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MODELING THE TURBULENT CROSS-HELICITY DISSIPATION RATE

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ABSTRACT

It has been recognized that the turbulent cross helicity (correlation between the velocity and magnetic-field fluctuations) can play an important role in several magnetohydrodynamic (MHD) plasma phenomena such as the global magnetic-field generation, turbulence suppression, etc. Despite its relevance to the cross-helicity evolution, little attention has been given to the dissipation rate of the turbulent cross helicity. In this paper, we consider the dissipation rate of the turbulent cross helicity and propose an algebraic model and an evolution equation of the cross-helicity dissipation rate (ε_W equation). We apply the model to the solar-wind turbulence, where several observations have been made on the turbulent cross helicity, and validate the model of cross-helicity dissipation. It is shown that, as far as the solar-wind application is concerned, the simplest possible algebraic model is useful enough to elucidate the spatial evolution of the solar-wind turbulence.

INTRODUCTION

In the magnetohydrodynamic (MHD) turbulent flow at high magnetic Reynolds number ($Rm \gg 1$), magnetic fields are considered to be frozen in plasmas, and move with the flow (Alfvén, 1950). In such a flow, the induced magnetic field is often much larger than the originally imposed field. Besides, MHD waves such as the Alfvén wave are ubiquitously observed. The cross helicity, defined by the correlation between the velocity $\mathbf{u}$ and magnetic field $\mathbf{b}$, is a possible describer of such MHD turbulence properties. Actually, the magnetic-field generation due to the turbulent cross helicity has been investigated (Yoshizawa, 1990, 1998; Yoshizawa and Yokoi, 1993; Yoshizawa et al., 2003; Yokoi, 1996, 1999).

As is well known, the total amount of cross helicity $\int_V \mathbf{u}\cdot\mathbf{b}dV$, as well as that of the MHD energy $\int_V (\mathbf{u}^2+\mathbf{b}^2)/2\ dV$, is an inviscid invariant of the incompressible MHD equations. Thanks to this conservative property, the turbulent MHD energy and cross-helicity densities, $K \equiv \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle/2$ and $W \equiv \langle \mathbf{u}'\cdot\mathbf{b}' \rangle$, play a central role in the turbulence modeling of MHD fluids ($\mathbf{u}'$: velocity fluctuation, $\mathbf{b}'$: magnetic-field fluctuation, $\langle\cdots\rangle$: ensemble average). The equations of K and W are similar in form and their mathematical structures are quite simple. They are written as

$$\frac{DG}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right) G = P_G - \varepsilon_G + \nabla\cdot\mathbf{T}_G, \qquad (1)$$

where $G = (K, W)$ and $\mathbf{U}$ is the mean velocity. Here, P_G, ε_G, and $\mathbf{T}_G$ are the production, dissipation, and transport rates of the turbulent statistical quantity G. They are defined by

$$P_K = -\mathcal{R}^{ab}\frac{\partial U^a}{\partial x^b} - \mathbf{E}_\mathrm{M}\cdot\mathbf{J}, \qquad (2a)$$

$$\varepsilon_K = \nu\left\langle\left(\frac{\partial u'^a}{\partial x^b}\right)^2\right\rangle + \lambda\left\langle\left(\frac{\partial b'^a}{\partial x^b}\right)^2\right\rangle \equiv \varepsilon, \qquad (2b)$$

$$\mathbf{T}_K = W\mathbf{B} - \left\langle\left(\frac{\mathbf{u}'^2+\mathbf{u}'^2}{2} + p'_\mathrm{M}\right)\mathbf{u}' + \left(\mathbf{u}'\cdot\mathbf{b}'\right)\mathbf{b}'\right\rangle, \qquad (2c)$$

$$P_W = -\mathcal{R}^{ab}\frac{\partial B^a}{\partial x^b} - \mathbf{E}_\mathrm{M}\cdot\boldsymbol{\Omega}, \qquad (3a)$$

$$\varepsilon_W = (\nu+\lambda)\left\langle\frac{\partial u'^a}{\partial x^b}\frac{\partial b'^a}{\partial x^b}\right\rangle, \qquad (3b)$$

$$\mathbf{T}_W = K\mathbf{B} - \left\langle\left(\mathbf{u}'\cdot\mathbf{b}'\right)\mathbf{u}' - \left(\frac{\mathbf{u}'^2+\mathbf{u}'^2}{2} - p'_\mathrm{M}\right)\mathbf{b}'\right\rangle. \qquad (3c)$$

Here, $\mathbf{B}$ is the mean magnetic field, $\boldsymbol{\Omega}$ the mean vorticity, ν the kinetic viscosity, λ the magnetic diffusivity, and p'_M the fluctuation part of the MHD pressure $p_\mathrm{M} (\equiv p + \mathbf{b}^2/2)$. The Reynolds stress $\boldsymbol{\mathcal{R}}$ and the turbulent electromotive force $\mathbf{E}_\mathrm{M}$ are defined by

$$\mathcal{R}^{\alpha\beta} \equiv \left\langle u'^\alpha u'^\beta - b'^\alpha b'^\beta\right\rangle, \qquad (4)$$

$$\mathbf{E}_\mathrm{M} \equiv \left\langle\mathbf{u}'\times\mathbf{b}'\right\rangle. \qquad (5)$$

In Eq. (3a), P_W represents the W production rate due to the coupling of the inhomogeneities of the mean fields with the fluctuation. It is mainly in the context of P_W associated with the turbulent dynamo that the cross-helicity effect has been investigated so far.

