THE TWO-POINT AVERAGE AND THE RELATED SUBGRID MODEL

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ABSTRACT

In this paper the properties of a simple two-point average have been examined. It results that the associated subgrid stress is expressed by the product of the velocity increments and that its statistical average is related to the second order structure function. It is shown that this result is not peculiar of this particular filtering operator but can be extended to discrete filters and generally to all filtering operator. The energy transfer between the resolved scale and the subgrid scale has been examined both from the point of view of the energy extracted by the subgrid stress and from the point of view of the statistical properties of the structure functions associated to the velocity increment. The two point average applied to a filtered field has been finally examined in the context of the theoretical interpretation of a real large eddy simulation.

INTRODUCTION

The large eddy simulation of turbulent flows is a topic of great interest for many reasons. First of all it is very important for the computation of turbulent fields that cannot be resolved in all details due to the large Reynolds numbers. In this case we are obliged to model the small scales, and the effects of the model on the results as concerns the statistical properties of the turbulent flow are very difficult to predict. Then we have to consider that the results are affected not only by the model but also by the numerical scheme that we apply. Obviously an ideal simulation should only depend on the model that we add to the Navier-Stokes equations and that represents the effect of the resolved scales, but in many cases there is an interference between the model and the numerical scheme with effects on the results which are of the same order. We remark that the *no model* approach is based on the numerical dissipation associated to the numerical scheme, and all that poses to the theoretical interpretation and to the mathematical formalization of the large eddy simulation techniques a real challenge. From the beginning, starting from the papers of Leonard (1974) and Schumann (1975), the basic idea is that the results that we obtain can be read as the real results modified by some operator, an analytical convolution or a truncating numerical discretisation applied to the ideal full representation of the turbulent field. Following this idea, in the succeeding thirty years a lot of work has been done. Many modeling approaches have been proposed and tested, see Sagaut (2005) for a recent review on the different numerical, physical and mathematical aspects, and a significative advancement has been done in the numerical simulation not only of turbulent fields of academic interest but also of turbulent flows relevant for the engineering applications.

In this complex context this contribution is related to a research devoted to explore the formal properties of filtering. Basic problems are to derive the subgrid stress associated to a given filtering average, and we refer to previous results concerning the subgrid stress associated to the product (Germano, 1992) and to the sum (Germano, 2004) of filtering operators. The main interest and the possible applications are obviously in the field of the Large Eddy Simulation of turbulent flows, and we refer in particular to the dynamic modeling procedures and to the hybrid models. Principally in this paper we have directed our interest to the discrete filters. They can be generally read as multipoint averages, and the simplest one is the centered two-point average defined as

$$G(\mathbf{x}-\boldsymbol{\xi}) = 0.5\delta(\mathbf{x}+\frac{\mathbf{r}}{2}-\boldsymbol{\xi}) + 0.5\delta(\mathbf{x}-\frac{\mathbf{r}}{2}-\boldsymbol{\xi}) \qquad (1)$$

where $\mathbf{r}$ is a constant length. We will derive the subgrid stress associated to this operator and we will extend this result to a general multipoint discrete filter. A new formulation of the subgrid stress (Germano, 2007) will be particularly useful to derive the subgrid stress associated to all filtering averages, owing to the fact that a continuous operator can always be reduced to a discretised form. The product of a discrete operator with a generic average will be examined formally and some conclusions will be derived as regards the practical interest of these results from the computational point of view in the context of the large eddy simulation of turbulent flows.

THE TWO-POINT AVERAGE

The analytical formulation of the large eddy simulation stands on the Leonard (1974) definition where a new filtered component of the velocity field $\bar{u}_i$ is introduced by the smoothing convolution

$$\bar{u}_i(\mathbf{x}) = \int G(\mathbf{x}-\boldsymbol{\xi}) u_i(\boldsymbol{\xi}) d\boldsymbol{\xi} \qquad (2)$$

