PURE LAGRANGIAN VORTEX METHODS FOR THE SIMULATION OF DECAYING ISOTROPIC TURBULENCE

Rio Yokota
Department of Mechanical Engineering,
Keio University
3-14-1 Hiyoshi, Yokohama, Japan
rioyokota@a5.keio.jp

Shinnosuke Obi
Department of Mechanical Engineering,
Keio University
3-14-1 Hiyoshi, Yokohama, Japan
obsn@mech.keio.ac.jp

ABSTRACT

The assessment of the vortex method for its application to turbulent flow simulation is performed through computation of homogeneous isotropic turbulence. In particular, the energy transfer across various scales of turbulence and the representation of the viscous dissipation are examined. The core spreading method and particle strength exchange were selected as the viscous diffusion scheme to be tested. The computations were accelerated by the use of a fast multipole method modified for periodic boundary conditions. Results show that both diffusion schemes are able to reproduce essential dynamics of turbulence including the decay of kinetic energy, though the spatial adaption is necessary even if the number of vortex elements is sufficiently large.

INTRODUCTION

The simulation of turbulence requires the accurate prediction of the production, transport, and dissipation of kinetic energy. In vortex methods, this is made possible by properly calculating the stretching term and diffusion term of the vorticity equation. The mesh-free nature of the pure Lagrangian vortex methods is itself a large advantage, but it is also the primary source of the viscous diffusion problem.

A straightforward application of the commonly used diffusion schemes in vortex methods to turbulent flows cause problems due to its Lagrangian nature. For example, heterogeneous distribution of vortex elements, which is inevitable in the course of calculation, is known to contaminate the accuracy of the diffusion. Furthermore, remeshing, adding, splitting, or merging the vortex elements to solve this problem either undermines the mesh-free nature or adds some ambiguity to the originally rigorous method. The modification of the existing treatments is necessary while retaining the advantage of the vortex method.

The following two methods are of primary interest in our study: The particle strength exchange (PSE) (Winckelmans, 2004) and the core spreading method (CSM) (Barba et al., 2005). The former permits the use of higher order kernels, does not require viscous splitting, and is a straightforward solution to the governing integral equation. It has also been successfully implemented in many applications. On the other hand, the latter has the potential of becoming a pure Lagrangian scheme. Unlike other viscous diffusion methods, the CSM does not require the use of any kind of mesh.

