Sixth International Symposium on Turbulence and Shear Flow Phenomena
Seoul, Korea, 22-24 June 2009

AN ATTACHED EDDY MODEL OF A BOUNDARY LAYER WITH NO FREE PARAMETERS

P. A. Davidson
Department of Engineering,
University of Cambridge
Cambridge, CB2 1PZ, United Kongdom
pad3@eng.cam.ac.uk

P. Å. Krogstad
Department of Energy and Process Engineering,
Norwegian University of Science and Technology
N-7491 Trondheim, Norway
per.a.krogstad@ntnu.no

ABSTRACT

We build on the work of Davidson et al. (2006) and propose an elementary model for the log-law region of a boundary-layer. The model contains only one free parameter (which we set equal to unity) and assumes very little about the shape of the boundary-layer eddies. The physical content of the model is simple: we assume that the two-point statistics of the streamwise velocity fluctuations know about the presence of the wall only to the extent that, over a range of eddy sizes, it imposes a kinetic energy scale proportional to the square of the shear velocity. Classic Kolmogorov phenomenology is assumed for the small scales. The model is an excellent fit to experimental data for the k^{-1} law of the one-dimensional, longitudinal spectrum, $\Phi_{uu}(k)$, and also to $\Phi_{uu}(k)$ in the inertial range. In addition, the model predicts the cross-stream variation of the variance of the streamwise velocity fluctuations, $\left\langle u_x^2 \right\rangle$. Our prediction of the cross-stream variation of $\left\langle u_x^2 \right\rangle$ differs from all other theories in that it incorporates a $\ln(P/\epsilon)$ correction, where P and ϵ are the production and dissipation of energy respectively.

INTRODUCTION

The attached-eddy model of Townsend and Perry et al.

Consider a zero-pressure-gradient boundary layer where x points in the streamwise direction and y is normal to the wall. We focus on the log-law region of such a boundary layer, where experiments suggest that there are eddies whose size s lies in the range $\ell < s < L$ and whose kinetic energy per unit mass scales as $K.E. \sim u_*^2$. (Here u_* is the shear velocity, ℓ is to be determined and L is an outer scale which we expect to be of the order of the boundary layer thickness, δ.) In addition, there are smaller eddies, $s < \ell$, whose energy distribution might be approximated by Kolmogorovs law, $K.E. \sim \epsilon^{2/3} s^{2/3}$, where ϵ is the energy dissipation rate. Noting that $\epsilon \sim P \sim u_*^3/y$ in the log layer, and that ℓ must satisfy $u_*^2 \sim \epsilon^{2/3}\ell^{2/3}$, the transition from one regime to the other occurs at around $\ell \sim y$.

Such a model of the log-layer was first proposed by Townsend (1976) on the basis of the attached eddy hypothesis, and later extended by Perry et al. (1986). Since then there have been a multitude of attached eddy models which often differ in detail from Townsend and Perry et al., but not in spirit. However, the attached eddy hypothesis is only one possible rationalisation of the experimental data, and perhaps the more fundamental point is that we observe a range of eddy sizes, $y < s < L$, whose kinetic energy scales as u_*^2. Consequently, in this paper we build upon the observed scaling, but hold back from interpreting the results in terms of attached eddies.

There are two other differences between our model and that of Perry et al. First, we develop the model in real space, whereas the general scaling arguements of Perry et al. were developed in Fourier space. It turns out that there is a crucial difference at the sort of Reynolds numbers encoutered in the laboratory. Second, Perry et al. had several free parameters in their model which were determined empirically. We have only one free parameter which we put equal to unity.

The boundary-layer model of Davidson et al.

Suppose that $\hat{E}_x(s)$ is the real-space, streamwise kinetic energy density of eddies of size s, defined by the requirement that $\int_{s_1}^{s_2} \hat{E}_x(s)\,ds$ gives the contribution to $\frac{1}{2}\left\langle u_x^2\right\rangle$ from eddies in the size range $s_1 < s < s_2$ (see Davidson, 2004, pp 419). Then we might express the observed energy distribution in the log-law region in the form,

$$s\hat{E}_x(s) = \frac{1}{4} B u_*^2 \, , \quad \ell < s < L \, , \tag{1}$$

$$s\hat{E}_x(s) = \frac{1}{3}\hat{\beta}\epsilon^{2/3}s^{2/3} \, , \quad \hat{\eta} < s < \ell \, , \tag{2}$$

where $\ell \sim y$, B and $\hat{\beta}$ are dimensionless coefficients of order unity, and $\hat{\eta}$ is the smallest eddy that makes a significant contribution to the two-thirds law, i.e. $\hat{\eta}$ is some multiple of the Kolmogorov scale, $\hat{\eta} = \alpha\eta = \alpha\left(\nu^3/\epsilon\right)^{1/4}$.

Now continuity of $\hat{E}_x(s)$ demands $\ell = \left(3B/4\hat{\beta}\right)^{3/2} \frac{u_*^3}{\epsilon}$ and so B, $\hat{\beta}$, and ℓ are not independent. Moreover, the rate of turbulent energy production in the log-law region of a boundary layer takes the form $P = u_*^3/\kappa y$, κ being Karman's constant, and so we can rewrite our expressions for ℓ and $\hat{\eta}$ as

$$\ell = \left(3B\kappa^{2/3}/4\hat{\beta}\right)^{3/2} \frac{P}{\epsilon} y \, . \tag{3}$$

and

$$\hat{\eta} = \alpha\kappa^{1/4}\left(\frac{u_* y}{\nu}\right)^{-3/4}\left(\frac{P}{\epsilon}\right)^{1/4} y \, . \tag{4}$$

Note that Eq.(3) can be rewritten as $\ell = \lambda (P/\epsilon)\, y$, where $\lambda \sim 1$. In a zero-pressure-gradient boundary layer we have $P \approx \epsilon$, though the ratio P/ϵ does exhibit a slow variation with y. Thus ℓ depends upon the distance from the wall partly through the linear term y , and partly because P/ϵ is a weak function of y.