The simplest case is obviously given by a kernel $G(\mathbf{x}-\boldsymbol{\xi})$ expressed by a delta function

$$G(\mathbf{x}-\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi}) \qquad (3)$$

where

$$\delta(\mathbf{x}-\boldsymbol{\xi}) = \prod_{k=1}^{3} \delta(x_k - \xi_k) \qquad (4)$$

that corresponds to the identity operator

$$\bar{u}_i(\mathbf{x}) = u_i(\mathbf{x}) \qquad (5)$$

and a first case of interest is given by the two-point average associated to the kernel

$$G(\mathbf{x}-\boldsymbol{\xi}) = 0.5\delta(\mathbf{x}+\frac{\mathbf{r}}{2}-\boldsymbol{\xi}) + 0.5\delta(\mathbf{x}-\frac{\mathbf{r}}{2}-\boldsymbol{\xi}) \qquad (6)$$

where $\mathbf{r}$ is a constant length and where

$$\delta(\mathbf{x}+\frac{\mathbf{r}}{2}-\boldsymbol{\xi}) = \prod_{k=1}^{3}\delta(x_k+\frac{r_k}{2}-\xi_k)$$
$$\delta(\mathbf{x}-\frac{\mathbf{r}}{2}-\boldsymbol{\xi}) = \prod_{k=1}^{3}\delta(x_k-\frac{r_k}{2}-\xi_k) \quad (7)$$

In this case we have

$$\bar{u}_i(\mathbf{x},\mathbf{r}) = 0.5u_i(\mathbf{x}+\frac{\mathbf{r}}{2}) + 0.5u_i(\mathbf{x}-\frac{\mathbf{r}}{2}) \quad (8)$$

and the basic problem as regards its possible application to the large eddy simulation of turbulent flows is to derive a subgrid model associated to the generalized central moment τ_{ij} of the two velocity components u_i and u_j given by

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i\ \bar{u}_j \quad (9)$$

If we write formally

$$\overline{u_i u_j} = 0.5u_i(\mathbf{x}+\frac{\mathbf{r}}{2})u_j(\mathbf{x}+\frac{\mathbf{r}}{2}) + 0.5u_i(\mathbf{x}-\frac{\mathbf{r}}{2})u_j(\mathbf{x}-\frac{\mathbf{r}}{2}) \quad (10)$$

we obtain

$$\tau_{ij} = \frac{d_{ij}}{4} \quad (11)$$

where

$$d_{ij} = [u_i(\mathbf{x}+\frac{\mathbf{r}}{2}) - u_i(\mathbf{x}-\frac{\mathbf{r}}{2})][u_j(\mathbf{x}+\frac{\mathbf{r}}{2}) - u_j(\mathbf{x}-\frac{\mathbf{r}}{2})] \quad (12)$$

and it is interesting to remark that the mean statistical value of the subgrid stress is directly related to the second order structure function $\langle d_{ij}\rangle$

$$\langle d_{ij}\rangle = < [u_i(\mathbf{x}+\frac{\mathbf{r}}{2}) - u_i(\mathbf{x}-\frac{\mathbf{r}}{2})][u_j(\mathbf{x}+\frac{\mathbf{r}}{2}) - u_j(\mathbf{x}-\frac{\mathbf{r}}{2})] > \quad (13)$$

where the angular brackets denote the ensemble average. We have directly

$$< \tau_{ij} > = \frac{\langle d_{ij}\rangle}{4} \quad (14)$$

Obviously this exact relation cannot be practically utilized in a large eddy simulation that only produces the filtered values $\bar{u}_i, \bar{u}_j$. In order to associate to τ_{ij} a subgrid model M_{ij} function of the filtered values $\bar{u}_i, \bar{u}_j$ we can use the Van Cittert deconvolution applied by Stolz and Adams (1999) as an approximate defiltering procedure. As a first approximation we can write

$$u_i(\mathbf{x}) \sim \bar{u}_i(\mathbf{x}) \quad , \quad u_j(\mathbf{x}) \sim \bar{u}_j(\mathbf{x}) \quad (15)$$

and we obtain as a first order subgrid model

$$M_{ij} = 0.25[\bar{u}_i(\mathbf{x}+\frac{\mathbf{r}}{2}) - \bar{u}_i(\mathbf{x}-\frac{\mathbf{r}}{2})][\bar{u}_j(\mathbf{x}+\frac{\mathbf{r}}{2}) - \bar{u}_j(\mathbf{x}-\frac{\mathbf{r}}{2})] \quad (16)$$

