , \tag{20}
$$

which is the key unclosed quantity in the Reynolds stress transport equation (5). Here $x_{(k)}$ and $x_{(l)}$ can be arbitrary vectors out of $x_{(0)},\dots,x_{(n)}$.

Also equation (19) implies all statistical information of the Navier-Stokes equations. However, apart from the latter simple relation to the Reynolds stress tensor it possesses the key disadvantage of being a *non-linear* infinite dimensional system of differential equations which make the extraction of Lie symmetries from this equation rather cumbersome. There are two essential sources of non-linearity in these equations. One is the known convection non-linearity which links the mean velocity to all correlation equations. The second source of non-linearity originates from the second row of equation (19). It is based on the fact that the gradient of the Reynolds stress tensor is contained in the equations of fluctuation. Hence, considering the following identity, this term is not equal to zero for turbulent flows and for multi-point correlation tensors of order higher than two

$$
R_{i_{\{1\}}[i_{(l)}\mapsto\emptyset]} = 0\ . \tag{21}
$$

As a direct consequence all multi-point correlation equations of order $n > 1$ are coupled to the two-point correlation equation.

From equation (6) a continuity equation for $R_{i_{\{n+1\}}}$ and $P_{i_{\{n\}}[l]}$ can be derived. They have identical form to (15) and (16) for $H_{i_{\{n+1\}}}$ and $I_{i_{\{n\}}}$ and hence will be omitted for brevity.

It is apparent that there is a unique relation between the instantaneous (**H**, **I**) and the fluctuation approach (**R**, **P**) though the actual crossover is somewhat cumbersome in particular with increasing tensor order because they may only be given in recursive form. Since needed later we give the first relations

$$\begin{aligned} H_{i_{(0)}} &= \bar{U}_{i_{(0)}} && (22)\\ H_{i_{(0)}i_{(1)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} + R_{i_{(0)}i_{(1)}} && (23)\\ H_{i_{(0)}i_{(1)}i_{(2)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\bar{U}_{i_{(2)}} + R_{i_{(0)}i_{(1)}}\bar{U}_{i_{(2)}} \\ &\quad + R_{i_{(0)}i_{(2)}}\bar{U}_{i_{(1)}} + R_{i_{(1)}i_{(2)}}\bar{U}_{i_{(0)}} + R_{i_{(0)}i_{(1)}i_{(2)}} && (24)\\ &\vdots \qquad \vdots \end{aligned}$$

where the indices also refer to the spatial points as indicated.

With the identities (22)-(24) and alike equations it becomes apparent that the rather compact **H**-notation in equation (14) leads to a highly non-linear and complex version of the MPC equation if written in **R**-notation i.e. based on the mean and fluctuating quantities.

As a special case of the equations (19) we consider $n = 1$ including the $(x, r)$-coordinate system (17) and we derive the equation for the two-point correlation tensor. To abbreviate the notation we introduce the following nomenclature:

$$R_{i_{\{2\}}} = R_{ii_{(1)}} = R_{ij} \quad . \tag{25}$$

In this case equation (19) reduces to

$$\begin{aligned} \mathscr{T}_{i_{\{2\}}} = & \frac{\bar{\mathrm{D}}R_{ij}}{\bar{\mathrm{D}}t} + R_{kj}\,\frac{\partial \bar{U}_i(x,t)}{\partial x_k} + R_{ik}\,\frac{\partial \bar{U}_j(x,t)}{\partial x_k}\bigg|_{x+r} \\ & + \left[\bar{U}_k\left(x+r,t\right) - \bar{U}_k\left(x,t\right)\right]\,\frac{\partial R_{ij}}{\partial r_k} + \frac{\partial \overline{pu}_j}{\partial x_i} - \frac{\partial \overline{pu}_j}{\partial r_i} \\ & + \frac{\partial \overline{u_i p}}{\partial r_j} - \nu\left[\frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\,\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + 2\,\frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}\right] \\ & + \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right] = 0 \quad . \end{aligned} \tag{26}$$

The vectors $\overline{pu}_j$ and $\overline{u_i p}$ are special cases of $P_{i_{\{n\}}[k]}$ and defined as

$$\begin{aligned} \overline{pu}_j(x,r,t) &= \overline{p(x_{(0)},t)\,u_j(x_{(1)},t)} \\ \overline{u_i p}(x,r,t) &= \overline{u_i(x_{(0)},t)\,p(x_{(1)},t)} \quad . \end{aligned} \tag{27}$$

For the two-point case the continuity equations take the form

$$\frac{\partial R_{ij}}{\partial x_i} - \frac{\partial R_{ij}}{\partial r_i} = 0 \ , \quad \frac{\partial R_{ij}}{\partial r_j} = 0 \tag{28}$$

and

$$\frac{\partial \overline{pu}_i}{\partial r_i} = 0 \ , \quad \frac{\partial \overline{u_j p}}{\partial x_j} - \frac{\partial \overline{u_j p}}{\partial r_j} = 0. \tag{29}$$

The non-locality of the two- and multi-point correlation equations is most obvious when we use the commutation of the two-point correlation tensor. Given $\overline{u_i(x_{(0)})u_j(x_{(1)})} = \overline{u_j(x_{(1)})u_i(x_{(0)})}$ with equation (17) leads to the functional relations

$$R_{ij}(x,r;t) = R_{ji}(x+r,-r;t) \tag{30}$$

and

$$\overline{pu}_j(x,r;t) = \overline{u_j p}(x+r,-r;t) \quad . \tag{31}$$

Analogous identities can be derived for all other two- and multi-point correlation tensors.