In Davidson et al. (2006) it is assumed that the eddies within the boundary layer have a particularly simple shape; that is, blobs of vorticity with a Gaussian profile. Moreover, it is supposed that, at any given level y, these vortex blobs are distributed in a random but locally isotropic manner. One may then calculate the longitudinal correlation function, $\langle u_x u_x' \rangle = \langle u_x(\mathbf{x})\, u_x(\mathbf{x} + r\hat{\mathbf{e}}_x)\rangle$, and its one-dimensional spectral counterpart, $\Phi_{uu}(k) = \frac{1}{\pi}\int_0^\infty \langle u_x(\mathbf{x})\, u_x(\mathbf{x} + r\hat{\mathbf{e}}_x)\rangle \cos(kr)\, dr$, for eddies of a given size s. These turn out to be $\langle u_x u_x' \rangle (r) = \left\langle u_x^2 \right\rangle^{(s)} f(r) = \left\langle u_x^2 \right\rangle^{(s)} \exp\left(-r^2/s^2\right)$ and $\Phi_{uu}(k) = \frac{1}{2\pi^{1/2}} \left\langle u_x^2 \right\rangle^{(s)} s\ \exp\left[-(ks)^2/4\right]$, where the superscript (s) indicates that we are considering eddies of size s only. In the model of Davidson et al. (2006), $\left\langle u_x^2 \right\rangle^{(s)}$ was taken to be a prescribed function of eddy size, with $\frac{1}{2}\left\langle u_x^2 \right\rangle^{(s)} = \hat{E}_s(s)\, ds$. Integration over s then yields

$$\Phi_{uu}(k) = \frac{1}{\pi^{1/2}} \int_0^\infty s\hat{E}_x(s)\ \exp\left[-(ks)^2/4\right] . \quad (5)$$

Combining Eq.(5) with Eq.(1) and (2) we find

$$\begin{aligned} \frac{k\Phi_{uu}(k)}{Bu_*^2} &= \frac{k}{4\pi^{1/2}} \int_{\hat{\eta}}^{\ell} (s/\ell)^{2/3} \exp\left[-(ks)^2/4\right] ds \\ &+ \frac{1}{4}\left[\operatorname{erf}(kL/2) - \operatorname{erf}(k\ell/2)\right] . \quad (6) \end{aligned}$$

In the limit of $\hat{\eta} \to 0$, Eq.(6) constitute the simple model of Davidson et al. (2006), where B and ℓ, or equivalently B and $\hat{\beta}$, are considered as free parameters.

We note that the distance from the wall, y, enters the analysis only through the spatial variation of $\epsilon(y)$, i.e. $\epsilon(y) = [\epsilon/P]\left[u_*^3/\kappa y\right] \sim u_*^3/\kappa y$, and so y always appears premultiplied by P/ϵ, as in Eq.(3). Second, the only role of ℓ, and hence y, in Eq.(6) is to define the point at which $\hat{E}_x(s)$ switches from $s\hat{E}_x(s) \sim u_*^2$ to the two-thirds law. The implication is that the two-point statistics of the streamwise component of the turbulence, knows about the presence of the wall only to the extent that it enforces an $s\hat{E}_x(s) \sim u_*^2$ energy scaling over a range of scales, and that y dictates the lower cut-off for this range. Third, the model does not incorporate the non-universal components of inactive motion which are associated with the very large-scale eddies in the outer part of the boundary layer. Fourth, the assumption of randomly distributed Gaussian vortex blobs is unlikely to be a realistic approximation in practice, but it is important only to the extent that it fixes the form of the longitudinal correlation function for eddies of size s as $\langle u_x u_x' \rangle = \left\langle u_x^2 \right\rangle^{(s)} \exp\left(-r^2/s^2\right)$, from which Eq.(5) follows. Finally, the model cannot predict the behaviour of the wall-normal fluctuations as it does not build in the so-called blocking effect of the wall, i.e. it does not satisfy $\mathbf{u} \cdot d\mathbf{S}$ at $y = 0$.

Now in the ranges $\ell << r << L$ and $\ell << k^{-1} << L$, both the dissipation and outer scales cease to play any role in Eq.(6), at least to leading order in $k\eta$ and $(kL)^{-1}$, and this expression simplifies to

$$\Phi_{uu}(k) = \frac{Bu_*^2}{4k}\left[1 - \frac{2}{5\pi^{1/2}} k\ell + O(k\ell)^3\right] , \quad (7)$$

Eq.(7) is the well-known k^{-1} law of Perry et al. (1986), with a first-order corrections in $k\ell$. It was suggested by Davidson et al. (2006) that B should be a universal constant, independent of the nature of the outer flow.

In this paper we adapt the model of Davidson et al., in such a way that: (i) there is only one free parameter (which we put equal to unity); (ii) we can enforce a lower cut-off in the two-thirds law; and (iii) the model is liberated from the unlikely assumption that the turbulence can be treated as an isotropic distribution of Gaussian vortex blobs. We shall see that the refined model yields an excellent fit to the experimental data for $\Phi_{uu}(k)$ in the ranges $\sqrt{3}y < r << L$ and $ky > 0.1$, respectively. The model also predicts the cross-stream variation of the streamwise velocity variance, $\left\langle u_x^2 \right\rangle$, and this prediction suggests that the classical formulation of Perry et al. (1986) needs to be modified to incorporate a $B\ln(P/\epsilon)$ correction.