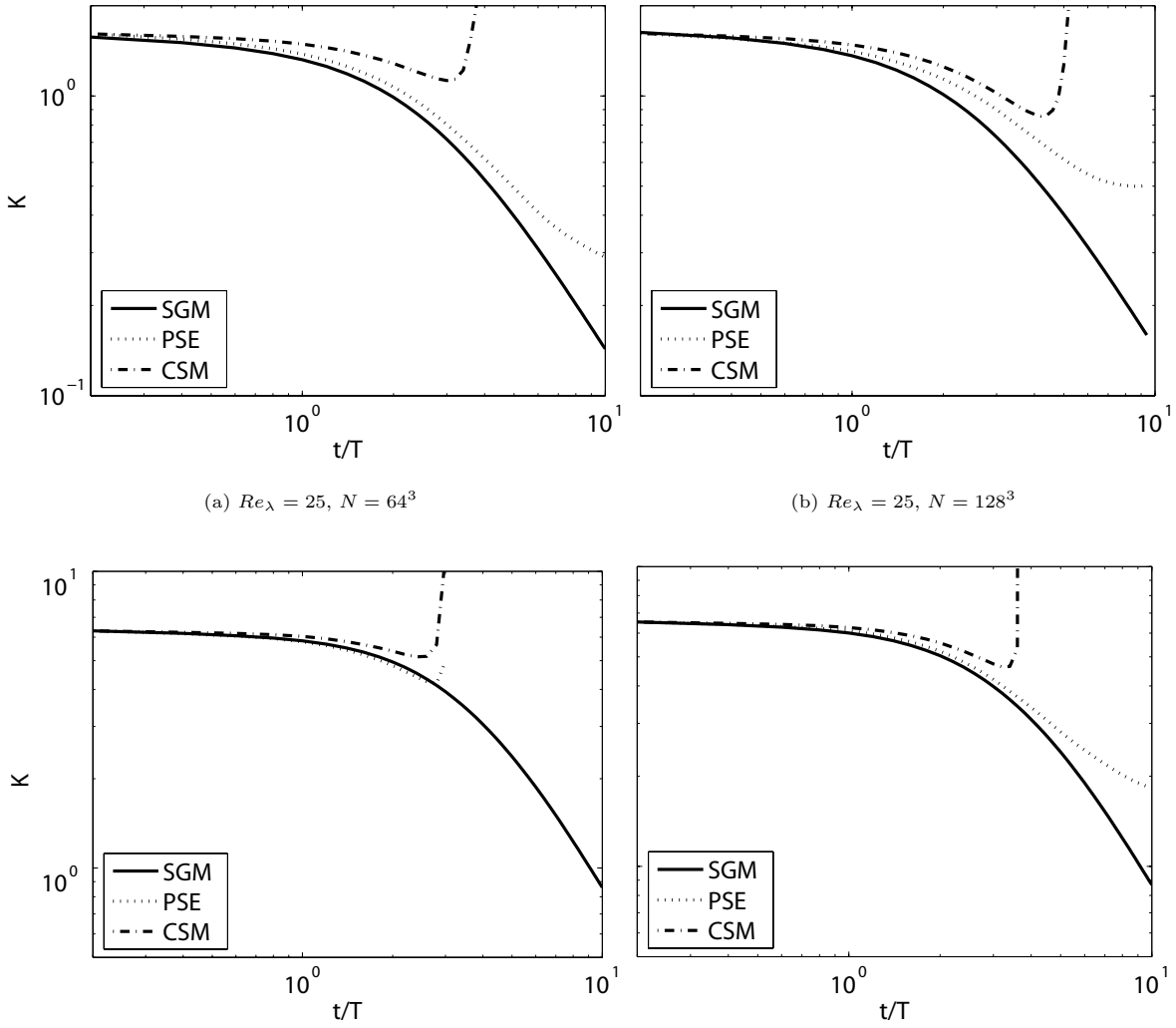
Both methods have been tested for a number of complex bluff body flows (Winckelmans, 2004), though only a few conduct a thorough investigation in canonical turbulent flows. The Lamb-Oseen vortex often serves as a standard benchmark for the viscous diffusion schemes, but this flow is too simple to represent the complex dynamics of vortices that occur in actual turbulent flows. A systematic assessment of these methods is desired.

The present study focuses on the performance of the vortex methods in simulating homogeneous isotropic turbulence. This problem is particularly suitable to test the ability of the method to reproduce fundamental features of turbulence due to the absence of mean shear, strain, rotation, or wall effects. It still involves, however, the energy cascade process from large to small scales, which is the key mechanism in the dynamics of turbulence. The suitability of the viscous diffusion schemes can also be examined by investigating the rate of energy dissipation.

There are only few preceding studies on the computation of homogeneous isotropic turbulence by the vortex methods. Cottet et al. (2002) used the vortex-in-cell method and compared with a spectral method for $N = 128^3$ grid points. The evolution of the energy spectrum, kinetic energy, dissipation, enstrophy and skewness were in excellent agreement. However, their method requires the use of a grid for the stretching, diffusion and velocity calculations.

Totsuka & Obi (2007) studied the performance of the core spreading method and a Laplacian model used in moving particle semi-implicit methods by computing two-dimensional homogeneous isotropic turbulence. The rate of energy decay could be reasonably well reproduced when special treatment was introduced. However, the effect of the stretching term which is another key of the vortex method calculation was left unexplored.

As a whole, the information on the ability of the vortex methods in the fundamental turbulent flow problems is limited. A possible reason is the comparative inefficiency of the vortex methods for the calculation of homogeneous turbulence. The calculation cost becomes high compared to spectral methods because vortex methods do not benefit from periodic boundary conditions, whereas the spectral methods enormously do.