The dissipation rate of the turbulent cross helicity, ε_W, represents the effects of molecular viscosity and magnetic diffusivity connected with the small-scale fluctuations. Some investigations have been made on ε_W in the homogeneous isotropic MHD turbulence studies (Grappin et al., 1982, 1983). However, almost no work has ever been done on ε_W in the context of the inhomogeneous turbulence. This situation may be related to the general tendency of ignoring the turbulent cross helicity itself. Almost sole exception lies in the solar-wind studies by satellite, where details of the spectrum of the cross helicity have been examined (Belcher and Davis, 1971; Roberts et al., 1987; Tu and Marsch, 1995). However, the arguments on the dissipation rate of W are still far from sufficient.

As we see from Eq. (1), in order to properly consider the evolution of the turbulent cross helicity W, it is indispensable to estimate the dissipation rate of W, ε_W. In this work, we will delve into this problem with the aid of the comparisons of the results of turbulence model with the satellite observations.

EQUATION FOR THE CROSS-HELICITY DISSIPATION RATE

From the equations of the fluctuation velocity and magnetic field, we construct the exact equation for the dissipation rate of the cross helicity, ε_W, as

$$\frac{D\varepsilon_W}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla \right) \varepsilon_W$$
$$= (\nu+\lambda) \left\langle \frac{\partial u'^a}{\partial x^c}\frac{\partial b'^b}{\partial x^c} - \frac{\partial b'^a}{\partial x^c}\frac{\partial u'^b}{\partial x^c} \right\rangle \frac{\partial U^a}{\partial x^b}$$
$$+(\nu+\lambda) \left\langle \frac{\partial u'^a}{\partial x^c} b'^b - u'^b \frac{\partial b'^a}{\partial x^c} \right\rangle \frac{\partial^2 U^a}{\partial x^b \partial x^c}$$
$$+(\nu+\lambda) \left\langle \frac{\partial u'^b}{\partial x^a}\frac{\partial^2 u'^b}{\partial x^a \partial x^c} + \frac{\partial b'^b}{\partial x^a}\frac{\partial^2 b'^b}{\partial x^a \partial x^c} \right\rangle B^c$$
$$+(\nu+\lambda) \left\langle \frac{\partial u'^c}{\partial x^a}\frac{\partial u'^c}{\partial x^b} - \frac{\partial b'^c}{\partial x^a}\frac{\partial b'^c}{\partial x^b} \right\rangle \frac{\partial B^a}{\partial x^b}$$
$$-(\nu+\lambda) \left\langle \frac{\partial u'^a}{\partial x^c}\frac{\partial u'^a}{\partial x^c} - \frac{\partial b'^a}{\partial x^c}\frac{\partial b'^a}{\partial x^c} \right\rangle \frac{\partial B^a}{\partial x^b}$$
$$-(\nu+\lambda) \left\langle \frac{\partial u'^a}{\partial x^c} u'^b - \frac{\partial b'^a}{\partial x^c} b'^b \right\rangle \frac{\partial^2 B^a}{\partial x^b \partial x^c}$$
$$-(\nu+\lambda) \left\langle \frac{\partial u'^b}{\partial x^a}\frac{\partial u'^c}{\partial x^a}\frac{\partial b'^b}{\partial x^c} \right\rangle - (\nu+\lambda) \left\langle \frac{\partial b'^b}{\partial x^a}\frac{\partial u'^c}{\partial x^a}\frac{\partial u'^b}{\partial x^c} \right\rangle$$
$$+(\nu+\lambda) \left\langle \frac{\partial u'^b}{\partial x^a}\frac{\partial b'^c}{\partial x^a}\frac{\partial b'^b}{\partial x^c} \right\rangle + (\nu+\lambda) \left\langle \frac{\partial b'^b}{\partial x^a}\frac{\partial b'^c}{\partial x^a}\frac{\partial b'^b}{\partial x^c} \right\rangle$$
$$-(\nu+\lambda) \left\langle (u'^c \pm b'^c) \frac{\partial}{\partial x^c} \left(\frac{\partial u'^b}{\partial x^a}\frac{\partial b'^b}{\partial x^a} \right) \right\rangle$$
$$+(\nu+\lambda) \frac{\partial}{\partial x^c} \left\langle \frac{1}{2} b'^c \frac{\partial}{\partial x^a} \left(u'^b \pm b'^b \right) \right\rangle$$
$$-(\nu+\lambda) \left\langle \frac{\partial b'^b}{\partial x^a}\frac{\partial^2 p'_{\rm M}}{\partial x^a \partial x^b} \right\rangle + (\nu+\lambda) \frac{\partial^2}{\partial x^c \partial x^c} \varepsilon_W$$
$$-(\nu+\lambda) \frac{\partial}{\partial x^c} \left[\nu \left\langle \frac{\partial u'^b}{\partial x^a}\frac{\partial^2 b'^b}{\partial x^a \partial x^c} \right\rangle + \lambda \left\langle \frac{\partial b'^b}{\partial x^a}\frac{\partial^2 u'^b}{\partial x^a \partial x^c} \right\rangle \right]$$
$$-(\nu+\lambda)^2 \left\langle \frac{\partial^2 u'^b}{\partial x^a \partial x^c}\frac{\partial^2 b'^b}{\partial x^a \partial x^c} \right\rangle. \qquad (6)$$

This equation, lacking the connection with a conservative law, has a considerably complicated structure. This is in sharp contrast to the K and W equations [Eq. (1)].