A REFINED MODEL

The weakest feature of the model detailed above is the assumption that locally, i.e. at a given level of y, the vorticity field can be modeled as an isotropic distribution of Gaussian vortex blobs. While this might be defended for the $K.E. \sim \epsilon^{2/3}s^{2/3}$ part of the spectrum, it is clearly inappropriate for the large eddies which contribute to the $K.E. \sim u_*^2$ law. However, as noted above, this assumption is used only to the extent that it prescribes the form of the longitudinal correlation function for eddies of size s:

$$\left\langle u_x u_x' \right\rangle (r) = \left\langle u_x^2 \right\rangle^{(s)} \exp\left(-r^2/s^2\right) . \quad (8)$$

It is this, and only this, which leads to Eq.(5). Consequently, it makes more sense to adopt Eq.(8) as the basis for our model. In our refined analysis, therefore, we make no assumption about the morphology of the vorticity field, or about the statistics of u_y or u_z. Rather, we simply assume that the streamwise longitudinal correlation coefficient corresponding to eddies of size s may be approximated by $f(r) = \exp\left(-r^2/s^2\right)$. Physically, Eq.(8) corresponds to a velocity signal, $u_x(x)$, which is statistically homogeneous in x and composed of a superposition of Gaussian disturbances of scale s, $u_x(x) \sim \sum A_i \exp\left[-2(x - x_i)^2/s^2\right]$, whose centers, x_i, and amplitudes, A_i, are randomly chosen. Each Gaussian may be thought of as representing a transverse slice through a vortical structure, say the leg of a hair-pin or a small-scale worm.

Since $\left\langle u_x^2 \right\rangle^{(s)}$ is an implicit function of s, we might rewrite Eq.(8) as

$$\left\langle u_x u_x' \right\rangle (s; r) = \left\langle u_x^2 \right\rangle^{(s)} (s) \exp\left(-r^2/s^2\right) . \quad (9)$$

As in the original analysis, we shall define $\hat{E}_x(y; s)$ through the expression

$$\frac{1}{2}\left\langle u_x^2 \right\rangle^{(s)} = \hat{E}_x(y; s)\, ds , \quad (10)$$

where the inclusion of y in $\hat{E}_x(y; s)$ implies that the way in which the streamwise energy is distributed across the scales may depend on y. With these definitions and assumptions we arrive back at Eq.(5). Finally, we adopt Eq.(1) and Eq.(2) as the simplest (and most plausible) approximation to $\hat{E}_x(y; s)$,

$$\begin{aligned} s\hat{E}_x(s) &= \frac{1}{4} Bu_*^2 , \quad \ell < s < L , \\ s\hat{E}_x(s) &= \frac{1}{3}\hat{\beta}\epsilon^{2/3}s^{2/3} , \quad \hat{\eta} < s < \ell , \quad (11) \end{aligned}$$

Sixth International Symposium on Turbulence and Shear Flow Phenomena
Seoul, Korea, 22-24 June 2009

and so our model equation is, as before, Eq.(6), with ℓ and $\hat{\eta}$ determined by Eq.(3) and (4). Expressions Eq.(9) - (11), along with the auxiliary relationships Eq.(3) and (4), represent the entire physical content of our refined model.

It turns out that the coefficient α in Eq.(4) is readily determined by demanding that the model spectrum integrates to give the correct dissipation (see section *"Evaluating the universal coefficients"*). This leaves ℓ as an unknown and we shall take the ratio

$$\lambda = \frac{\ell}{(P/\epsilon)\, y} \;, \qquad (12)$$

in Eq.(3) as our one free parameter. We shall see shortly that specifying λ is sufficient to determine all of the other coefficients in the model, and we shall choose $\lambda = 1$.

The blocking effect of the wall

Another shortcoming of Davidson et al. (2006) is that it fails to incorporate the kinematic requirement that $\mathbf{u}\cdot d\mathbf{S} = 0$ at $y = 0$. This is a problem for our refined model only if, for some kinematic reason, it invalidates approximation Eq.(9). However, it is easy to find counter examples which show that Eq.(9) is not excluded by the presence of wall blocking. In this respect it is useful to consider the following model problem which is relevant to the $K.E. \sim \epsilon^{2/3} s^{2/3}$ part of the spectrum, if not the larger scales.

Let us go back to our isotropic velocity field, $\mathbf{u}$, associated with a random distribution of Gaussian eddies of size s. Let $\hat{\mathbf{u}}$ be the homogeneous velocity distribution obtained from $\mathbf{u}$ through a reflection in the plane $y = 0$, and $\overline{\mathbf{u}}$ be the inhomogeneous velocity field defined by $\overline{\mathbf{u}} = \frac{1}{2}(\mathbf{u} + \hat{\mathbf{u}})$. Evidently $\overline{\mathbf{u}}$ satisfies the wall-blocking requirement that $\overline{\mathbf{u}} \cdot d\mathbf{S} = 0$ on $y = 0$. Moreover, we can determine the statistical properties of $\overline{\mathbf{u}}$ from those of the homogenous velocity field $\mathbf{u}$. In particular, $\langle \overline{u}_x \overline{u}'_x \rangle (s; y; r) = \frac{1}{4}\left[\langle u_x u'_x\rangle + \langle \hat{u}_x \hat{u}'_x\rangle + \langle \hat{u}_x u'_x\rangle + \langle u_x \hat{u}'_x\rangle\right] = \frac{1}{2}\left[Q_{xx}(r\hat{\mathbf{e}}_x) + Q_{xx}(r\hat{\mathbf{e}}_x \pm 2y\hat{\mathbf{e}}_y)\right]$, where Q_{ij} is the velocity correlation tensor associated with the homogeneous velocity field $\mathbf{u}$. Next, isotropy of $\mathbf{u}$ allows us to substitute for Q_{ij} in terms $\langle u_x u'_x\rangle$, which, in turn, is given by $\langle u_x u'_x\rangle = \left\langle u_x^2\right\rangle^{(s)} \exp\left(-r^2/s^2\right)$. After some algebra we find