## 3 Symmetries of statistical transport equations

In the present section we first revisit the Lie symmetries of the Euler and Navier-Stokes equations. In turn they will all be transferred to its corresponding ones for the MPC equations. In the second part we show that the MPC equations admit even more Lie symmetries which are not reflected in the original Euler and Navier-Stokes equations.

Both sets of symmetries will finally be employed in section 4 to show that classical and new scaling laws may not be determined from the classical symmetries alone but essentially rely on the new symmetries which we will call *statistical symmetries*.

In order to appreciate the analysis on Lie symmetries below we will define its basic concepts including that of *invariant solutions* which in the fluid mechanics community is usually referred to as self-similar solution though this in principle is limited to invariant solutions with certain scaling properties involved. In the turbulence community these types of solutions are usually denoted *turbulent scaling laws* though there they are in most cases not solutions of equations derived from first principles.

Suppose the system of partial differential equations under investigation is given by

$$F(y, z, z^{(1)}, z^{(2)}, \dots) = 0, \tag{32}$$

where $y$ and $z$ are the independent and the dependent variables respectively and $z^{(n)}$ refers to all n$^{th}$-order derivatives of any component of $z$ with respect to any component of $y$. A transformation

$$y = \phi(y^*, z^*) \quad \text{and} \quad z = \psi(y^*, z^*) \tag{33}$$

is called a *symmetry* or *symmetry transformation* of the equation (32) if the following equivalence holds

$$F(y,z,z^{(1)},\dots)=0 \quad \Leftrightarrow \quad F(y^*,z^*,z^{*(1)},\dots)=0, \quad (34)$$

i.e. the transformation (33) substituted into (32) does not change the form of equation (32) if written in the new variables $y^*$ and $z^*$.

In order to make this concept more coming alive we consider the 1D heat equation

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2} \quad (35)$$

which, beside other symmetries, admits the scaling symmetry

$$t^*=\mathrm{e}^{2\alpha_1}t,\ x^*=\mathrm{e}^{\alpha_1}x,\ T^*=\mathrm{e}^{\alpha_2}T, \quad (36)$$

with the two independent group parameter $(\alpha_1,\alpha_2)\in\mathbb{R}$. For brevity the underlying two symmetries are presently combined into one transformation (36). It is rather apparent that condition (34) is fulfilled since the implementation of (36) into (35) leaves the equation invariant if written in the new variables denoted by $*$.

A second concept which will be heavily relied on is that of an *invariant*. It refers to quantities that do not change structure under a given symmetry i. e.

$$I(y,z)=I(\phi(y^*,z^*),\psi(y^*,z^*))=I(y^*,z^*) \quad (37)$$

or in other words the form of $I$ is invariant under the transformation. This may easily be clarified using the two-parameter symmetry (36). Beside others, in fact infinitely many, we may define and easily prove the existence of the two subsequent invariants

$$\delta=\frac{x}{\sqrt{t}}=\frac{x^*}{\sqrt{t^*}}, \quad \Delta=\frac{T}{t^{\frac{\alpha_2}{2\alpha_1}}}=\frac{T^*}{t^{*\frac{\alpha_2}{2\alpha_1}}}. \quad (38)$$

The final concept in this context is that of an *invariant solution*. It designates to the remarkable finding that the invariants may be taken as new dependent and independent variables which in turn leads to a reduction of the number of the independent variables often referred to as symmetry reduction. It is exactly this property of self-similar solutions which is profitable for some further analysis since less dimensions are involved.

For the example of the heat equation above we introduce $\delta$ and $\Delta$ as new independent variables, i.e. we implement its definitions (38) into (35) and obtain the reduced differential equation

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}\delta^2}+\frac{1}{2}\delta\frac{\mathrm{d}\Delta}{\mathrm{d}\delta}-\frac{\alpha_2}{2\alpha_1}\Delta=0. \quad (39)$$

It should be noted that the actual computation of the symmetries, the invariants and the invariant solutions is extremely simplified if the infinitesimal form is invoked (see Bluman *et al.*, 2009), which has been left out in the present contribution.

## 3.1 Symmetries of the Euler and Navier-Stokes equations

The Euler equations, i.e. equation (1) and (3) with $\nu=0$ admit a ten-parameter symmetry group,

$$\begin{aligned}
T_1:&\ t^*=t+a_1,\ x^*=x,\ U^*=U,\ P^*=P,\\
T_2:&\ t^*=t,\ x^*=\mathrm{e}^{a_2}x,\ U^*=\mathrm{e}^{a_2}U,\ P^*=\mathrm{e}^{2a_2}P,\\
T_3:&\ t^*=\mathrm{e}^{a_3}t,\ x^*=x,\ U^*=\mathrm{e}^{-a_3}U,\ P^*=\mathrm{e}^{-2a_3}P,\\
T_4-T_6:&\ t^*=t,\ x^*=\mathbf{a}\cdot x,\ U^*=\mathbf{a}\cdot U,\ P^*=P,\\
T_7-T_9:&\ t^*=t,\ x^*=x+f(t),\ U^*=U+\frac{\mathrm{d}f}{\mathrm{d}t},\\
&\ P^*=P-x\cdot\frac{\mathrm{d}^2 f}{\mathrm{d}t^2},\\
T_{10}:&\ t^*=t,\ x^*=x,\ U^*=U,\ P^*=P+f_4(t)\ ,
\end{aligned} \quad (40)$$