In the hydrodynamic case with an electrically non-conducting fluid, the energy dissipation rate

$$\epsilon \equiv \nu \left\langle \frac{\partial u'^a}{\partial x^b}\frac{\partial u'^a}{\partial x^b} \right\rangle, \qquad (7)$$

as well as the turbulent energy $k \equiv \left\langle \mathbf{u}'^2 \right\rangle /2$, plays a central role in turbulence modeling (Launder and Spalding, 1972). From the equation of the velocity fluctuation, we write the ϵ equation exactly as

$$\frac{D\epsilon}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla \right) \epsilon$$
$$= -2\nu \left\langle \frac{\partial u'^b}{\partial x^a}\frac{\partial u'^c}{\partial x^a}\frac{\partial u'^b}{\partial x^c} \right\rangle - 2 \left\langle \left(\nu \frac{\partial^2 u'^b}{\partial x^a \partial x^c} \right)^2 \right\rangle$$
$$-2\nu \left\langle \frac{\partial u'^a}{\partial x^c}\frac{\partial u'^b}{\partial x^c} - \frac{\partial u'^a}{\partial x^c}\frac{\partial u'^b}{\partial x^c} \right\rangle \frac{\partial U^a}{\partial x^b}$$
$$-2\nu \left\langle u'^b \frac{\partial u'^a}{\partial x^c} \right\rangle \frac{\partial^2 U^a}{\partial x^b \partial x^c}$$
$$+\frac{\partial}{\partial x^c} \left[-\nu \left\langle \left(\frac{\partial u'^b}{\partial x^a} \right)^2 \right\rangle - 2\nu \left\langle \frac{\partial p'}{\partial x^b}\frac{\partial u'^a}{\partial x^b} \right\rangle \right]$$
$$+\nu \frac{\partial^2}{\partial x^c \partial x^c} \epsilon. \qquad (8)$$

The mathematical structure of this equation is also complicated because of the lack of the connection with a conservative law.

The energy dissipation ϵ itself is dominant at small scales. Using this, the length and velocity scales are estimated as

$$|\mathbf{x}| \sim \nu^{3/4}\epsilon^{-1/4}, \quad |\mathbf{u}'| \sim \nu^{1/4}\epsilon^{1/4}, \qquad (9)$$

respectively. Using Eq. (9), we can estimate each term in Eq. (8). In the flow at high Re ($\nu \to 0$), two terms behaving as $O(\nu^{-1/2}\epsilon^{3/2})$ are dominant, and these two terms should balance each other (Tennekes and Lumley, 1972; Yoshizawa, 1998)

$$-2\nu \left\langle \frac{\partial u'^a}{\partial x^b}\frac{\partial u'^a}{\partial x^c}\frac{\partial u'^b}{\partial x^c} \right\rangle \sim 2 \left\langle \left(\nu \frac{\partial^2 u'^a}{\partial x^b \partial x^c} \right)^2 \right\rangle. \qquad (10)$$

In other word, in the modeling of the ϵ equation, it is of crucial importance to properly estimate the two terms in Eq. (10).

In the hydrodynamic turbulence modeling, an empirical model equation for ϵ:

$$\frac{D\epsilon}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla \right) \epsilon = C_{\epsilon 1}\frac{\epsilon}{k}P_k - C_{\epsilon 2}\frac{\epsilon}{k}\epsilon + \nabla \left(\frac{\nu_{\rm T}}{\sigma_\epsilon}\nabla\epsilon \right) \qquad (11)$$

was proposed and has been widely accepted as useful.

Here, $C_{\epsilon 1}$, $C_{\epsilon 2}$, and σ_ϵ are model constants. The values of these constants have been optimized through various applications of the $k - \epsilon$ model. Usually, the values of

$$C_{\epsilon 1} = 1.4, \quad C_{\epsilon 2} = 1.9, \quad \sigma_\epsilon = 1.0 \qquad (12)$$

are adopted (Launder and Spalding, 1972).

Equation (11) is based on an empirical inference that the dissipation should be larger where the turbulence is larger, and is derived by dimensional analysis. However, it is also pointed out that the system of constants given by Eq. (12) is not a unique combination for the model constants because ϵ equation can take an infinite number of self-similar states (Rubinstein and Clark, 2005).