(a) $Re_\lambda = 25,\ N = 64^3$

(b) $Re_\lambda = 25,\ N = 128^3$

(c) $Re_\lambda = 50,\ N = 64^3$

(d) $Re_\lambda = 50,\ N = 128^3$

Figure 1: Decay of Turbulent Kinetic Energy.

In the present study, an acceleration technique for the Biot-Savart calculation in a periodic domain is developed and validated. With the help of this acceleration technique, cost effective calculation of the vortex method in three-dimensional homogeneous turbulence has become possible.

The computations are made using a pure Lagrangian vortex method and compared with a spectral method calculation. The number of elements used for both methods are set equal to each other in order to provide more or less the same spatial resolution and also to facilitate the assessment on the cost efficiency.

NUMERICAL METHOD

Periodic Fast Multipole Method

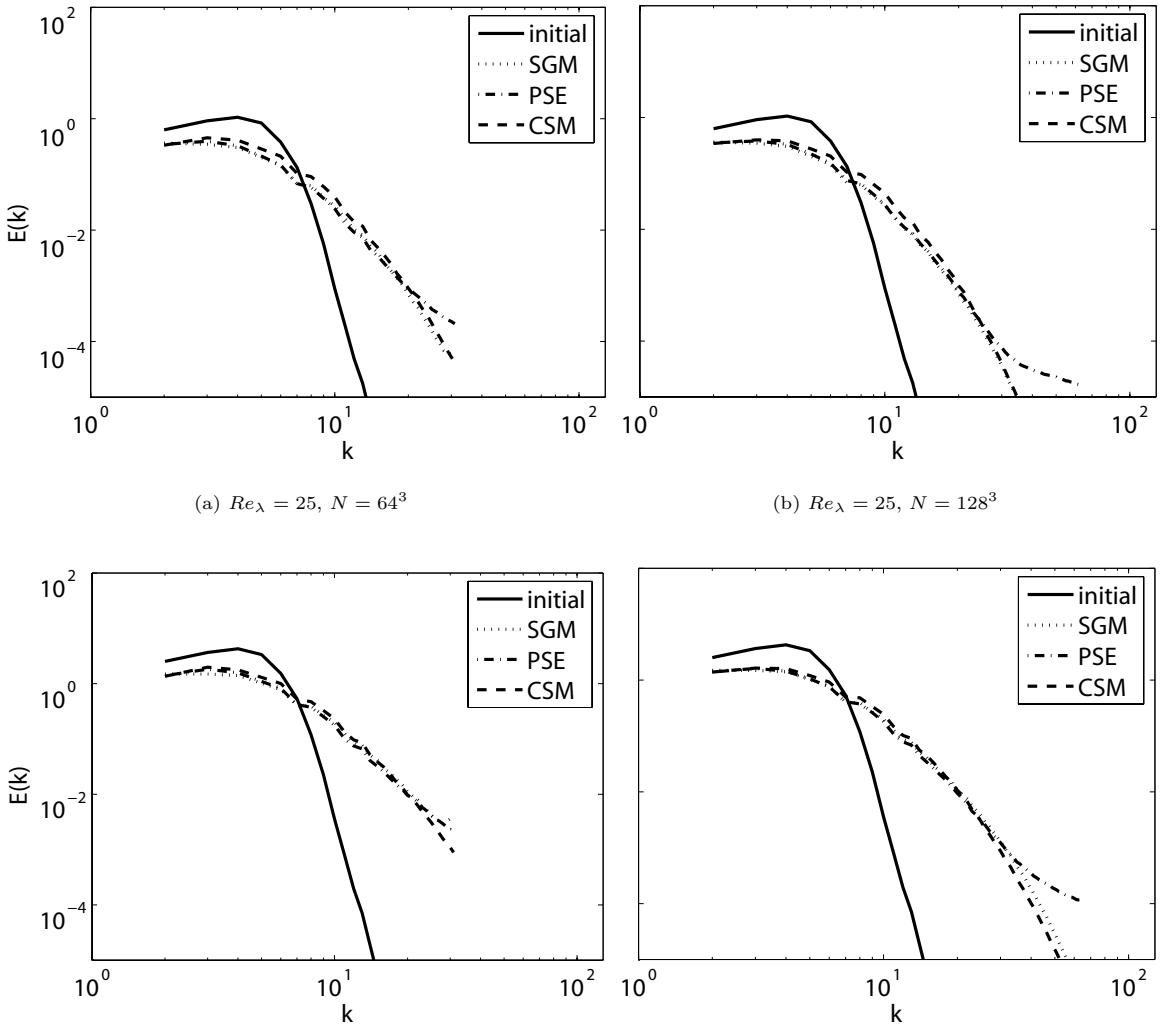
In the present study the fast multipole method (FMM) by Cheng et al. (1999) is modified to include the effect of periodic boundary conditions. Previous attempts to use the FMM in a periodic domain have a $3^k \times 3^k \times 3^k$ structure (Lambert et al., 1996), where k is the number of periodic image cells per dimension. The present method forms a $2^k \times 2^k \times 2^k$ structure, by grouping the cells in a different manner.