where $a_1$-$a_3$ are independent group-parameters, $\mathbf{a}$ denotes a constant rotation matrix with the properties $\mathbf{a}\cdot\mathbf{a}^\mathsf{T}=\mathbf{a}^\mathsf{T}\cdot\mathbf{a}=\mathbf{I}$ and $|\mathbf{a}|=1$. Moreover $f(t)=(f_1(t),f_2(t),f_3(t))^\mathsf{T}$ with twice differentiable functions $f_1$-$f_3$ and $f_4(t)$ may have arbitrary time dependence.

Each of the symmetries has a distinct physical meaning. $T_1$ means time translation i.e. any physical experiment is independent of the actual starting point. $T_4$-$T_6$ designate rotation invariance which refers to the possibility to let an experiment undergo a fixed rotation without changing physics. Note, that this does mean moving into a rotating system since this does significantly change physics and hence is not a symmetry. The symmetries $T_7$-$T_9$ comprise translational invariance in space for constant $f_1$-$f_3$ as well as the classical Galilei group if $f_1$-$f_3$ are linear in time. These are key properties of classical mechanics referring to the fact that physics is independent of the location or if moved at a constant speed. In its rather general form $T_7$-$T_9$ and $T_{10}$ are direct consequences of an incompressible flow and do not have a counterpart in the case of compressible flows. The complete record of all point-symmetries (40) was first published by Pukhnachev Pukhnachev (1972).

Invoking a formal transfer from Euler to the Navier-Stokes equations symmetry properties change and a recombination of the two scaling symmetries $T_2$ and $T_3$ is observed

$$T_{NaSt}:\ t^*=\mathrm{e}^{2a_4}t,\ x^*=\mathrm{e}^{a_4}x,\ U^*=\mathrm{e}^{-a_4}U,\ P^*=\mathrm{e}^{-2a_4}P, \quad (41)$$

while the remaining groups stay unaltered.

It should be noted that additional symmetries exist for dimensional restricted cases such as plane or axisymmetric flows (see Andreev & Rodionov, 1988; Cantwell, 1978).

## 3.2 Symmetries of the MPC implied by Euler and Navier-Stokes symmetries

Adopting the classical Reynolds notation first, where the instantaneous quantities are split into mean and fluctuating values, we may directly derive from (40)

$$
\begin{aligned}
\bar{T}_1: &\ t^* = t + a_1,\ x^* = x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^* = \bar{U},\\
&\bar{P}^* = \bar{P},\ \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\ \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}},\\
\bar{T}_2: &\ t^* = t,\ x^* = \mathrm{e}^{a_2} x,\ r^*_{(l)} = \mathrm{e}^{a_2} r_{(l)},\ \bar{U}^* = \mathrm{e}^{a_2}\bar{U},\\
&\bar{P}^* = \mathrm{e}^{2a_2}\bar{P},\ \mathbf{R}^*_{\{n\}} = \mathrm{e}^{na_2}\mathbf{R}_{\{n\}}, \mathbf{P}^*_{\{n\}} = \mathrm{e}^{(n+2)a_2}\mathbf{P}_{\{n\}},\\
\bar{T}_3: &\ t^* = \mathrm{e}^{a_3} t,\ x^* = x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^* = \mathrm{e}^{-a_3}\bar{U},\\
&\bar{P}^* = \mathrm{e}^{-2a_3}\bar{P},\ \mathbf{R}^*_{\{n\}} = \mathrm{e}^{-na_3}\mathbf{R}_{\{n\}},\\
&\mathbf{P}^*_{\{n\}} = \mathrm{e}^{-(n+2)a_3}\mathbf{P}_{\{n\}},\\
\bar{T}_4 - \bar{T}_6: &\ t^* = t,\ x^* = \mathbf{a}\cdot x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^* = \mathbf{a}\cdot\bar{U},\ \bar{P}^* = \bar{P},\\
&\mathbf{R}^*_{\{n\}} = \mathbf{A}_{\{n\}} \otimes \mathbf{R}_{\{n\}},\ \mathbf{P}^*_{\{n\}} = \mathbf{A}_{\{n\}} \otimes \mathbf{P}_{\{n\}},\\
\bar{T}_7 - \bar{T}_9: &\ t^* = t,\ x^* = x + f(t),\ r^*_{(l)} = r_{(l)},\ \bar{U}^* = \bar{U} + \frac{\mathrm{d}f}{\mathrm{d}t},\\
&\bar{P}^* = \bar{P} - x\cdot\frac{\mathrm{d}^2 f}{\mathrm{d}t^2},\ \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\ \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}},\\
\bar{T}_{10}: &\ t^* = t\ ,\ x^* = x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^* = \bar{U},\\
&\bar{P}^* = \bar{P} + f_4(t),\ \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\ \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}}, \quad (42)
\end{aligned}
$$

where all function and parameter definitions are adopted from 3.1 and $\mathbf{A}$ is a concatenation of rotation matrices as $A_{i_{(0)}j_{(0)}i_{(1)}j_{(1)}\ldots i_{(n)}j_{(n)}} = a_{i_{(0)}j_{(0)}} a_{i_{(1)}j_{(1)}} \ldots a_{i_{(n)}j_{(n)}}$.