A similar argument using Eq. (9) can be applied to the exact equation for the dissipation rate of the turbulent cross helicity [Eq. (6)]. The difference between the molecular viscosity ν and the magnetic diffusivity λ is expressed by the magnetic Prandtl number:

$$Pm \equiv \nu/\lambda. \qquad (13)$$

If we estimate each term in Eq. (6) under the simplest possible condition of $Pm = 1$, we have a dominant balance between the terms expressed by

$$(\nu+\lambda) \left\langle \frac{\partial u'^b}{\partial x^a}\frac{\partial u'^c}{\partial x^a}\frac{\partial b'^b}{\partial x^c} \right\rangle \sim \nu^{-1/2}\varepsilon^{3/2}, \qquad (14a)$$

$$(\nu+\lambda)^2 \left\langle \frac{\partial^2 u'^b}{\partial x^a \partial x^c}\frac{\partial^2 b'^b}{\partial x^a \partial x^c} \right\rangle \sim \nu^{-1/2}\varepsilon^{3/2} \qquad (14b)$$

in the ε_W equation at the high Reynolds number flow.

MODELS FOR THE DISSIPATION OF THE TURBULENT CROSS HELICITY

Algebraic model

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As was mentioned in the previous section, the equation governing the dissipation rate of W, ε_W, has very complicated form. In such a situation, the simplest possible model for ε_W is the algebraic approximation as follows.

Using the turbulent MHD energy K and its dissipation rate ε, we construct a characteristic time scale of turbulence as

$$\tau = K/\varepsilon. \tag{15}$$

With the aid of this time scale, the dissipation rate of W can be modeled as

$$\varepsilon_W = C_W \frac{\varepsilon}{K} W, \tag{16}$$

where C_W is the model constant. Namely, we consider the dissipation of the turbulent cross helicity is proportional to the turbulent cross helicity divided by the time scale.

It is worth noting a mathematical constraint on the cross helicity: the magnitude of the turbulent cross helicity W is bounded by the magnitude of the turbulent MHD energy K as

$$\frac{|W|}{K} = \frac{|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle|}{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2} \le 1. \tag{17}$$

This relation constrains the value of ε_W. Actually, from Eq. (1), the turbulent cross helicity scaled by the turbulent MHD energy, W/K, is subject to

$$\begin{aligned} \frac{D}{Dt}\frac{W}{K} &= \frac{W}{K}\left(\frac{1}{W}\frac{DW}{Dt} - \frac{1}{K}\frac{DK}{Dt}\right) \\ &= \frac{W}{K}\left(\frac{1}{W}P_W - \frac{1}{K}P_K\right) - \frac{W}{K}\left(\frac{1}{W}\varepsilon_W - \frac{1}{K}\varepsilon\right) \\ &+ \frac{W}{K}\left(\frac{1}{W}T_W - \frac{1}{K}T_K\right). \end{aligned} \tag{18}$$

In the homogeneous turbulence, where the production and transport rate vanish, Eq. (18) is reduced to

$$\frac{\partial}{\partial t}\frac{W}{K} = -\left(\frac{1}{W}\varepsilon_W - \frac{1}{K}\varepsilon\right)\frac{W}{K}. \tag{19}$$

It follows that the condition for $|W|/K$ being below the unity [Eq. (17)] is expressed as

$$\frac{|\varepsilon_W|}{\varepsilon} > \frac{|W|}{K}. \tag{20}$$

Model equation for the cross-helicity dissipation rate (ε_W equation)

We see from Eq. (1) that the equations of W and K are written in a similar form. So, it is natural to consider that the equation of the dissipation rate of W, ε_W, can be expressed in a form similar to the equation of the K dissipation rate ε. Then we consider the equation for ε_W as

$$\frac{D\varepsilon_W}{Dt} = C_{W1}\frac{\varepsilon}{K}P_W - C_{W2}\frac{\varepsilon}{K}\varepsilon_W + \nabla \cdot \left(\frac{\nu_K}{\sigma_{\varepsilon W}}\nabla \varepsilon_W\right), \tag{21}$$

where C_{W1}, C_{W2}, and $\sigma_{\varepsilon W}$ are the model constants.

In the equation of the K dissipation rate, ε equation [Eq. (38) below], the model constants are same as the ones appearing in the dissipation rate of the hydrodynamic (HD) or non-MHD turbulence energy, ϵ equation [Eq. (11)]. This is a consequence of the requirement that the ε equation for MHD turbulence should be reduced to the ϵ equation for the HD in the limit of the vanishing magnetic field ($\mathbf{B} = 0, \mathbf{b}' = 0$). In contrast, the cross helicity, defined by the correlation between the velocity and magnetic field, vanishes in the limit of the vanishing magnetic field. As this result, as far as the ε_W equation is concerned, we can not make use of the knowledge accumulated in the history of HD turbulence modeling. In this sense, at present, we do not have enough information regarding the model constants C_{W1}, C_{W2}, and $\sigma_{\varepsilon W}$ in Eq. (21).