$$\left\langle \overline{u}_x \overline{u}'_x \right\rangle (y; s; r) = \left\langle \overline{u}_x^2 \right\rangle^{(s)} (y; s) \exp\left(-r^2/s^2\right) \;. \qquad (13)$$

and

$$\Phi_{uu}(y; s; k) = \frac{1}{2\pi^{1/2}} \left\langle \overline{u}_x^2 \right\rangle^{(s)} (y; s)\;\; s\;\exp\left[-(ks)^2/4\right] \;, \qquad (14)$$

where $\left\langle \overline{u}_x^2 \right\rangle^{(s)} (y; s)$ is related to the streamwise energy of the original isotropic field, $\frac{1}{2}\left\langle u_x^2\right\rangle^{(s)}$, by

$$\left\langle \overline{u}_x^2 \right\rangle^{(s)} (y; s) = \frac{1}{2}\left\langle u_x^2\right\rangle^{(s)} G(2y/s) \;, \qquad (15)$$

$$G(\chi) = 1 + \left(1 - \chi^2\right)\exp\left(-\chi^2\right) \;. \qquad (16)$$

The important result here is Eq.(13). For a given distance from the wall the streamwise longitudinal correlation function takes the form

$$\left\langle \overline{u}_x \overline{u}'_x \right\rangle (s; r) = \left\langle \overline{u}_x^2 \right\rangle^{(s)} (s) \exp\left(-r^2/s^2\right) \;, \qquad (17)$$

or $f(r) = \exp\left(-r^2/s^2\right)$. In this respect the inhomogeneous velocity field created by wall blocking is no different to the initial isotropic velocity field.

Of course, this simple example is only relevant to the $K.E. \sim \epsilon^{2/3} s^{2/3}$ part of the spectrum, for which $G(\chi) = 1$. However, it does at least show that there is no kinematic reason to abandon Eq.(9) because of wall blocking. So we shall adopt Eq.(9)-(11) as the basis of our model and explore its consequences, comparing prediction with experiment.

Evaluating the universal coefficients

Let us now evaluate all of the coefficients in our refined model: i.e. B, ℓ, L and $\hat{\eta}$ (or equivalently B, $\hat{\beta}$, L and $\hat{\eta}$). Our starting point is to return to Eq.(3). Let us make the *ad hoc* assumption that, when $P = \epsilon$, we have $\ell = y$. This amounts to setting our free parameter in Eq.(12), λ, to one. Of course any value of λ of order unity is consistent with our model, however, taking $\ell = y$ seems a natural choice for ℓ. If we now allow for the weak dependence of P/ϵ on y, we must replace Eq.(3) with

$$\ell = \frac{P}{\epsilon} y \;, \qquad (18)$$

which, in turn, requires

$$\left(3B/4\hat{\beta}\right)^{3/2} \kappa = 1 \;. \qquad (19)$$

It is demonstrated in Davidson and Krogstad (2008) that $\hat{\beta}$ and β are related through the expression $\hat{\beta} = \frac{\beta}{2\Gamma(2/3)}$, where $\beta \approx 2.0$ is Kolmogorov's universal constant in the second order structure function, ($\left\langle [\Delta v]^2 \right\rangle = \beta \epsilon^{2/3} r^{2/3}$), and Γ the usual gamma function. Expression Eq.(19) now requires B to have the value

$$B = \frac{2}{3\Gamma(2/3)} \frac{\beta}{\kappa^{2/3}} \;. \qquad (20)$$

This fixes our two model parameters, B and $\hat{\beta}$, in terms of the universal constants β and κ. Note that Eq.(20) requires B to be universal, which is consistent with the suggestion of Perry et al. (1986).

We now determine the coefficient α in $\hat{\eta} = \alpha\eta$. To this end we note that, in isotropic turbulence, $\epsilon = \nu\left\langle \omega^2\right\rangle = 30\nu\int_0^\infty k^2 \Phi_{uu}(k)\, dk$, and since the integral is dominated by the small scales, this is also a reasonable approximation in anisotropic turbulence. Substituting for $\Phi_{uu}(k)$ using Eq.(6), dividing throughout by ϵ and discarding terms of order Re^{-1} and smaller, we obtain $\alpha^{4/3} = 15\beta/2\Gamma(2/3)$. This, in turn, yields $\alpha = 6.07$ for $\beta = 2$.