The latter symmetries may also be transformed into the $\mathbf{H}$-notation. This will be omitted for briefness and also because the turbulent scaling laws to be derived and discussed below are rarely considered in this notation.

### 3.3 Statistical symmetries of the MPC equations

First hints towards a considerably extended set of symmetries for the MPC equation in the form (14) or (19) may e.g. be taken from Oberlack (2000) and Khujadze & Oberlack (2004). Its importance was not observed therein - rather it was stated that they may be mathematical artifacts of the averaging process and probably physically irrelevant. The set of new symmetries was first presented and its key importance for turbulence recognized in Oberlack & Rosteck (2010) and later extended in Rosteck & Oberlack (2011).

The actual finding of symmetries of the non-rotating MPC is rather difficult since an infinite system of equations has to be analyzed. For this task, however, it is considerably easier to investigate the linear $\mathbf{H}$-$\mathbf{I}$-system (14)-(16) rather than the non-linear $\mathbf{R}$-$\mathbf{P}$-system (19) extended by its corresponding continuity equations. However, since the latter formulation is more common, the symmetries will finally be rewritten in this notation.

This entire new set of symmetries for the $\mathbf{H}$-$\mathbf{I}$-system

$$
\begin{aligned}
\bar{T}'_1: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)} + a_{(l)},\\
&\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}},\ \mathbf{I}^*_{\{n\}} = \mathbf{I}_{\{n\}}, && (43)\\
\bar{T}'_{2_{\{n\}}}: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)},\\
&\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}} + \mathbf{C}_{\{n\}},\ \mathbf{I}^*_{\{n\}} = \mathbf{I}_{\{n\}} + \mathbf{D}_{\{n\}}, && (44)\\
\bar{T}'_s: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)},\\
&\mathbf{H}^*_{\{n\}} = \mathrm{e}^{a_s}\mathbf{H}_{\{n\}},\ \mathbf{I}^*_{\{n\}} = \mathrm{e}^{a_s}\mathbf{I}_{\{n\}}, && (45)
\end{aligned}
$$

can be seperated in three sets of symmetries. In the translation of the relative coordinates (43) $a_{(l)}$ represents the related set of group parameters. Note that this group is not related to the classical translation group in usual $x$-space (here $T_7 - T_9$ in equation (40) with $f = const.$).

The second set of statistical symmetries was in fact already partially identified in Oberlack (2000), however, falsely taken for the Galilean group, where $\mathbf{C}_{\{n\}}$ and $\mathbf{D}_{\{n\}}$ refer to group parameters. These symmetries have been extended in Rosteck & Oberlack (2011), so that $\mathbf{C}_{\{n\}}$ is a function of time and then a derivative of $\mathbf{C}_{\{n\}}$ appears also for $\mathbf{I}^*_{\{n\}}$.

It is essentially a part of this latter new group that is one of the key ingredients for he logarithmic law of the wall which in fact constitutes a solution of the infinite set of MPC equations to be shown below.

The third statistical group (45) that has been identified denotes simple scaling of all MPC tensors.

Furthermore there exists at least one more symmetry, which consists of a combination of multi-point velocity and of pressure-velocity correlations (see Rosteck & Oberlack, 2011). Its concrete form is omitted at this point because it is not needed for the further considerations.

It should be finally added that due to the linearity of the MPC equation (14) another rather generic symmetry is admitted. This is in fact featured by all linear differential equations (see Bluman *et al.*, 2009). It merely reflects the super-position principle of linear differential equations though usually cannot directly be adopted for the practical derivation of group invariant solutions.

Transforming (43)-(45) into classical notation we have

$$
\begin{aligned}
\bar{T}'_1: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)} + a_{(l)},\ \bar{U}^* = \bar{U},\\
&\bar{P}^* = \bar{P},\ \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\ \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}}, && (46)\\
\bar{T}'_{2_{\{1\}}}: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^*_{i_{(0)}} = \bar{U}_{i_{(0)}} + C_{i_{(0)}},\\
&R^*_{i_{(0)}i_{(1)}} = R_{i_{(0)}i_{(1)}} + \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} -\\
&\left(\bar{U}_{i_{(0)}} + C_{i_{(0)}}\right)\left(\bar{U}_{i_{(1)}} + C_{i_{(1)}}\right),\ \cdots && (47)\\
\bar{T}'_{2_{\{2\}}}: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^*_{i_{(0)}} = \bar{U}_{i_{(0)}},\\
&R^*_{i_{(0)}i_{(1)}} = R_{i_{(0)}i_{(1)}} + C_{i_{(0)}i_{(1)}},\ \cdots && (48)\\
\bar{T}'_s: &\ t^* = t,\ x^* = x,\ r^*_{(l)} = r_{(l)},\ \bar{U}^*_{i_{(0)}} = \mathrm{e}^{a_s}\bar{U}_{i_{(0)}},\\
&R^*_{i_{(0)}i_{(1)}} = \mathrm{e}^{a_s}\left[R_{i_{(0)}i_{(1)}} + (1-\mathrm{e}^{a_s})\bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\right],\ \cdots, && (49)
\end{aligned}
$$

where for the translation symmetry (44) only $n = 1$ and $n = 2$ were considered because despite of the fact that each of these groups appear to be almost trivial, since they are simple translational groups in the dependent coordinates, they exhibit an increasingly complexity with increasing tensor order if written in the $(\bar{U}, \mathbf{R})$ formulation.