A MODEL FOR MHD TURBULENCE

One of the prominent features of turbulence is its wide ranges of scales. We have continuous scales of motions ranging from the energy-containing scale, in which energy is injected to turbulence through the inhomogeneity of mean fields, to the dissipation or Kolmogorov scale where the dissipation plays a dominant role. Owing to this breadth of scales, it is impossible in the foreseeable future to simultaneously solve all the scales of turbulence at the high Reynolds number encountered in the real-world flow of interests. In such a situation, the notion of turbulence modeling provides a useful tool for analyzing the real-world turbulence phenomena. In the turbulence modeling, small-scale motions are modeled, and turbulence effects are incorporated into the analysis of the large-scale or mean motions.

In an MHD fluid, the mean fields obey
Equation of continuity:

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\bar{\rho}\mathbf{U}) = 0, \tag{22}$$

Momentum equation:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{UU} - \mathbf{BB}) = -\nabla P_{\rm M} + \nabla \cdot \boldsymbol{\mathcal{R}} + \nu \nabla^2 \mathbf{U}, \tag{23}$$

Magnetic induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \mathbf{E}_{\rm M} + \lambda \nabla^2 \mathbf{B}, \tag{24}$$

Solenoidal condition for the magnetic field:

$$\nabla \cdot \mathbf{B} = -\frac{1}{2}(\mathbf{B} \cdot \nabla) \ln \bar{\rho}. \tag{25}$$

Here, $\bar{\rho}$ is the mean density. For the sake of simplicity, the fluctuation part of the density has been neglected ($\rho' = 0$). Note that the magnetic field etc. are measured in the Alfvén speed unit. They are related to the ones measured in the original unit (asterisked) as

$$\mathbf{b} = \frac{\mathbf{b}_*}{(\mu_0 \rho)^{1/2}}, \quad \mathbf{j} = \frac{\mathbf{j}_*}{(\rho/\mu_0)^{1/2}}, \quad \mathbf{e} = \frac{\mathbf{e}_*}{(\mu_0 \rho)^{1/2}}. \tag{26}$$

Measured in the Alfvén speed unit, the solenoidal condition of the original mean magnetic field:

$$\nabla \cdot \mathbf{B}_* = 0 \tag{27}$$

is expressed as Eq. (25).

Equations (23) and (24) show that the Reynolds stress $\boldsymbol{\mathcal{R}}$ and the turbulent electromotive force $\mathbf{E}_{\rm M}$ defined by Eqs. (4) and (5), represent the turbulence effects in the mean-field equations. Estimating these correlations are of central importance in the study of inhomogeneous turbulence. In order to close the system of equations they should be modeled in terms of the mean-field quantities. Here, we adopt

$$\mathcal{R}^{\alpha\beta} = \frac{2}{3}K_{\rm R}\delta^{\alpha\beta} - \nu_{\rm K}\mathcal{S}^{\alpha\beta} + \nu_{\rm M}\mathcal{M}^{\alpha\beta}, \tag{28}$$

$$\mathbf{E}_{\rm M} = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U}, \tag{29}$$

where $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{M}}$ are the strain rates of the mean velocity and the mean magnetic field:

$$\mathcal{S}^{\alpha\beta} = \frac{\partial U^a}{\partial x^b} + \frac{\partial U^b}{\partial x^a} - \frac{2}{3}\delta^{\alpha\beta}\nabla\cdot\mathbf{U}, \tag{30a}$$

$$\mathcal{M}^{\alpha\beta} = \frac{\partial B^a}{\partial x^b} + \frac{\partial B^b}{\partial x^a} - \frac{2}{3}\delta^{\alpha\beta}\nabla\cdot\mathbf{B}. \tag{30b}$$

We should note that the transport coefficients appearing in Eqs. (28) and (29), $\nu_{\rm K}$, $\nu_{\rm M}$, α, β, and γ, are not adjustable parameters, but should be determined self-consistently through the dynamic properties of turbulence. They in general depend on location and time. In this work, we adopt model expressions for them as

$$\alpha = C_\alpha \tau H, \quad \beta = \frac{5}{7}\nu_{\rm K} = C_\beta \tau K, \quad \gamma = \frac{5}{7}\nu_{\rm M} = C_\gamma \tau W, \tag{31}$$

where τ denotes the time scale of turbulence. The model constants C_α, C_β, and C_γ are estimated as

$$C_\alpha \simeq 0.02, \quad C_\beta \simeq 0.05, \quad C_\gamma \simeq 0.04 \tag{32}$$

(Yoshizawa, 1998).

In the Reynolds-averaged turbulence model, properties of turbulence are represented by some statistical quantities. We adopt the turbulent MHD energy K, its dissipation rate ε, the turbulent cross helicity W, and the turbulent residual energy $K_{\rm R}$ as the turbulence statistical quantities. They are defined by

$$K \equiv \frac{1}{2}\left\langle \mathbf{u}'^2 + \mathbf{b}'^2 \right\rangle, \tag{33}$$

$$\varepsilon \equiv \nu \left\langle \left(\frac{\partial u'^a}{\partial x^b} \right)^2 \right\rangle + \lambda \left\langle \left(\frac{\partial b'^a}{\partial x^b} \right)^2 \right\rangle, \tag{34}$$

$$W \equiv \left\langle \mathbf{u}'\cdot\mathbf{b}' \right\rangle, \tag{35}$$

$$K_{\rm R} \equiv \frac{1}{2}\left\langle \mathbf{u}'^2 - \mathbf{b}'^2 \right\rangle. \tag{36}$$

In this work, we assume the α- or turbulent residual-helicity [$H(\equiv \langle -\mathbf{u}'\cdot\boldsymbol{\omega}' + \mathbf{b}'\cdot\mathbf{j}'\rangle)$]-related term in Eq. (29) is much smaller than the γ- or cross-helicity-related term, and neglect the former. So, we do not include the evolution equation of H. As for the equation of H, you are referred to Yoshizawa (1998; 1999) and Yokoi et al. (2008).