while a second order approximations is given by

$$u_i(\mathbf{x}) \sim 2\bar{u}_i(\mathbf{x}) - \bar{\bar{u}}_i(\mathbf{x}) = = 2\bar{u}_i(\mathbf{x}) - 0.5\bar{u}_i(\mathbf{x}+\frac{\mathbf{r}}{2}) - 0.5\bar{u}_i(\mathbf{x}-\frac{\mathbf{r}}{2})$$

$$u_j(\mathbf{x}) \sim 2\bar{u}_j(\mathbf{x}) - \bar{\bar{u}}_j(\mathbf{x}) = = 2\bar{u}_j(\mathbf{x}) - 0.5\bar{u}_j(\mathbf{x}+\frac{\mathbf{r}}{2}) - 0.5\bar{u}_j(\mathbf{x}-\frac{\mathbf{r}}{2}) \quad (17)$$

It is interesting to remark that this reconstruction model associated to the two-point average is similar to the velocity increment model introduced and applied by Brun and Friedrich (2001) and Brun, Friedrich and da Silva (2006) following different arguments.

THE ENERGY TRANSFER

Let us now examine the energy transfer between the resolved scale represented by the two point average and the subgrid scale represented by the velocity difference. In order to do that we apply the analytical procedure used to derive the velocity structure functions, see Hill (2001) and Hill (2002). By definition we have

$$\bar{u}_i(\mathbf{y},\mathbf{z}) = 0.5u_i(y_k) + 0.5u_i(z_k) =$$
$$= 0.5u_i(x_k+\frac{r_k}{2}) + 0.5u_i(x_k-\frac{r_k}{2}) = \bar{u}_i(\mathbf{x},\mathbf{r}) \quad (18)$$

where

$$y_k = x_k + \frac{r_k}{2}$$
$$z_k = x_k - \frac{r_k}{2} \quad (19)$$

and the Navier-Stokes equations for the velocity components $u_i(y_k)$ and $u_i(z_k)$ are respectively

$$\frac{\partial u_i(\mathbf{y})}{\partial t} + u_n(\mathbf{y})\frac{\partial u_i(\mathbf{y})}{\partial y_n} = -\frac{\partial p(\mathbf{y})}{\partial y_i} + \nu\frac{\partial^2 u_i(\mathbf{y})}{\partial y_n \partial y_n} \quad (20)$$

$$\frac{\partial u_i(\mathbf{z})}{\partial t} + u_n(\mathbf{z})\frac{\partial u_i(\mathbf{z})}{\partial z_n} = -\frac{\partial p(\mathbf{z})}{\partial z_i} + \nu\frac{\partial^2 u_i(\mathbf{z})}{\partial z_n \partial z_n} \quad (21)$$

coupled with the incompressibility conditions

$$\frac{\partial u_n(\mathbf{y})}{\partial y_n} = 0$$
$$\frac{\partial u_n(\mathbf{z})}{\partial z_n} = 0 \quad (22)$$

where ν and p are the kinematic viscosity and pressure. Let us now consider the independent variables x_k and r_k given explicitly by

$$x_k = \frac{y_k + z_k}{2}$$
$$r_k = y_k - z_k \quad (23)$$

The relationships among the partial derivatives are

$$\frac{\partial}{\partial y_i} = \frac{\partial}{\partial r_i} + \frac{1}{2}\frac{\partial}{\partial x_i}$$
$$\frac{\partial}{\partial z_i} = -\frac{\partial}{\partial r_i} + \frac{1}{2}\frac{\partial}{\partial x_i}$$
$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} + \frac{\partial}{\partial z_i}$$
$$\frac{\partial}{\partial r_i} = \frac{1}{2}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)$$
$$\frac{\partial^2}{\partial y_n \partial y_n} + \frac{\partial^2}{\partial z_n \partial z_n} = 2\frac{\partial^2}{\partial r_n \partial r_n} + \frac{1}{2}\frac{\partial^2}{\partial x_n \partial x_n} \quad (24)$$

and we have the incompressibility relations

$$\frac{\partial \bar{u}_n(\mathbf{x},\mathbf{r})}{\partial x_n} = 0$$
$$\frac{\partial \bar{u}_n(\mathbf{x},\mathbf{r})}{\partial r_n} = 0 \quad (25)$$