Spectral Method

The spectral Galerkin method with primitive variable formulation (Rogallo, 1981) is used in the present study. A pseudo-spectral method was used to compute the convolution sums, and the aliasing error was removed by the 3/2-rule. The time integration was performed using the fourth-order Runge-Kutta method for all terms.

Initial Condition

The initial condition was generated in Fourier space as a solenoidal isotropic velocity field with random phases and a prescribed energy spectrum, and transformed to physical space (Rogallo, 1981). The spectral method calculation used this initial condition directly. The strength of the vortex elements was calculated from the vorticity field on the grid by solving a system of equations (Barba, 2005). The core ra-



(a) $Re_\lambda = 25$, $N = 64^3$

(b) $Re_\lambda = 25$, $N = 128^3$

(c) $Re_\lambda = 50$, $N = 64^3$

(d) $Re_\lambda = 50$, $N = 128^3$

Figure 2: Kinetic Energy Spectra at $t/T = 2$.

dius of the vortex elements were set to $2\pi/N$ and the overlap ratio was 1.

Computational Details

Calculations were performed for $Re_\lambda = 25$ and 50. The Reynolds number was changed by adjusting the strength of the prescribed energy spectrum, and the viscosity remains constant. The number of elements was $N = 64^3$ and 128^3. The time increment was $\Delta t = 0.005$ and 0.0025 for the calculations of $Re_\lambda = 25$ and 50, respectively. This corresponds to a quarter of the Kolmogorov time scale.

RESULTS AND DISCUSSION

Energy Decay

First, the decay of kinetic energy is examined in order to quantify the global feature of the vortex methods. N and Re_λ are changed independently to observe the relation between the spatial resolution and Reynolds number. The four different cases are shown in Figure. 1. SGM, PSE, and CSM stand for the spectral Galerkin method, particle strength exchange and core spreading method. The time t is normalized by T, with T being the eddy turnover time defined as $T = L/u'$, where L is the integral scale and u' is the velocity fluctuation.

The results of SGM indicate that the rate of decay in total kinetic energy is relatively small at the beginning and increases afterwards. The change of the decay rate occurs slightly later for the larger Re_λ.

The PSE fails to capture the decay of the kinetic energy at later time steps. On the other hand, the CSM calculation diverges at $t/T \geq 3$ for all four conditions. It is understood that the condition shown in (c) represents the most low spatial resolution against Re_λ. On the other hand, the comparison between (a) and (b) shows that the increase of spatial resolution does not always improve the performance of PSE, which is seen in the earlier departure of the plot from that of the SGM.

Energy Spectra

Next, the kinetic energy spectra are examined in order

to validate the energy balance at one instant. The kinetic energy spectra at $t/T = 2$ are shown in Figure. 2.