If we take the effects of mean-density variation into account, the evolution equations for the statistical quantities are written as

$$\begin{aligned}\frac{\partial K}{\partial t} =& -(\mathbf{U}\cdot\nabla)K - \frac{1}{6}(3K + K_{\rm R})\nabla\cdot\mathbf{U} \\ &+ \frac{1}{2}(\nu_{\rm K}\boldsymbol{\mathcal{S}}^2 - \nu_{\rm M}\boldsymbol{\mathcal{S}}:\boldsymbol{\mathcal{M}}) + \beta\mathbf{J}^2 - \gamma\boldsymbol{\Omega}\cdot\mathbf{J} \\ &- \varepsilon + \nabla\cdot(W\mathbf{B}) + \nabla\cdot(\nu_{\rm K}\nabla K),\end{aligned} \tag{37}$$

$$\frac{\partial \varepsilon}{\partial t} = -(\mathbf{U}\cdot\nabla)\varepsilon + C_{\varepsilon 1}\frac{\varepsilon}{K}P_K - C_{\varepsilon 2}\frac{\varepsilon}{K}\varepsilon + \nabla\cdot\left(\frac{\nu_K}{\sigma_\varepsilon}\nabla\varepsilon\right), \tag{38}$$

$$\begin{aligned}\frac{\partial W}{\partial t} =& -(\mathbf{U}\cdot\nabla)W - \frac{1}{2}W\nabla\cdot\mathbf{U} \\ &+ \frac{1}{2}\left(\nu_{\rm K}\boldsymbol{\mathcal{S}}:\boldsymbol{\mathcal{M}} - \nu_{\rm M}\boldsymbol{\mathcal{M}}^2\right) + \beta\boldsymbol{\Omega}\cdot\mathbf{J} - \gamma\boldsymbol{\Omega}^2 \\ &- \varepsilon_W + \nabla\cdot(K\mathbf{B}) + \nabla\cdot\left(\frac{\nu_K}{\sigma_W}\nabla W\right),\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{\partial K_{\rm R}}{\partial t} =& -(\mathbf{U}\cdot\nabla)K_{\rm R} - \frac{1}{6}(K + 3K_{\rm R})\nabla\cdot\mathbf{U} \\ &+ \frac{1}{2}\nu_{\rm R}(\boldsymbol{\mathcal{S}}^2 - \boldsymbol{\mathcal{M}}^2) - \varepsilon_{\rm R} + \nabla\cdot\left(\frac{\nu_{\rm K}}{\sigma_{\rm R}}\nabla K_{\rm R}\right).\end{aligned} \tag{40}$$

Here $\nu_{\rm R}$ denotes the residual turbulent viscosity defined by

$$\nu_{\rm R} = \nu_{\rm K}\frac{K_{\rm R}}{K}. \tag{41}$$

The dissipation rate of the turbulent cross helicity, ε_W [Eq. (3b)], is estimated by Eq. (16) or alternatively, by Eq. (21).

The dissipation rate of $K_{\rm R}$, $\varepsilon_{\rm R}$, can be expressed as

$$\varepsilon_{\rm R} = \left(1 + C_r\frac{\mathbf{B}^2}{K}\right)\frac{\varepsilon}{K}K_{\rm R} \tag{42}$$

(Yokoi 2006). Here, C_r and $\sigma_{\rm R}$ are positive model constants. The large-scale behavior of $K_{\rm R}$ depends on the choice of these constants. At this stage of modeling, we do not insist on fine tuning of the model constants, but roughly put them

$$C_r = 0.01, \quad \sigma_{\rm R} = 1.0. \tag{43}$$

Equation (42) as a whole will destruct the turbulent residual energy, and return MHD turbulence to equipartition. The first part in the parentheses of Eq. (42) represents the $K_{\rm R}$ destruction due to the eddy distortion. The second part represents the destruction of $K_{\rm R}$ due to the mean magnetic field. The latter corresponds to the Alfvén effect, in which the presence of the mean magnetic field leads MHD turbulence to an equipartition state between the kinetic and magnetic energies (Iroshnikov, 1964; Kraichnan, 1965; Pouquet et al., 1976; Yokoi, 2006).

Equation (42) may be reinterpreted as the modulation of MHD-turbulence time scale due to the mean magnetic field:

$$\tau = \frac{K}{\varepsilon} \quad \rightarrow \quad \left(1 + \chi\frac{\mathbf{B}^2}{K}\right)^{-1}\frac{K}{\varepsilon}, \tag{44}$$

where $\chi(= C_r)$ represents the synthesization ratio of time scales. Equation (43) infers that we should put the ratio as

$$\chi = O(10^{-2}). \tag{45}$$

As for an example considering the synthesized time scale of MHD turbulence, you are referred to Yokoi et al. (2008) and references cited therein.