If we now introduce the velocity difference $v_i(\mathbf{y},\mathbf{z})$ defined as

$$v_i(\mathbf{y},\mathbf{z}) = u_i(y_k) - u_i(z_k) =$$
$$= u_i(x_k + \frac{r_k}{2}) - u_i(x_k - \frac{r_k}{2}) = v_i(\mathbf{x},\mathbf{r}) \qquad (26)$$

the two-point mean pressure $\bar{p}(\mathbf{y},\mathbf{z})$

$$\bar{p}(\mathbf{y},\mathbf{z}) = 0.5p(y_k) + 0.5p(z_k) =$$
$$= 0.5p(x_k + \frac{r_k}{2}) + 0.5p(x_k - \frac{r_k}{2}) = \bar{p}(\mathbf{x},\mathbf{r}) \qquad (27)$$

and the pressure difference $q(\mathbf{x},\mathbf{r})$ given by

$$q(\mathbf{y},\mathbf{z}) = p(y_k) - p(z_k) =$$
$$= p(x_k + \frac{r_k}{2}) - p(x_k - \frac{r_k}{2}) = q(\mathbf{x},\mathbf{r}) \qquad (28)$$

it is easy to obtain from (20) and (21) the following equations for the two point velocity mean

$$\frac{\partial \bar{u}_i(\mathbf{x},\mathbf{r})}{\partial t} + \bar{u}_n(\mathbf{x},\mathbf{r})\frac{\partial \bar{u}_i(\mathbf{x},\mathbf{r})}{\partial x_n} + \frac{\partial \tau_{in}(\mathbf{x},\mathbf{r})}{\partial x_n} =$$
$$= -\frac{\partial \bar{p}(\mathbf{x},\mathbf{r})}{\partial x_i} + \nu\frac{\partial^2 \bar{u}_i(\mathbf{y},\mathbf{z})}{\partial y_n \partial y_n} + \nu\frac{\partial^2 \bar{u}_i(\mathbf{y},\mathbf{z})}{\partial z_n \partial z_n} \qquad (29)$$

and the two point velocity difference

$$\frac{\partial v_i(\mathbf{x},\mathbf{r})}{\partial t} + \bar{u}_n(\mathbf{x},\mathbf{r})\frac{\partial v_i(\mathbf{x},\mathbf{r})}{\partial x_n} + 4\frac{\partial \tau_{in}(\mathbf{x},\mathbf{r})}{\partial r_n} =$$
$$= -\frac{\partial q(\mathbf{x},\mathbf{r})}{\partial x_i} + \nu\frac{\partial^2 v_i(\mathbf{y},\mathbf{z})}{\partial y_n \partial y_n} + \nu\frac{\partial^2 v_i(\mathbf{y},\mathbf{z})}{\partial z_n \partial z_n} \qquad (30)$$

where we remark that

$$\tau_{in}(\mathbf{x},\mathbf{r}) = \frac{v_i(\mathbf{x},\mathbf{r})v_n(\mathbf{x},\mathbf{r})}{4} \qquad (31)$$

and where we temporarily retain the dependence of the viscous terms on $\mathbf{y}$ and $\mathbf{z}$. We notice that the two point means and differences are related by the derivation rules

$$\frac{\partial \bar{u}_i(\mathbf{x},\mathbf{r})}{\partial x_n} = \frac{\partial v_i(\mathbf{x},\mathbf{r})}{\partial r_n} \quad ; \quad \frac{\partial \bar{u}_i(\mathbf{x},\mathbf{r})}{\partial r_n} = \frac{1}{4}\frac{\partial v_i(\mathbf{x},\mathbf{r})}{\partial x_n}$$
$$\frac{\partial \bar{p}(\mathbf{x},\mathbf{r})}{\partial x_n} = \frac{\partial q(\mathbf{x},\mathbf{r})}{\partial r_n} \quad ; \quad \frac{\partial \bar{p}(\mathbf{x},\mathbf{r})}{\partial r_n} = \frac{1}{4}\frac{\partial q(\mathbf{x},\mathbf{r})}{\partial x_n} \qquad (32)$$

and also in this case we have the incompressibility relations

$$\frac{\partial v_n(\mathbf{x},\mathbf{r})}{\partial x_n} = 0$$
$$\frac{\partial v_n(\mathbf{x},\mathbf{r})}{\partial r_n} = 0 \qquad (33)$$