Next we focus on the ranges $\ell << r << L$ and $\ell << k^{-1} << L$, where it may be shown that, to within the appropriate orders in $k\ell$ and $(kL)^{-1}$, Eq.(6) simplifies to

$$\begin{aligned} \frac{4k\Phi_{uu}(k)}{Bu_*^2} &= 1 - \frac{2}{5\pi^{1/2}} k\ell + O(k\ell)^3 \\ &- \frac{\exp\left[-(kL/2)^2\right]}{\sqrt{\pi}kL/2}\left(1 + O(kL)^{-1}\right) \end{aligned} \qquad (21)$$

It is readily confirmed that

$$\Phi_{uu}(k) = \frac{Bu_*^2}{4k}\left[1 - \frac{2}{5\pi^{1/2}}k\ell - \frac{\exp\left[-(kL/2)^2\right]}{\sqrt{\pi}kL/2}\right] \qquad (22)$$

is accurate to within 1% for the range $0.1 < k\ell < 1.0$.Thus all the model parameters in Eq.(7), i.e. B and ℓ, are uniquely determined by the universal constants β and κ, and by the measured distribution of P/ϵ. Adopting the generally accepted values of $\kappa = 0.40$ and $\beta = 2.0$, yields $B = 1.81$.

It remains to fix L. To this end we note that the streamwise extent of the large-scale hair-pin packets observed in boundary layers is thought to be of the order of $2.5\delta \rightarrow 3\delta$. Hence we shall take $L = 2.7\delta$. Thus our model should capture the influence of the large-scale hair-pin packets (assuming their kinetic energy scales as u_*^2), but perhaps not the effects of the non-universal very-large-scale structures which are thought to reside in the outer regions of the boundary layer.

Inner and outer scaling for $\Phi_{uu}(k)$

Let us now compare the different forms of the compensated spectrum, $k\Phi_{uu}(k)/u_*^2$, for the inner and outer ranges, $y << k^{-1} << L$ and $k^{-1} \sim L$, respectively. For $y << k^{-1} << L$, which encompasses the k^{-1} range, we have already seen that the model predicts

$$\frac{k\Phi_{uu}(k)}{u_*^2} = \frac{B}{4}\left[1 - \frac{2}{5\pi^{1/2}}\frac{P}{\epsilon}ky + O(k\ell)^3\right], \quad (23)$$

which, for $P = \epsilon$, is a function of the inner variable, ky, only. However, when $P \neq \epsilon$ we have the complication that P/ϵ is a function of y/δ, which involves the outer variable, δ. Thus our model predicts that $k\Phi_{uu}(k)/u_*^2$ exhibits mixed scaling in the inner, k^{-1} region. Of course, one may argue that the $O(ky)$ term in Eq.(23) is negligible for $ky << 1$. However, we shall see shortly that, at typical values of Re, there is almost no purely $\Phi_{uu}(k) \sim k^{-1}$ region, and that the $O(ky)$ term is always significant.

Note also that Eq.(22) shows that, according to our model, the value of $k\ell$ (and hence ky) at which $k\Phi_{uu}(k)/u_*^2$ is a maximum is a simple, monotonically increasing function $\ell/L \sim y/\delta$. Thus, for fixed $u_*\delta/\nu$, the peak in $k\Phi_{uu}(k)/u_*^2$ moves to higher ky as u_*y/ν increases. We shall illustrate this in section *"Comparison with experimental data"* where it will become evident that one consequence of this behaviour is that the width of the region governed by Eq.(23) is predicted to decrease with increasing $y^+ = u_*y/\nu$. In any event, that part of $k\Phi_{uu}(k)/u_*^2$ which surrounds the maximum in $k\Phi_{uu}$ (and hence lies immediately to the left of the $\Phi_{uu} \sim k^{-1}$ region) is clearly a mixed function of inner and outer variables, at least according to our model.

For $y << k^{-1} \sim L$, on the other hand, Eq.(6) yields the simpler result $\frac{k\Phi_{uu}(k)}{u_*^2} = \frac{B}{4}\left[\mathrm{erf}(kL/2) + O(k\ell)\right]$, which is a function of the outer variable, kL, only. In summary, then, our model predicts mixed scaling for the $\Phi_{uu} \sim k^{-1}$ region, and outer scaling for $k^{-1} \sim L$. We shall see shortly that the experimental data displays the same scalings in these two regimes, and that our model does a good job of predicting the distribution of $k\Phi_{uu}(k)/u_*^2$ for the inner regime.

The prediction for $\langle u_x^2 \rangle$

Davidson and Krogstad (2008) showed that the variance of the streamwise velocity fluctuations, $2\langle u_x^2\rangle_{r\rightarrow\infty} = u_*^2 B\left[\frac{3}{2} + \ln(L/\ell) - \frac{3}{2}\left(\frac{\hat{\eta}}{\ell}\right)^{2/3}\right]$, and on substituting for $\hat{\eta}$ and ℓ using $\hat{\eta} = 6.07\eta$ and Eq.(18) we obtain

$$\begin{aligned}\frac{\langle u_x^2\rangle}{u_*^2\frac{1}{2}B} &= \frac{3}{2} - \ln(P/\epsilon) + \ln(L/y) \\ &- 4.29(P/\epsilon)^{-1/2}\left(\frac{u_*y}{\nu}\right)^{-1/2}. \quad (24)\end{aligned}$$

We might compare Eq.(24) with the classic prediction of Townsend (1976) and Perry et al. (1986). Adopting the notation of Perry et al., and omitting any viscous correction, this takes the form

$$\frac{\langle u_x^2\rangle}{u_*^2} = B_1 + A_1 \ln(L/y), \quad (25)$$

where A_1 is assumed to be a universal constant and B_1 is a non-universal constant which depends on the nature of the outer flow. Since B_1 is non-universal, the coefficient of 3/2 in Eq.(24) must be regarded with suspicion: certainly our model does not incorporate such detailed outer-layer dynamics as long, streamwise rolls. Comparing the $\ln(L/y)$ terms, on the other hand, we see that $A_1 = B/2$ in our model.