## 4 Turbulent scaling laws

The rather old idea of a turbulent scaling law (see Pope, 2000) usually refers to two distinct facts:

(i) Introducing a certain set of parameters to non-dimensionalize statistical turbulence variables such as the mean velocity leads to a collapse of data if one external parameter is varied such as the Reynolds number.

(ii) An explicit mathematical function is given for statistical turbulence variables such as the mean velocity, Reynolds stresses, etc..

Presently we primarily contemplate with the second definition while the normalization according to (i) will be introduced on dimensional reasons as well as employing classical arguments. In order to rigorously derive such laws directly from the MPC equations we employ the previously defined idea of a group invariant solution.

*This appears to be the driving mechanism for statistical turbulence quantities which have the strong tendency to establish invariant solutions of the MPC equations while at the same time maximizing the number of involved symmetries being only limited by the boundary condtion.*

In the remaining two subsections we adopt the latter condition for the derivation of the accordant invariant solution alternatively later also named turbulent scaling laws, which is the usual phrase in the turbulence literature.

### 4.1 Stationary wall-bounded turbulent shear flows

Due to its eminent practical importance wall-bounded shear flows are by far the most intensively investigated turbulent flow thereby employing a vast number of numerical, experimental and modeling approaches and this, in fact, for more than a century.

From all the theoretical approaches the universal law of the wall is the most widely cited and also accepted approach with its essential ingredient being the logarithmic law of the wall. Though a variety of different approaches have been put forward for its derivation neither of them have employed the full multi-point equations, which are the basis for statistical turbulence, nor do they solve an equation that is related to the Navier-Stokes equations.

In the following we demonstrate that the log-law is an invariant solution of the infinite set of multi-point equations and further it is shown that it essentially relies on one of the new symmetry groups (44) or more specifically (47).

Already in Oberlack (1995) it was observed that in the limit of high Reynolds numbers and $|r| \gg \eta_K$ the logarithmic wall law allows for a self-similar solution of the two-point correlation equation (26). This is rather remarkable since for inhomogeneous flows equation (26) is not a partial differential equation in the classic sense but a non-local differential equation. Non-locality is denoted by the fact that for a given point $x$ and $r$ not only the dependent variables and derivatives are connected but also terms "at the point" $x+r$ contribute to the equation. In equation (26) this is given by the last term in the first row and the first term in the second row.

Indeed, the terms mentioned above were the major cause for the limitation of the two-point correlation equation to be only applicable to homogeneous flows and, even more challenging, if the Fourier transformed version was considered. In particular the above mentioned non-local terms may not be transformed into Fourier space. It is important to note that this limitation is not present for the symmetry approach.

Within this subsection we exclusively examine wall-parallel turbulent flows only depending on the wall-normal coordinate $x_2$. Further, we only explicitly write the two-point correlation $R_{ij}$ though all results are also valid for all higher order correlations. This finally yields

$$\bar{U}_1 = \bar{U}_1(x_2), \;\; R_{ij} = R_{ij}(x_2, r), \;\; \dots \tag{50}$$

With these geometrical assumptions we identify a reduced set of groups. Necessary for the calculation of the subsequent scaling laws are the two scaling groups $\bar{T}_2$ and $\bar{T}_3$ and the translation invariance form $\bar{T}_7 - \bar{T}_9$ in $x_2$-direction in (42). Additionally, it is necessary to use the above-mentioned statistical symmetries, especially the translational group in correlation space (46), the translational group (47) for $\bar{U}_1$, the translational group (48) for $R_{ij}$ and finally the scaling group (49)

With the know symmetries of the MPC equations we may in a final step generate invariants and as a consequence invariant solutions as have been done for the heat equation above in equation (38). As was noted above the actual calculations may be done invoking infinitesimal transformations (see Bluman *et al.*, 2009) leading to an equivalent form of the invariance condition.

For the reason of briefness we skip the lengthy computations and obtain the invariance condition for the MPC equation

$$\begin{aligned}
&\frac{\mathrm{d}x_2}{k_2 x_2 + k_{x_2}} = \frac{\mathrm{d}r_{[k]}}{k_2 r_{[k]} + k_{r_{[k]}}} = \frac{\mathrm{d}\bar{U}_1}{(k_2 - k_3 + k_s)\bar{U}_1 + k_{\bar{U}_1}} \\
&= \frac{\mathrm{d}R_{[ij]}}{\xi_{R_{[ij]}}} = \cdots \;\text{ with} \\
&\xi_{R_{ij}} = (2k_2 - 2k_3 + k_s)R_{ij} - \big(k_s \bar{U}_1(x_2)\bar{U}_1(x_2 + r_2) + \\
&\quad k_{\bar{U}_1}\left(\bar{U}_1(x_2) + \bar{U}_1(x_2 + r_2)\right)\big)\,\delta_{i1}\delta_{j1} + k_{R_{ij}}\,,
\end{aligned} \tag{51}$$

where no summation is implied by the indices in square brackets and instead a concatenation is implied where the indices are consecutively assigned its values. For brevity explicit dependencies on the independent variables are only given where there is an unambiguity. Any solution of (51) for an arbitrary set of parameters $k_i$ generates a set of invariants which are in fact invariant solutions and hence if implemented into the MPC equation leads to a symmetry reduction.