We can now derive the balance equations for the resolved turbulent kinetic energy K

$$K = \frac{\bar{u}_i\bar{u}_i}{2} \qquad (34)$$

and the subgrid turbulent kinetic energy k given by

$$k = \frac{\tau_{ii}}{2} = \frac{v_i v_i}{8} \qquad (35)$$

We remark that the mean kinetic energy $\overline{E}$ is given by

$$\overline{E} = \frac{\overline{u_i u_i}}{2} = K + k \qquad (36)$$

due to the fact that by definition

$$\overline{u_i u_j} = \bar{u}_i\bar{u}_j + \tau_{ij} \qquad (37)$$

and we notice that in the case of homogeneous turbulence we have

$$\begin{aligned}
\langle \overline{E} \rangle &= \langle E \rangle \\
\langle k \rangle &= 0 \qquad \mathbf{r} = 0 \\
\langle k \rangle &\to \frac{\langle E \rangle}{2} \qquad \mathbf{r} \to \infty \\
\langle K \rangle &= \langle E \rangle \qquad \mathbf{r} = 0 \\
\langle K \rangle &\to \frac{\langle E \rangle}{2} \qquad \mathbf{r} \to \infty
\end{aligned} \qquad (38)$$

From the previous equations it is now easy to write

$$\frac{\partial K}{\partial t} + \frac{\partial(\bar{u}_n K + \bar{u}_i \tau_{in})}{\partial x_n} - \frac{\partial(v_n k)}{\partial r_n} =$$
$$= -\frac{\partial(\bar{p}\bar{u}_n)}{\partial x_n} + 2\nu\frac{\partial^2 K}{\partial r_n \partial r_n} + \frac{\nu}{2}\frac{\partial^2 (K - \bar{p})}{\partial x_n \partial x_n} - \frac{\bar{\varepsilon}}{2} \qquad (39)$$

$$\frac{\partial k}{\partial t} + \frac{\partial(\bar{u}_n k)}{\partial x_n} + \frac{\partial(v_n k)}{\partial r_n} =$$
$$= -\frac{\partial}{\partial x_n}\left(\frac{q v_n}{4}\right) + 2\nu\frac{\partial^2 k}{\partial r_n \partial r_n} + \frac{\nu}{2}\frac{\partial^2 (k - \bar{p})}{\partial x_n \partial x_n} - \frac{\bar{\varepsilon}}{2} \qquad (40)$$

where we have applied the identity

$$\tau_{in}\frac{\partial \bar{u}_i}{\partial x_n} = \frac{\partial(v_n k)}{\partial r_n} \qquad (41)$$

and where $\bar{\varepsilon}$ is the two-point average of the local dissipation, see the appendix I,

$$\bar{\varepsilon} = \frac{\varepsilon(\mathbf{y}) + \varepsilon(\mathbf{z})}{2} \qquad (42)$$

We remark that the equation for k is the equation for the velocity difference recently derived by Hill (2002), related both to the structure functions and to the statistical inertial and viscous range laws of Kolmogorov (1941). We notice that the dissipation is equally distributed between the resolved turbulent kinetic energy K and the subgrid turbulent kinetic energy k and we see that the reciprocal energy transfer F is given by the term

$$F = \frac{\partial(v_n k)}{\partial r_n} = \frac{1}{8}\frac{\partial d_{iin}}{\partial r_n}$$

where

$$d_{iin} = v_i v_i v_n$$

and we can finally conclude that the statistical transfer of energy between the resolved scale and the subgrid scale is in the case of the two-point average simply related to the third order structure function $\langle d_{iin} \rangle$ by the expression

$$\langle F \rangle = \frac{1}{8}\frac{\partial \langle d_{iin} \rangle}{\partial r_n}$$