Summary of model predictions

Let us now summarize the model predictions. B is a universal constant, independent of the nature of the outer flow, and given by

$$B = \frac{2}{3\Gamma\left(\frac{2}{3}\right)}\frac{\beta}{\kappa^{2/3}} = 1.81. \quad (26)$$

The spectrum is given by

$$\frac{4k\Phi_{uu}(k)}{Bu_*^2} = 1 - \frac{2}{5\pi^{1/2}}\frac{P}{\epsilon}ky - \frac{\exp\left[-(kL/2)^2\right]}{\sqrt{\pi}kL/2}, \quad (27)$$

which holds for $0.1 < ky < 1.0$. For the broader range $ky > 0.01$ we have to go back to the more general result

$$\begin{aligned}\frac{k\Phi_{uu}(k)}{Bu_*^2} &= \frac{k}{4\pi^{1/2}}\int_{\hat{\eta}}^{\ell}(s/\ell)^{2/3}\exp\left[-(ks)^2/4\right]ds \\ &+ \frac{1}{4}\left[\mathrm{erf}(kL/2) - \mathrm{erf}(k\ell/2)\right], \quad (28)\end{aligned}$$

where $\ell = (P/\epsilon)y$, $\hat{\eta}$ is given by $\hat{\eta} = 6.07\left(\nu^3/\epsilon\right)^{1/4}$, and $L = 2.7\delta$.

The variance of the streamwise velocity fluctuations, on the other hand, is predicted by Eq.(24), and on substituting for B this becomes

$$\begin{aligned}\frac{\langle u_x^2\rangle}{u_*^2} &= 1.36 - 0.91\ln(P/\epsilon) + 0.91\ln(L/y) \\ &- 3.88(P/\epsilon)^{-1/2}\left(\frac{u_*y}{\nu}\right)^{-1/2}. \quad (29)\end{aligned}$$

One novelty of Eq.(27)-(29) is that, having set $\lambda = 1$ and $L = 2.7\delta$, there are no free parameter in these expressions, with all of the coefficients determined by the universal constants β and κ. Another is that there is a $\ln(P/\epsilon)$ correction to B_1 in Eq.(29) which does not appear in other theories. It is natural to ask if this correction is a genuine feature of near-wall turbulence, or if it is merely an artefact of our model. We shall see shortly that the experimental evidence is ambiguous on this point, but tends to favour the former interpretation.

Comparison with other theoretical studies

Before confronting Eq.(26)-(29) with the experimental data, it is natural to compare them with other theoretical models. In Davidson et al. (2006) B was found to be $B = 1.83$. This values is very close to the one given in Eq.(26), i.e. $B = 1.81$. Since we have made only one *ad hoc* assumption in the analysis, i.e. that $\ell = y$ when $P = \epsilon$, it is remarkable that the empirical estimates of both coefficients are so close to the model predictions.

Sixth International Symposium on Turbulence and Shear Flow Phenomena
Seoul, Korea, 22-24 June 2009

Next we note that expression Eq.(29) has the same structural form as that proposed by Perry et al. (1986) for near-wall turbulence in which $P = \epsilon$. In particular, Perry et al. used the attached eddy hypothesis, in conjunction with general scaling arguments, to suggest

$$\frac{\langle u_x^2 \rangle}{u_*^2} = B_1 + A_1 \ln (L/y) - C \left(\frac{u_* y}{\nu} \right)^{-1/2} \qquad (30)$$

for the log-law region. Here A_1, B_1 and C are constants which are treated as free parameters to be determined by the experimental data, with A_1 and C assumed to be universal, but B_1 dependant on the type of outer flow (i.e. pipe, duct or boundary layer) and on the choice of outer scale, L. Perry et al. (1986) empirically found $A_1 = 0.90$ and $C = 6.06$, comparet to our estimates $A_1 = 0.91$ and $C = 3.88$ in Eq. (29). Given that our predicted value of C depends on the value of α in accordance with $C \sim \alpha^{2/3}$, and that our choice of α is a little arbitrary (we have chosen to approximate the dissipation scales by a truncated two-thirds law), the comparison between our predictions of A_1 and C and the empirical estimates of Perry et al. (1986) is surprisingly good. Note that Eq.(29) contains a $(P/\epsilon)^{-1/2}$ correction to C, which is absent in other models.

The study of Perry et al. was subsequently refined by many others, and the results are summarized in Marusic et al. (1997). For example, Spalart (1988) suggests adopting $C = 4.37$ in Eq.(30), which is close to our estimate of $C = 3.88$. Moreover, some authors have developed a more elaborate viscous correction to replace $C\,(u_* y/\nu)^{-1/2}$, while Marusic et al. added a so-called wake function to Eq.(30) which depends on y/δ but is negligible in the log-law region. In the refined model of Marusic et al. the value of A_1 is taken as $A_1 = 1.03$, which is also close to our predicted value of $A_1 = 0.91$. The measured values of B_1, on the other hand, vary somewhat, with Perry et al. (1986) reporting $B_1 = 2.67$ for pipe flow and Marusic et al. suggesting $B_1 = 2.2$ for zero-pressure-gradient boundary layers.

In summary, then, our predictions of the universal constants B and C, are all close the empirically determined values. The key structural differences between our model and that of earlier theories are that: (i) our real-space formulation turns out to have a significant advantage over the equivalent spectral formulation of Perry et al. (1986) at finite Re (see next section); and (ii) within the log-law region, the non-universal coefficient B_1 is predicted to be a function of y/δ in accordance with

$$B_1 = 1.36 - 0.91 \ln (P/\epsilon) \ . \qquad (31)$$

We shall show that this is probably (but not definitely) a genuine feature of near-wall turbulence.