In fact, with a distinct combinations of parameters $k_2$, $k_3$ and $k_s$ a multitude of flows may be described where here we first focus on the log-law. We may keep in mind that $\bar{U}_1$ exclusively depends on $x_2$ and not on $r$.

Considering the classic case of the logarithmic wall law the reason of the appearing symmetry breaking can be found by revisiting the key idea of von Kármán. He assumed that close to the wall the wall-friction velocity $u_\tau$ is the only parameter determining the flow. This condition causes a symmetry breaking of the form $k_2 - k_3 + k_s = 0$ (see Oberlack & Rosteck, 2010).

Under this assumption (51) leads to the extended classical functional form of the mean velocity

$$\bar{U}_1 = \frac{k_{\bar{U}_1}}{k_2} \ln\left(x_2 + \frac{k_{x_2}}{k_2}\right) + C_{\log} \tag{52}$$

and the invariant correlations read

$$\begin{aligned}
\tilde{r}_k &= \frac{r_k}{x_2 + \frac{k_{x_2}}{k_2}}, \\
R_{ij} &= \left(x_2 + \frac{k_{x_2}}{k_2}\right)^{-\frac{k_s}{k_2}} \tilde{R}_{ij}(\tilde{r}) + \frac{k_{R_{ij}}}{k_s} \quad \text{for } ij \neq 11 \\
R_{11} &= \tilde{R}_{11}(\tilde{r}) \left(x_2 + \frac{k_{x_2}}{k_2}\right)^{-k_s/k_2} - \frac{k_{\bar{U}_1}^2}{k_2^2} \ln^2\left(x_2 + \frac{k_{x_2}}{k_2}\right) \\
&\quad - \left(2\frac{k_{\bar{U}_1}}{k_2} C_{log} + \frac{k_{\bar{U}_1}^2}{k_2^2} \ln(\tilde{r}_2 + 1)\right) \ln\left(x_2 + \frac{k_{x_2}}{k_2}\right) \\
&\quad - \left(\frac{C_{log} k_{\bar{U}_1}}{k_2} \ln(\tilde{r}_2 + 1) + C_{log}^2 - \frac{k_{R_{11}}}{k_s}\right), \ \dots\ ,
\end{aligned} \tag{53}$$

where $C_{\log}$ in (52) and the variables marked with "~" in (53) are constants of integration. These are the new invariant coordinates solely depending on $\tilde{r}$. $C_{\log}$ is an exception since according to (50) $\bar{U}_1$ depends on $x_2$ alone.

Obviously equation (52) is a slightly generalized form of the classic logarithmic wall law since by the term $\frac{k_{x_2}}{k_2}$ a displacement of the origin is admitted. In its classical dimensionless form it reads

$$u^+ = \frac{1}{\kappa} \ln(x_2^+ + A^+) + C \ . \tag{54}$$

Moreover, the two-point correlation can be reduced to Reynolds stresses by taking the limit $r \to 0$ so that we gain

$$R_{ij} = (x_2^+ + A^+)^{\gamma} D_{ij} + B_{ij}\,, \quad ij \neq 11 \tag{55}$$

$$\begin{aligned}
R_{11} &= D_{11}(x_2^+ + A^+)^{\gamma} - \frac{1}{\kappa^2} \ln^2(x_2^+ + A^+) \\
&\quad - 2\frac{1}{\kappa} C \ln(x_2^+ + A^+) + B_{11}
\end{aligned} \tag{56}$$

where the new constants $\gamma$, $D_{ij}$ and $B_{ij}$ are combinations of the $k_\alpha$ in equation (53). It is remarkable to note, that $\gamma$ is the same constant in all higher moments, so that the main behavior of these scaling laws only depends on a reduced set of parameters.

As betokened above the non-locality of the two- and multi-point correlation equations appears not to hinder the analysis and, of course, a reduced form of the equation may be given for any multi-point correlation tensor equation.

The invariant variable $\tilde{r}$ in (53) for the logarithmic wall law was already given by Hunt *et al.* (1987) though in a less general form and has been numerically verified by DNS data of a turbulent channel flow. Especially for the two-point correlation tensor $R_{22}$ this data matches very well (not shown here).

## 4.2 Rotating channel

The second application to be focussed on is the rotating channel flow, where different rotational axes will be considered.

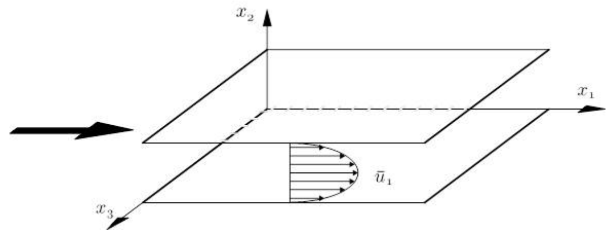


Figure 1. Flow geometry of the pressure driven channel flow.