DISCRETE FILTERS

Let us now extend the results to a generic discrete filter given by

$$G(\mathbf{x}-\boldsymbol{\xi}) = \sum_\alpha g_\alpha \delta(\mathbf{x}+\mathbf{r}_\alpha-\boldsymbol{\xi}) = = \sum_\alpha g_\alpha \prod_{k=1}^{3} \delta(x_k + r_{\alpha k} - \xi_k) \quad (43)$$

where

$$\sum_\alpha g_\alpha = 1 \quad (44)$$

In this case we have

$$\bar{u}_i(\mathbf{x}) = \int G(\mathbf{x}-\boldsymbol{\xi}) u_i(\boldsymbol{\xi}) d\boldsymbol{\xi} = \sum_\alpha g_\alpha u_{i\alpha} \quad (45)$$

where

$$u_{i\alpha} = u_i(\mathbf{x}+\mathbf{r}_\alpha) \quad (46)$$

and we remark that discrete filters have been studied and applied in the context of the large eddy simulation of turbulent flows by many authors, see Sagaut and Grohens (1999) and Stolz, Adams and Kleiser (2001), in particular as regards their application as test filters for the dynamic model, see Najjar and Tafti (1996). Here we will derive explicitly the subgrid stress associated to a discrete filter. If we write

$$\overline{u_i u_j} = \sum_\alpha g_\alpha u_{i\alpha} u_{j\alpha} \quad (47)$$

it is easy to obtain after simple manipulations

$$\tau_{ij} = \frac{1}{2} \sum_{\alpha,\beta} g_\alpha g_\beta d_{ij\alpha\beta} \quad (48)$$

where

$$d_{ij\alpha\beta} = \left(u_{i\alpha} - u_{i\beta}\right)\left(u_{j\alpha} - u_{j\beta}\right) \quad (49)$$

A NEW FORMULATION OF THE SUBGRID STRESS.

The formulation of the subgrid stress in terms of the velocity increments is not peculiar of the two-point average and the discrete filters but can be extended to all filtering operators. It is easy to see that starting from the classical definition of the subgrid stress we can write an equivalent formulation in terms of the local velocity increments, Germano (2007). This expression suggests that the subgrid stress is the sum of local contributions due to local velocity increments.

We recall that the subgrid stress $\tau_{ij}(\mathbf{x})$ can be explicitly written as

$$\tau_{ij}(\mathbf{x}) = \overline{u_i u_j} - \bar{u}_i \bar{u}_j = = \int G(\mathbf{x}-\boldsymbol{\xi}) u_i(\boldsymbol{\xi}) u_j(\boldsymbol{\xi}) d\boldsymbol{\xi} - - \int G(\mathbf{x}-\boldsymbol{\xi}) u_i(\boldsymbol{\xi}) d\boldsymbol{\xi} \int G(\mathbf{x}-\boldsymbol{\xi}') u_j(\boldsymbol{\xi}') d\boldsymbol{\xi}' \quad (50)$$

and it is interesting to notice that we can equivalently write, see the Appendix II,

$$\tau_{ij}(\mathbf{x}) = \frac{1}{2} \iint G(\mathbf{x}-\boldsymbol{\xi}) G(\mathbf{x}-\boldsymbol{\xi}') d_{ij}(\boldsymbol{\xi},\boldsymbol{\xi}') d\boldsymbol{\xi} d\boldsymbol{\xi}' \quad (51)$$

where

$$d_{ij}(\boldsymbol{\xi},\boldsymbol{\xi}') = \left(u_i(\boldsymbol{\xi}) - u_i(\boldsymbol{\xi}')\right)\left(u_j(\boldsymbol{\xi}) - u_j(\boldsymbol{\xi}')\right) \quad (52)$$

due to the fact that

$$\int G(\mathbf{x}-\boldsymbol{\xi}) d\boldsymbol{\xi} = 1 \quad (53)$$

If we now introduce the coordinates $\mathbf{r}$ and $\mathbf{s}$ defined as

$$\mathbf{r} = \boldsymbol{\xi} - \boldsymbol{\xi}' \qquad \mathbf{s} = \frac{\boldsymbol{\xi} + \boldsymbol{\xi}'}{2} \quad (54)$$