For all four conditions, the PSE matches the SGM for low wave numbers but departs at higher wave numbers. The Kolmogorov wave numbers are $k_\eta \approx 20$ and 30 for $Re_\lambda = 25$ and 50, respectively. The spectrum of the PSE diverges from the SGM at a wave number higher than k_η for all calculation conditions except for $\{Re_\lambda, N\} = \{50, 64^3\}$. Furthermore, increasing the spatial resolution shifts the point of departure of the PSE to a higher wave number, irrespective of the Reynolds number.

On the other hand, the CSM slightly overpredicts the energy in the lower k, and agrees better with the SGM at higher k. This is most clearly observed in Figure. 2 (b), where the CSM spectrum is obviously larger than the other two for intermediate wave numbers.

We suspect the problem does not lie in the treatment of the diffusion term only but in the combined effect with the stretching term, because the calculation of two-dimensional isotropic turbulence by Totsuka & Obi (2007) have already shown that the decay rate of the kinetic energy can be reproduced when the diffusion terms are properly handled. In the next section we will evaluate the spectral energy transfer and shed light on the stretching term from a different perspective.

Spectral Energy Transfer

A dynamical equation for the energy spectrum $E(k)$ is expressed by

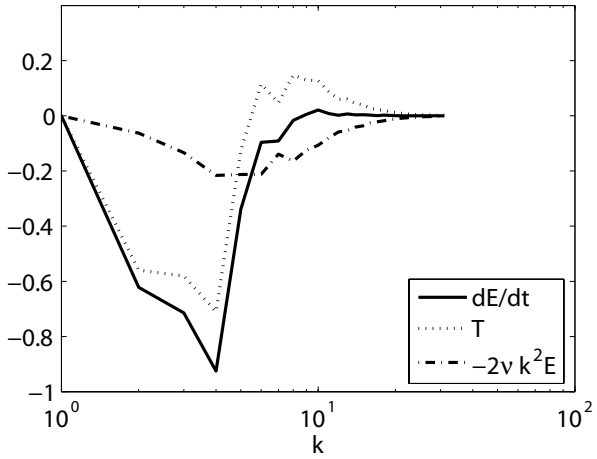
$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k) \tag{1}$$

The first term on the right hand side is the transfer term, which expresses the amount of energy being transferred between the wave numbers. The second term is the dissipation, which accounts for the energy being dissipated at that particular wave number. The balance of these two determine the rate of change in kinetic energy contained in each wave number.

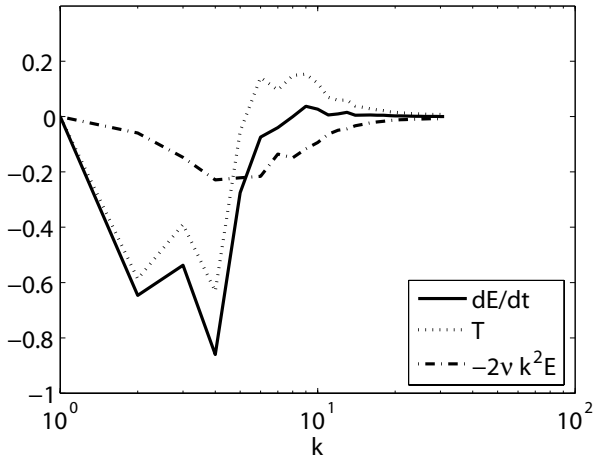
The terms of the energy spectrum equation for $Re_\lambda = 25$, $N = 64^3$ at $t/T = 2$ are plotted in Figure. 3. Although there are certain differences, the qualitative behavior is quite similar for the three cases. The transfer term T is negative at lower k and positive at higher k, indicating that the energy is being transferred to smaller scales. The dissipation has a negative peak at intermediate range of k rather than at smaller scales, presumably because the viscous decay does not yet take place at this time, cf.− Figure. 1.