Using our symmetry analysis in order to gain scaling laws, the calculated symmetries have to be transformed into the coordinate system of a rotating frame. Then the invariant system can be developed and for each rotational axis the symmetries used in this case must be determined.

This leads to a rather complex and involved form of the operator (51) so details have to be omitted and only results for the mean flow will be given.

We first assume that the rotational axis lies along the $x_3$ direction i.e. only $\Omega_3$ is non-zero. Applying Lie symmetry analysis the classical symmetries i.e. scaling in space and the Galilei invariance are used and they are extended by the action of the new scaling symmetry (49) and the translation of the velocities (47). Solving the resulting system for the invariant solution, we gain

$$\bar{U}_1(x_2) = C x_2^{\beta} + A\,, \tag{57}$$

where $C$, $\beta$ and $A$ are constants

Assuming that $\beta = 1$ and re-scaling based on $\Omega_3$ we obtain the well-known scaling law for a rotating channel about the $x_3$-axis (see Oberlack, 2001)

$$\bar{U}_1(x_2) = \alpha_{rot} \Omega_3 x_2 + \bar{U}_{cl}\,. \tag{58}$$

A clear validation of (58) is given in figure 2 for various $\Omega_3$ taken from the DNS of Kristoffersen & Andersson (1993). Interesting enough the value for $\alpha_{rot}$ appears to be very close to 2.

Next, assuming rotation about $x_2$, two velocity components $\bar{U}_1$ and $\bar{U}_3$ have to be taken into consideration since the Coriolis force induces a cross flow. Again, both averaged velocities may only depend on $x_2$. Different to the first case is that one additional symmetry appears, namely translation in time i.e. $\bar{T}_1'$ in (43). From this we derive the new $\Omega_2$ depend-

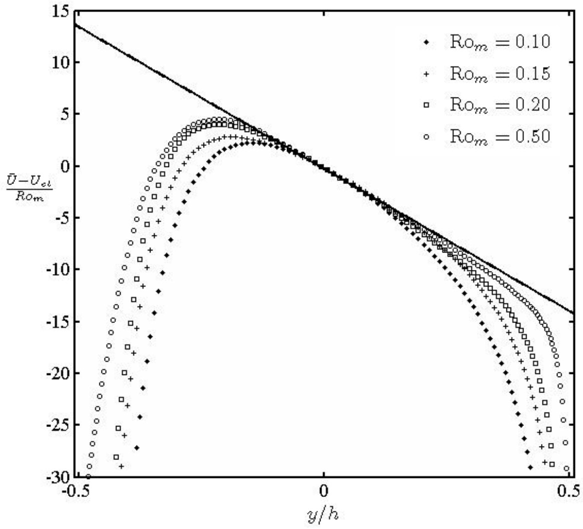


Figure 2. Comparison of the scaling law ($-$) in (58) with the DNS data ($\cdots$) of Kristoffersen & Andersson (1993) at various rotation rates $Ro_m = \frac{\Omega_3 h}{u_m}$ and $Re_\tau = 194$.

ing scaling laws

$$
\begin{aligned}
\bar{U}_1 = \left(\frac{y}{h}\right)^b & \left[a_1 \cos\left(cRo_2 \cdot \ln\frac{y}{h}\right)\right. \\
& \left. + a_2 \sin\left(cRo_2 \cdot \ln\frac{y}{h}\right)\right] + d_1(Ro_2) \\
\bar{U}_3 = \left(\frac{y}{h}\right)^b & \left[a_1 \sin\left(cRo_2 \cdot \ln\frac{y}{h}\right)\right. \\
& \left. - a_2 \cos\left(cRo_2 \cdot \ln\frac{y}{h}\right)\right] + d_2(Ro_2)\,. \qquad (59)
\end{aligned}
$$

A first comparison to DNS data is done for the non-rotating case. Here we employ the DNS data of Hoyas & Jimenez (2006) at $Re_\tau = 2003$ and compare them the scaling law (59a) with $\Omega_2 = 0$. Rewriting it in defect scaling we obtain

$$
\frac{U_{cl} - \bar{U}_1}{u_\tau} = a\left(\frac{y}{h}\right)^b \qquad (60)
$$

where $U_{cl}$ is the velocity at the center of the channel and $u_\tau$ is the friction velocity. Figure 3 shows an almost perfect agreement of the scaling law (60) with the latter DNS where the parameters are fitted to $a = 6.43$ and $b = 1.93$.

Finally we consider the rotating case and compare the DNS data of Mehdizadeh & Oberlack (2010) at $Re_\tau = 360$ with the scaling law (59). Results are depicted for two different rotation numbers $Ro_2$ with

$$
Ro_2 = \frac{2\Omega_2 h}{u_{\tau 0}} \qquad (61)
$$

in the figures 4 and 5 exhibiting an excellent fit in the center of the channel for all cases. $u_{\tau 0}$ refers to the friction velocity of the non-rotating case. It is to note from all the DNS data