Comparison with experimental data

In the following we shall take $L = 2.7\delta$. Figure 1 shows a comparison of Eq.(27) with two sets of zero-pressure-gradient boundary-layer data taken from Davidson et al. (2006). The Reynolds number is $Re_\theta = 12800$. For this set P/ϵ was found to fall continuously with increasing y, from $P/\epsilon = 1.4$ at $u_* y/\nu = 100$ to $P/\epsilon = 0.9$ at $u_* y/\nu = 500$. (See Davidson et al. for the details of how the data was acquired and for the distribution of P/ϵ.) Evidently, the fit between theory and experiment is excellent. Note that there is a negligible k^{-1} range in Figure 1, with the $O(k\ell)$ correction term in Eq.(27) playing an important role.

The absence of a k^{-1} region in Figure 1 is interesting and it highlights one weakness of the spectral formulation of Perry et al. (1986), which would predict a plateau in Figure 1 of the form $k\Phi_{uu}(k)/u_*^2 \sim constant$. In our real-space formulation, on the other hand, the equivalent statement is $s\hat{E}_x(s)/u_*^2 \sim 1$, and this has led to the correct functional form for $\Phi_{uu}(k)$ in the k^{-1} region. The key point is this; the $O(ky)$ correction in Eq.(27) is important at terrestrial values of Re, and this correction arises naturally in a real-space formulation of the problem, but is absent in the equivalent spectral model.

Next, Figure 2 shows the predicted one-dimensional spectrum in compensated form for the wider range $ky > 0.01$, which includes the inertial and dissipation ranges, as well as the low-k end of the spectrum. Figure 2(a) shows $k\Phi_{uu}(k)/u_*^2$ plotted against the inner variable ky, while Figure 2(b) shows the spectrum as a function of the outer variable, $k\delta$. The model prediction is compared with the same data set as in Figure 1 (i.e.$Re_\theta = 12800$) and, although there is some discrepancy at around $k\delta \sim 2$, the comparison is favorable elsewhere. In the light of the discussion above, we might interpret the difference between the measured and predicted spectra at $k\delta \sim 2$ as representative of the energy of the so-called very-large-scale streamwise structures in the outer layer. There is also some difference between Eq.(28) and the experimental data in the dissipation range, but this is to be expected, as we have modeled the dissipation range in a simplistic manner, as an extended $s^{2/3}$ region truncated at $s = 6.07\eta$. Note that the width of the k^{-1} region decreases as $y^+ = u_* y/\nu$ increases, as anticipated in section *"Inner and outer scaling for $\Phi_{uu}(k)$"*.

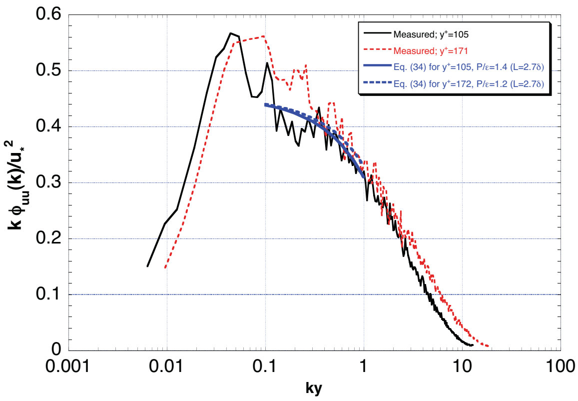

Figure 1: Comparison of Eq.(27) with data. ky is restricted to the k^{-1} region.

Finally, Figure 3 compares our prediction of $\langle u_x^2 \rangle$ with the $Re_\theta = 12800$ data set. The comparison is restricted to the log-law range. Three curves are shown, one corresponding to Eq.(29), a second corresponding to Eq.(29) in which the additive constant of 1.36 has been changed to 1.70, and a third in which the $\ln (P/\epsilon)$ term is omitted from Eq.(29) and 1.36 is replaced by 1.70. The logic behind changing the additive constant to 1.70 is that this constant is non-universal and we have not incorporated a sufficiently detailed description of the outer-flow dynamics in our model to be able to capture its value. Moreover, if our interpretation of Figure 2 is correct, the predicted value of the additive constant in Eq.(29) should lie below the measured one, since we have not captured the energy of the very-large-scale structures in the outer layer. Note that the curve in Figure 3, in which the $\ln (P/\epsilon)$ term is omitted from B_1, is a less good fit to the data, suggesting that the $\ln (P/\epsilon)$ correction is a genuine feature of near-wall turbulence.

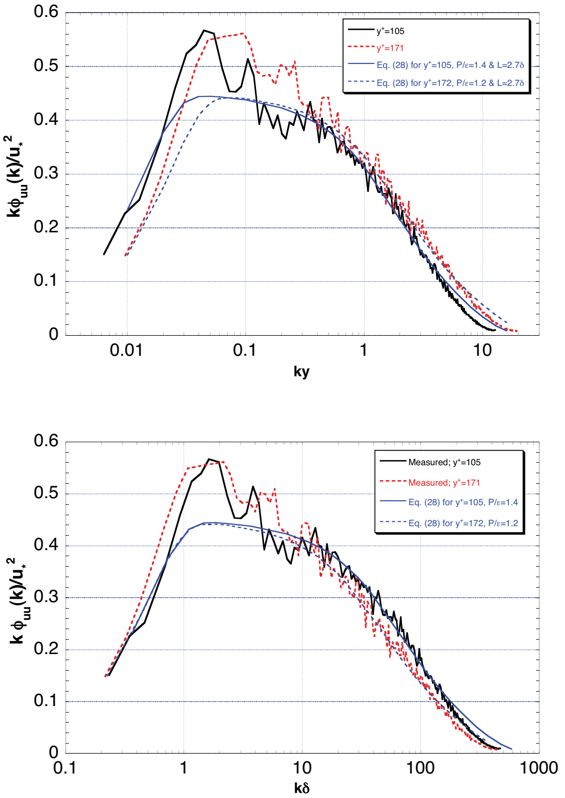

Figure 2: As for Figure 1, but with the theoretical curve extended to $ky > 0.01$. (a) $k\Phi_{uu}(k)/u_*^2$ plotted against ky, (b) $k\Phi_{uu}(k)/u_*^2$ plotted against $k\delta$.