The comparison among the different schemes reveals that the transfer term in Figure. 3 (c) shows a larger discrepancy than the other two methods. The over-proportional energy transfer at the higher k rapidly increases after a few time steps, and results in the blow up of CSM computation as shown in Figure. 2(a).

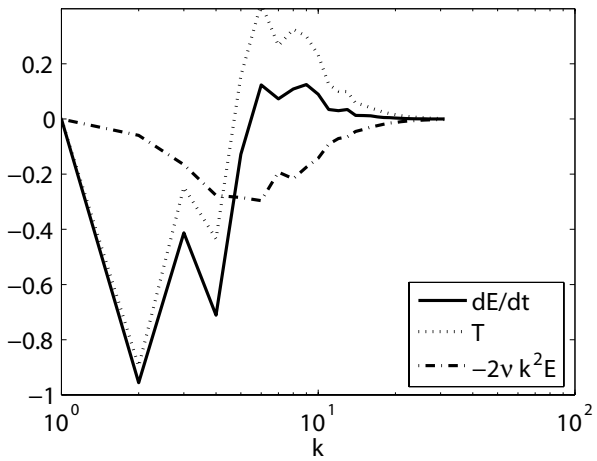
The erroneous behavior of the straightforward implementation of the PSE and CSM is somewhat expected. The PSE calculation uses the neighboring particles as quadrature points, and requires a homogeneous particle distribution for an accurate calculation. On the other hand, a straightforward implementation of the CSM lacks convergence due to the fact that the ever-expanding Gaussian distribution moves with the velocity at its center (Greengard, 1985). In the following two sections, rigorous measures are taken to improve the accuracy of both methods.



(a) SGM

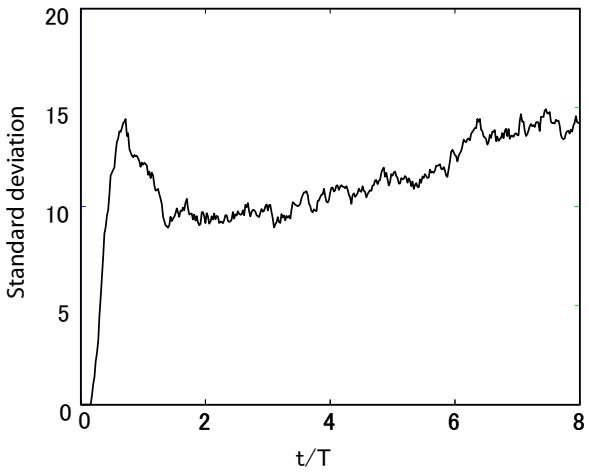


(b) PSE

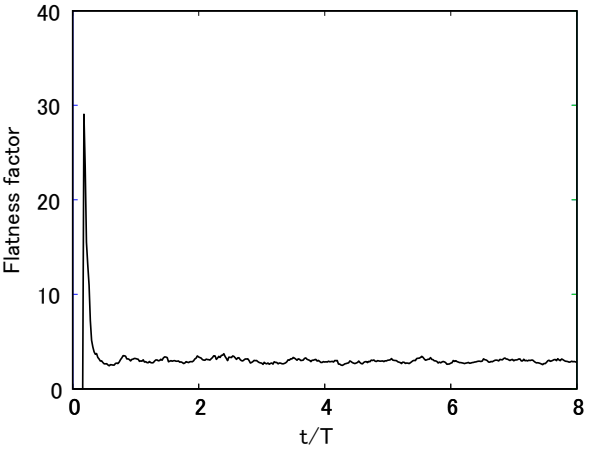


(c) CSM

Figure 3: Energy Spectrum Equation Budget at t/T=2.



(a) Standard Deviation



(b) Flatness Factor

Figure 4: Statistics of the Particle Distribution.