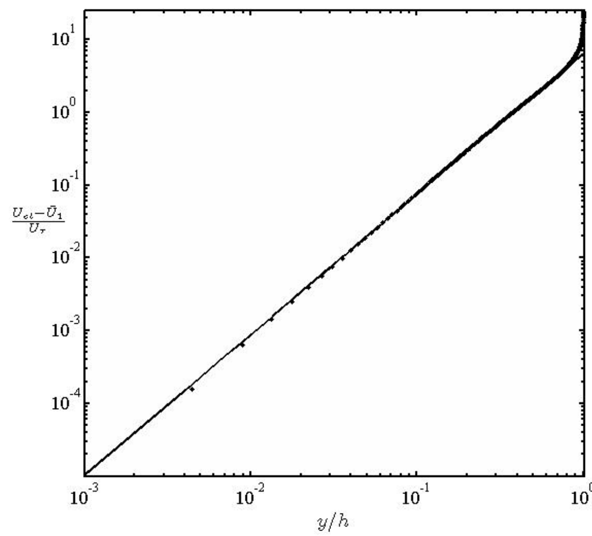


Figure 3. Comparison of the scaling law ($-$) in (60) with the DNS data ($\cdots$) of Hoyas & Jimenez (2006) at $Re_\tau = 2003$.

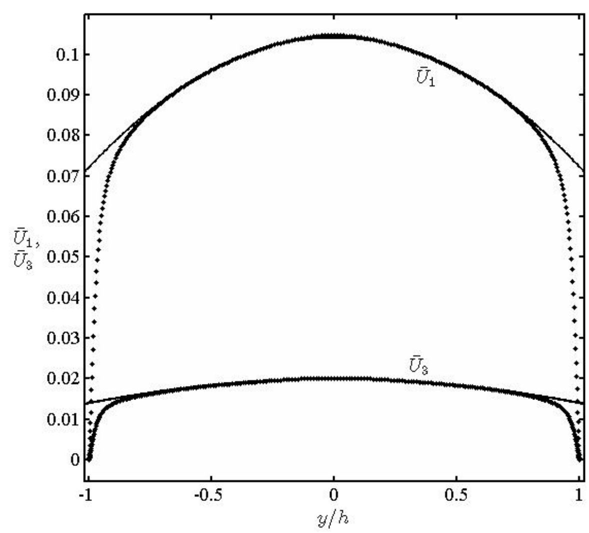


Figure 4. Comparison of the scaling law ($-$) in (59) with the DNS data ($\cdots$) of Mehdizadeh & Oberlack (2010) at $Ro_2 = 0.011$.

sets in Mehdizadeh & Oberlack (2010) we find that with an increasing $\Omega_2$ the magnitude of $\bar{U}_1$ and $\bar{U}_3$ switch position since with increasing rotation rates $\bar{U}_1$ is suppressed while $\bar{U}_3$ increases up to a certain point and decreases again though to a smaller extend compared to $\bar{U}_1$. This behavior is exactly described by the scaling law (59).

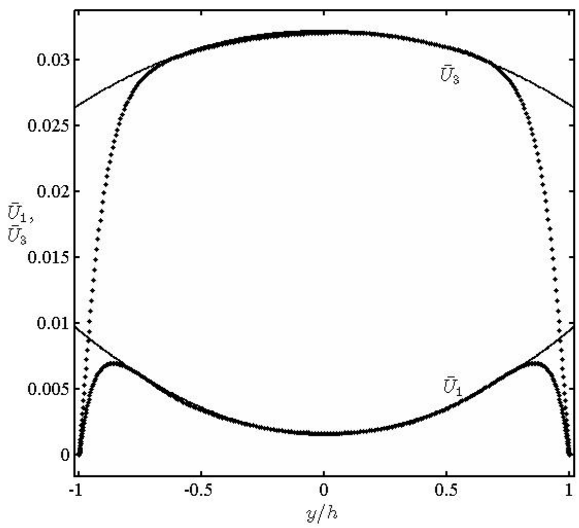


Figure 5. Comparison of the scaling law ($-$) in (59) with the DNS data ($\cdots$) of Mehdizadeh & Oberlack (2010) at $Ro_2 = 0.18$.

## 5 Summary and outlook

Within the present contribution it was shown that the admitted symmetry groups of the infinite set of multi-point correlation equations are considerable extended by three classes of groups compared to those originally stemming from the Euler and the Navier-Stokes equations. In fact, it was demonstrated that it is exactly these symmetries which are essentially needed to validate certain classical scaling laws such as the log-law from first principles and also to derive a large set of new scaling laws.

Implicitly, symmetries have been used in turbulence modeling for several decades since essentially all symmetries of Euler and the Navier-Stokes equations have been made part of modern turbulence models. Still, this is only partially true for the new statistical symmetries. In fact, some of them have been employed even in very early turbulence models since many of them where calibrated against the log-law. Many other symmetries, however, have never been made use of and in fact, it might be even impossible to make turbulence models consistent with some of the symmetries such as the new scaling symmetry (45).

Still, even with these new symmetry groups at hand which give a much deeper understanding on turbulence statistics there are still some key open questions to be answered. (i) So far completeness of all admitted symmetries of the MPC equation has not been shown. This appears to be necessary not only from a theoretical point of few but rather essential to generate scaling for all higher moments. (ii) From turbulence data it is apparent that the appearing group parameters do have certain decisive values which are to be determined. In some very rare cases such as the classical decaying turbulence case values such as the decay exponent may be determined from integral invariants. Still, a general scheme is unknown. (iii) Finally, we clearly observe that certain scaling laws such as the log law only cover certain regions of a turbulent flow and are usually embedded within other layers of turbulence. Still, the matching of turbulent scaling laws is still an open question.

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