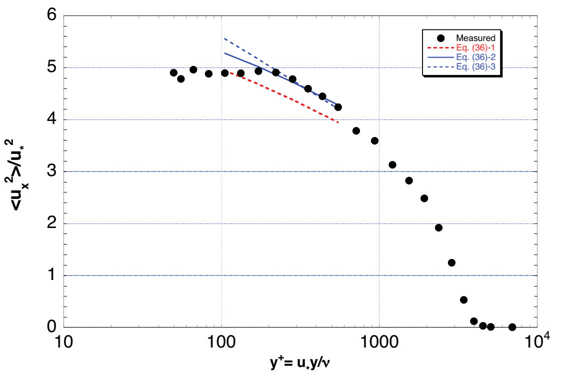

Figure 3: Comparison of Eq.(29) with the data (lower doted line); with the additive constant 1.36 replaced by 1.70 (intermediate line); and without the $\ln(P/\epsilon)$ term, but 1.36 replaced by 1.70 (upper dotted line).

DISCUSSION AND CONCLUSIONS

We have reinterpreted and extended the model of Davidson et al. (2006) by : (i) dispensing with the assumption that the turbulence is locally isotropic; (ii) incorporating a lower cut-off in the two-thirds law, and (iii) showing how all the coefficients in the model can be determined from first principles in terms of the universal constants β and κ. The only *ad hoc* ingredient of our model lies in Eq.(18), where we take $\ell = y$ when $P = \epsilon$, rather than $\ell \sim y$.

The resulting model predictions are in excellent agreement with $\Phi_{uu}(k)$ in the k^{-1} region, and a reasonable fit to $\Phi_{uu}(k)$ elsewhere. The model also predicts the streamwise velocity variance, $\langle u_x^2 \rangle$, and the predicted values of A_1 and C in Eq.(30) are close to those measured by Perry et al. (1986) and others (Spalart, 1988, Marusic et al.,1997).

Despite the naivety of our model, it does a reasonable job of predicting $\Phi_{uu}(k)$, except at low k. There are two possible explanations for this: (i) the apparent success could simply be a combination of dimensional necessity plus coincidence; or (ii) long, outer layer structures apart, perhaps $\Phi_{uu}(k)$ care about the presence of the wall only to the extent that it imposes the energy scale u_*^2 over a range of eddy sizes, and that the lower cut-off for this range is set by y (which are the key physical ingredients of our model).

Option (i) is certainly a possibility,but perhaps unlikely, if only because we have managed to pin down all universal constants, B and C, and such information does not follow from dimensional arguments. We are forced then to entertain option (ii), that is, our model has captured the essential physics of the log layer, at least as far as $\Phi_{uu}(k)$ is concerned. But it too seems a little hard to believe, since our model is extremely primitive yet many studies have shown that the vorticity field is surprisingly organized in the log-layer. These must certainly influence the statistics of u_x, yet our model, which does a reasonable job of predicting $\Phi_{uu}(k)$ at moderate to high k, includes none of this detailed information. So if we are to entertain option (ii), then we must explain why the rich and subtle dynamics of the log-layer go largely undetected by $\Phi_{uu}(k)$.

This apparent failure of $\Phi_{uu}(k)$ can be understood, at least in part, by the fact that they are both time averages, so that they average over many events. However, the problem goes deeper than this: it turns out that $\Phi_{uu}(k)$ average over many scales of structure, representing cumulative statistical averages of all scales below r or π/k, with $\Phi_{uu}(k^*)$ representing a weighted average of all the energy held in structures below scale $s = \pi/k^*$.

REFERENCES

P.A. Davidson, P.-Å. Krogstad & T.B. Nickels "A refined interpretation of the logarithmic structure function law in wall layer turbulence." *Phys. Fluids* **18**(6), pp. 065112-1–065112-7 (2006).

P.A. Davidson & P.-Å. Krogstad "A simple model for the log-law region of a boundary layer." Submitted to *Phys. Fluids* (2008).

Townsend, A.A. *The Structure of Turbulent Shear Flows.* 2nd edition, (Cambridge University Press, 1976).

A.E. Perry, S.M. Henbest & M.S. Chong "A theoretical and experimental study of wall turbulence." *J. Fluid Mech.* **165**, pp. 163–199 (1986).

Davidson, P.A. *Turbulence, An Introduction for Scientists and Engineers.* (Oxford University Press, 2004).

I. Marusic, A.K.M. Uddin & A.E. Perry "Similarity law for the streamwise turbulence intensity in zero-pressure-gradient turbulence boundary layers." *Phys. Fluids* **9**(12), pp. 00212-2 (1997).

P.R. Spalart "Direct simulation of turbulent boundary layer up to $Re_\theta = 1410$." *J. Fluid Mech.* **187**, pp. 61–98 (1988).

P.A. Davidson & P.-Å. Krogstad "On the deficiency of even-order structure functions as inertial range diagnostics." *J. Fluid Mech.* **602**, pp. 287–302 (2008).