Spatial Adaption in PSE

To begin with, the particle density distribution throughout the flow field is examined by sequentially subdividing the domain into smaller boxes and monitoring the number of particles in these boxes. We will consider the case for $Re_\lambda = 25$ and $N = 64^3$, for which the domain is divided into 16^3 boxes. One box contains an average of 64 particles. We also consider particles in the 26 neighboring boxes so the total is $27 \times 64 = 1728$. This value differs between dense regions and sparse regions.

Figure. 4 shows the standard deviation and flatness factor of the particle density distribution. The standard deviation is 0 for the initially uniform distribution, and grows rapidly at the earlier stages of the calculation. Though, the increment becomes moderate at later stages and the final value is still less than 1% of the mean value that is constantly 1728. The flatness factor jumps at the first few time steps but remains close to 3 otherwise.

We will use a remeshing technique to maintain the uniformity of particles (Koumoutsakos & Cottet, 2000). The

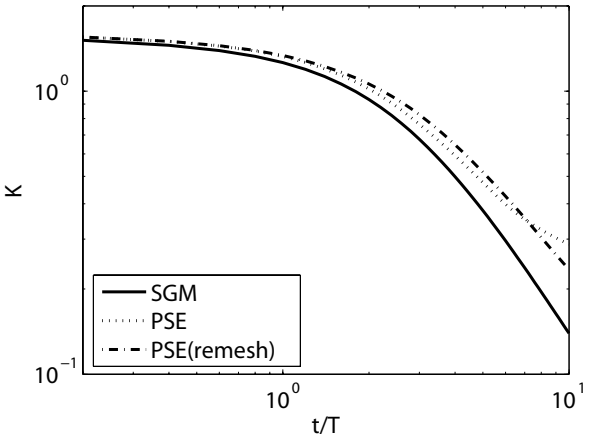


Figure 5: Kinetic Energy Decay for PSE.

M_4' function is used as the interpolation formula.

$$M_4'(x) = \begin{cases} 0 & \text{if} \quad |x| > 2 \\ \frac{1}{2}(2-|x|)^2(1-|x|) & \text{if} \quad 1 \le |x| \le 2 \\ 1 - \frac{5x^2}{2} + \frac{3|x|^3}{2} & \text{if} \quad |x| \le 1 \end{cases} \tag{2}$$

The remeshing is performed every 10 time steps, which corresponds to $t/T = 0.2$. The decay of kinetic energy for the PSE with and without remeshing are plotted in Figure. 5. The result with remeshing is closer to the SGM at the end. The slope at the end is different from the case without remeshing. However, the decay rate of the kinetic energy is still insufficient. Judging from the fact that the standard deviation of the particle density is less than 1% of the mean value, the homogenous isotropic turbulence is not a practical case for which the remeshing improves the results of the PSE.

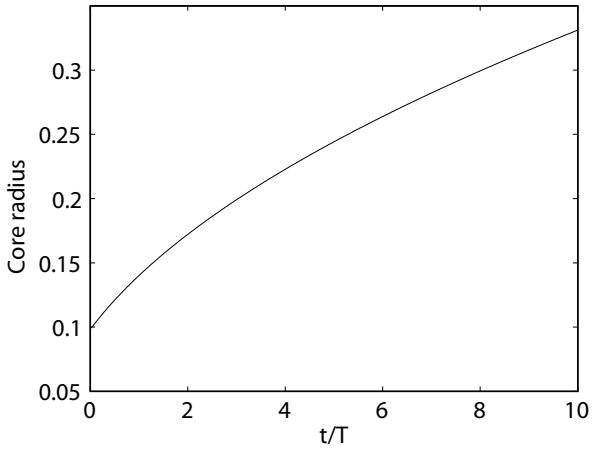
Spatial Adaption in CSM

The CSM is known to converge if spatial refinement is performed either by splitting elements (Huang, 2005) or recalculating the vortex strengths for smaller sized blobs (Barba, 2005). In our case, we could not use element splitting methods since the calculation cost would become overwhelming for an already spatially well resolved three-dimensional simulation.