AN APPLICATION TO THE SOLAR-WIND TURBULENCE

Solar-wind turbulence

Solar wind is a continuous plasma flow blown away from the coronal bases to the solar-system space. Its origin is considered to be the violent magnetic activities on the solar surface. The streams from the low- or mid-latitude region whose typical speed is 400 km s^{-1} are called the slow wind. On the other hand, the streams from the high-latitude region such as coronal hole whose typical speed is 800 km s^{-1} are called the fast wind. It is known that the slow wind has a large velocity shear, while the fast wind has substantially no velocity shears. The influence of an explosive magnetic activity on the solar surface is conveyed to the magnetosphere of the Earth within several days, and terrestrial environments may be affected much. A direct exposure to a high-energy particle convected by the solar wind is quite harmful to astronauts and electric devices on the satellite. So it is highly desirable to predict the behavior of the solar wind: A problem of the space weather (Figure 1).

Satellite observations have revealed the statistical properties of the solar-wind turbulence including the velocity,

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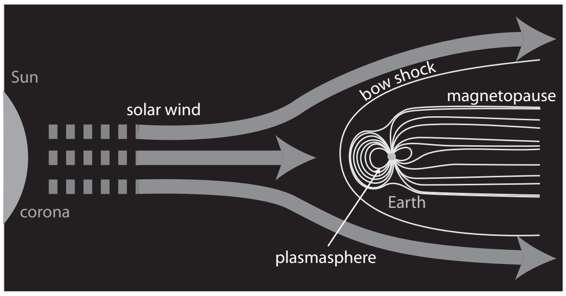

Figure 1: Solar wind and terrestrial environments. The solar wind, a high speed plasma flow of several hundreds km s^{-1}, interacts with the terrestrial magnetosphere to induce several phenomena such as substorms, auroras.

magnetic field, density, etc. It has been investigated how MHD turbulence evolves in the large-scale velocity and magnetic-field structures (Belcher and Davis, 1971; Roberts, et al., 1987; Tu and Marsch, 1995). According to the observations, the solar-wind turbulence shows a strong Alfvénicity near the Sun. Namely, there is a strong correlation between the velocity and magnetic-field fluctuations, and equipartition between the kinetic and magnetic turbulent energies is realized. At the same time, it is pointed out that this Alfvénicity decays as the heliocentric distance increases (Tu and Marsch, 1995). One of the main unsolved problems in this field is the spatial evolutions of the turbulent cross helicity W and the Alfvén ratio r_A (Roberts et al., 1990).

The magnitude of the scaled turbulent cross helicity, $|W|/K$, whose value is almost unity near the Sun, decreases as the heliocentric distance increases. In the region with larger mean-velocity shear, the decay rate of $|W|/K$ is larger. The value of $|W|/K$ far from the Sun is small $(0 - 0.2)$. On the other hand, in the region with smaller mean-velocity shear, the decay of $|W|/K$ is suppressed. The value of $|W|/K$ remains to be large $(0.4 - 0.7)$ even far from the Sun (Goldstein et al., 1990). The previous models could not reproduce this large value of the scaled W in the region with small or almost no mean-velocity shear.

On the other hand, the Alfvén ratio r_A is defined by the ratio of the turbulent kinetic and magnetic energies:

$$r_\mathrm{A} \equiv \frac{\langle \mathbf{u}'^2 \rangle}{\langle \mathbf{b}'^2 \rangle}. \qquad (46)$$

This ratio is almost unity near the Sun, then decreases with the heliocentric distance. At about 3 AU (astronomical unit: 1 AU $= 1.5\times 10^{11}$ m) from the Sun, r_A reaches 0.5 and thereafter remains to be the same (~ 0.5) with the heliocentric distance as long as the observations exist:

$$r_\mathrm{A} \simeq 0.5 \quad \text{for} \quad r \geq 3 \text{ AU}. \qquad (47)$$

Previous research could not properly explain this stationary value of 0.5 (Tu and Marsch, 1995).

In this work, we address these problems from the viewpoint of the turbulence model. In particular, we examine how the large-scale spatial evolution varies as the model for the turbulent cross-helicity dissipation rate changes.

Numerical simulations

We examine the evolution of the turbulent statistical quantities under the prescribed mean velocity and magnetic field. For the choice of the mean fields, we fully utilize the current theory and modeling of the solar wind. The magnetic rotator model of the solar wind takes the effects of rotation and magnetic field into consideration (Parker, 1958; Weber and Davis, 1967), and known to be a good approximation for the mean velocity and magnetic field of the solar wind. However, its large-scale velocity and magnetic-field shears are weak as compared with the large-scale shears in the slow–fast-wind and magnetic-sector boundaries. So, the adopted mean-field profile is suitable for representing relatively calm flow fields within one magnetic sector.

First, we adopt the algebraic expression [Eq. (16)] for the dissipation of the turbulent cross helicity. In this case, Eq. (20) is rewritten as

$$C_W > 1. \qquad (48)$$

Namely, the model constant C_W for ε_W in the W equation should be larger than the counterpart $(= 1)$ for ε in the K equation.