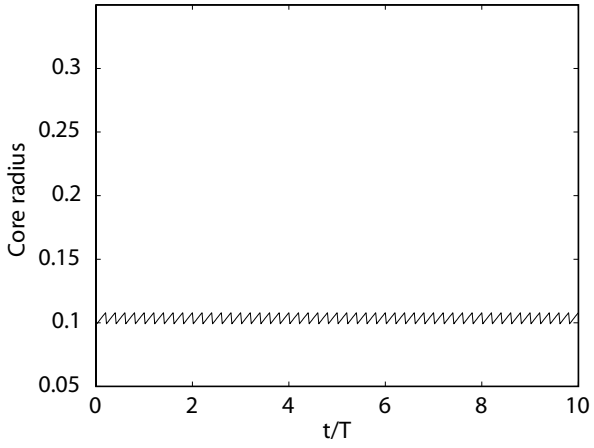
In the present study, spatial adaption is performed by using the radial basis function interpolation (Barba, 2005) for smaller sized blobs to reproduce the vorticity field without increasing the number of elements. The BICGSTAB method without preconditioning is used for the iteration, and calculated until the L^2 norm error was less than 10^{-3}. The FMM box structure was also used to calculate efficiently inside the BICGSTAB iteration.

The growth of the core radius is shown for the cases with and without spatial adaption in Figure. 6. The initial core radius is $2\pi/64 \approx 0.1$ and triples by the time $t/T = 10$ without spatial adaption. However, with the spatial adaption, the core radius increases merely 1%.

The kinetic energy decay is shown in Figure. 7 for the same calculations. The CSM with spatial adaption (noted as rbf) matches the result of the SGM. These results are quite encouraging, and it is fair to say that the core spreading method with spatial adaption is a viscous diffusion scheme



(a) Without Spatial Adaption



(b) With Spatial Adaption

Figure 6: Growth of the Core Radius.

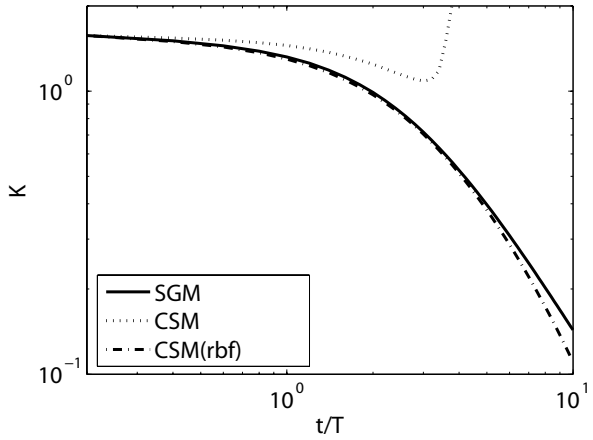


Figure 7: Kinetic Energy Decay for CSM.

that calculates the kinetic energy decay properly. The reason for the discrepancy between the PSE with remeshing and CSM with spatial adaption requires further investigation.

At this point we have not found a way to significantly accelerate this procedure any further, and one spatial adaption step takes the same amount of time as 5 vortex method time steps. Hence, if we perform the spatial adaption every 5 time steps, it will double the total calculation time.

CONCLUSIONS

The vortex method is applied to the calculation of a decaying homogeneous isotropic turbulence in order to quantify the error involved in the cascade and dissipation of kinetic energy. The core spreading method and particle strength exchange were selected as the viscous diffusion scheme to be examined. The effect of spatial resolution was examined along with Reynolds number dependence and the effect of spatial adaption of elements. The following conclusions are drawn from the results of our calculations.

For the particle strength exchange, the kinetic energy spectrum agrees well with the spectral method up to the Kolmogorov wave number. However, the number of elements required is larger than that of the spectral method. Also, the use of remeshing has minimal effect in an isotropic turbulence, where the vortex elements remain evenly distributed.

The core spreading method without spatial adaption is valid only for the primary stage of decay. However, the use of the radial basis function interpolation allows the core spreading method to calculate the kinetic energy decay at an accuracy close to the spectral methods until the kinetic energy decreases an order of magnitude.

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