In this work, we adopt

$$C_W = 1.4, \quad \sigma_W = 1.0. \qquad (49)$$

Here, we should note the following point. In Yokoi and Hamba (2008), $C_W = 1.8$ was adopted for the algebraic model constant for ε_W. As was mentioned above, the assumed mean fields correspond to a weak-shear case in which W is expected to show no considerable decay. If we consider this fact, we see that the less dissipative value for C_W [Eq. (49)] is more suitable than the more dissipative value of $C_W = 1.8$ previously adopted.

Results

Alfvén ratio. The evolution of Alfvén ratio r_A does not depend so much on the value of C_W. In the case of higher velocity shear, the large scale evolution of r_A shows a more stationary behavior, which is in better agreement with the observations. As was already mentioned, if we made the fine tuning of C_r value, the agreement of the simulation results and observations becomes better. In this work, however, we do not do such fine tuning, and made only a rough estimate of the constants as in Eq. (43).

Cross helicity. The simulated cross helicity W is shown in Figure 2 with the comparisons with the observations and with the previous work.

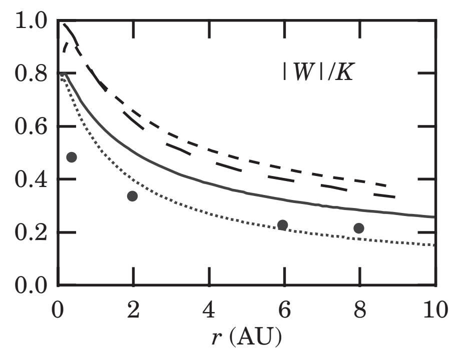

Figure 2: The radial evolution of the scaled turbulent cross helicity W. ○: observations (Roberts et al. 1987); ——: the present model; · · · · · ·: the present model with enhanced shear. Previous work, - - -: Zhou and Matthaeus (1990); — — —: Tu and Marsch (1993).

The radial evolution of the cross helicity in the higher velocity shear shows better agreement with the observations. The mean fields adopted in this work correspond to a weak-shear case, while the observations were performed in situations with stronger shears. Then, the tendency that the simulation results in the high-shear case are in better agreement is preferable. This agrees with the observational findings that in the fast solar wind, which has a small velocity shear, the scaled turbulent cross helicity is kept large without decaying. In the simulation with a weak shear, the scaled turbulent cross helicity, $|W|/K$, is about 0.5 at 3 AU and about 0.3 at 10 AU, which are kept relatively large values.

From this agreement of the numerical simulation with the observations, we see that the algebraic-model expression for ε_W [Eq. (16)] is good enough, as far as the application to the solar wind is concerned.

Cross helicity and velocity shear

The magnitude of the scaled turbulent cross helicity $|W|/K$ decays much in the slow-wind region where the velocity shear is strong, while $|W|/K$ decays less in the fast-wind region where the shear is weak (Goldstein et al., 1995). This basic tendency can be elucidated from the viewpoint of turbulence model through a simple argument.

The equation for $|W|/K$ is given by Eq. (18). Among the terms, we pick up the production-related terms which are directly connected to the mean-velocity shear as

$$\frac{D}{Dt}\frac{W}{K} = \left(\frac{1}{W}P_W - \frac{1}{K}P_K\right)\frac{W}{K} - \cdots. \qquad (50)$$

We substitute the expressions for P_W [Eq. (3a)] and P_K [Eq. (2a)] with Eqs. (28) and (29) into Eq. (50). Then, we have a contribution from the mean-velocity shear as

$$\frac{D}{Dt}\frac{W}{K} = \cdots - \frac{7}{10}C_\beta\frac{K}{\varepsilon}\boldsymbol{\mathcal{S}}^2\frac{W}{K} + \cdots. \qquad (51)$$

Since both C_β and K/ε are positive, in the presence of the mean-velocity shear, irrespective of the sign of W/K, it works for a decrease in the magnitude of W/K.

If we scrutinize the production rate of the turbulent cross helicity W, we see that whether W may increase or decrease depends on the coupling of the mean velocity and magnetic-field shears. At the same time, the turbulent MHD energy K always increases in the presence of the mean-velocity shear. As this result, the magnitude of the turbulent cross helicity scaled by the turbulent MHD energy, $|W|/K$, will decrease in the presence of the mean-velocity shear in the primary sense. The present model reflects this mechanism properly. So, this model is particularly promising for describing the evolution of the solar-wind turbulence.

CONCLUSION

A turbulence model for plasmas with four one-point turbulent statistical quantities (the turbulent MHD energy K, its dissipation rate ε, the turbulent cross helicity W, and the turbulent residual energy K_R) is proposed. The model was applied to the solar-wind turbulence. In particular, two possibilities of expressing the dissipation rate of W, ε_W, are examined. Namely, (i) the algebraic model and (ii) the evolution equation for ε_W. It is shown that, as far as the application to the solar-wind turbulence is concerned, the algebraic model for ε_W with one model constant gives results plausible enough.

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