we can write

$$\tau_{ij}(\mathbf{x}) = \frac{1}{2} \int\int G(\mathbf{x}-\mathbf{s}-\frac{\mathbf{r}}{2}) G(\mathbf{x}-\mathbf{s}+\frac{\mathbf{r}}{2}) d_{ij}(\mathbf{r},\mathbf{s}) d\mathbf{r} d\mathbf{s} \quad (55)$$

and it is easy to verify that the subgrid stress force f_i

$$f_i = \frac{\partial \tau_{ij}}{\partial x_j} \quad (56)$$

is given by

$$f_i = \frac{1}{2} \int\int G(\mathbf{x}-\mathbf{s}-\frac{\mathbf{r}}{2}) G(\mathbf{x}-\mathbf{s}+\frac{\mathbf{r}}{2}) \frac{\partial d_{ij}}{\partial s_j} d\mathbf{r} d\mathbf{s} \quad (57)$$

We remark finally that this formulation of the turbulent stress in terms of velocity increments can also be extended to the Reynolds stresses R_{ij}. If we define as usual the statistical mean in terms of a long time average we can write

$$R_{ij} = \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle = = \lim_{T\to\infty} \frac{1}{T} \int_{t-T}^{t} u_i(t') u_j(t') dt' - - \lim_{T\to\infty} \frac{1}{T} \int_{t-T}^{t} u_i(t') dt' \lim_{T\to\infty} \frac{1}{T} \int_{t-T}^{t} u_j(t'') dt'' \quad (58)$$

and we can equivalently write both the relation

$$R_{ij} = \lim_{T\to\infty} \frac{1}{2T^2} \int_{t-T}^{t} \int_{t-T}^{t} d_{ij}(t',t'') dt' dt'' \quad (59)$$

and the relation

$$R_{ij} = \lim_{T\to\infty} \frac{1}{2T^2} \int_{t-T}^{t} \int_{t-T}^{t} \langle d_{ij}(t',t'') \rangle dt' dt'' \quad (60)$$

where

$$d_{ij}(t',t'') = (u_i(t') - u_i(t''))(u_j(t') - u_j(t'')) \quad (61)$$

and $\langle d_{ij}(t',t'') \rangle$ is the second order temporal structure function associated to the velocity components u_i and u_j.

THE TWO-POINT AVERAGE APPLIED TO A FILTERED FIELD

The two-point average till now considered is applied to the original unfiltered velocity field. It is interesting, in the context of the theoretical interpretation of a real large eddy simulation, to consider also the application of the two-point average to a previously filtered field. We remark that in a real numerical computation we can represent the combined effect of an explicit subgrid model and a numerical scheme as due to the product of an implicit *functional* filter $\mathcal{F}$, following the terminology introduced by Sagaut (2003), and an additional *structural* operator $\mathcal{D}$ applied to the filtered equation, whose exact nature is usually unknown, see Carati, Winckelmans and Jeanmart (2001). The first filter is usually dissipative and usually implicit, and its effect is to produce an additive force, the divergence of the subgrid stress. The second filter mainly is the discretization operator, and we here assume that it can be represented as a first approximation by a two-point average along a characteristic length $\mathbf{r}$ of the same order of the grid length. We remark that the characteristic length associated to the dissipative physical filter, typically the characteristic length associated to an eddy viscosity, can be different of $\mathbf{r}$, so that it is reasonable to assume that some reversibility is associated to the discretization filter, while nothing usually can be recovered from the first filter that truncates the information related to the unrepresented scales. In conclusion let us now consider formally the product $\mathcal{P}$ of our discrete two-point averaging operator $\mathcal{D}$ with a generic filter $\mathcal{F}$

$$\mathcal{P} = \mathcal{D}\mathcal{F} \tag{62}$$

In this case we have formally

$$\tau_p(u_i, u_j) = \langle \tau_f(u_i, u_j) \rangle_d + \tau_d(\langle u_i \rangle_f, \langle u_j \rangle_f) \tag{63}$$

where now the brackets stand for the filtering operation

$$\begin{aligned} \langle \cdots \rangle_f &= \mathcal{F}(\cdots) \\ \langle \cdots \rangle_d &= \mathcal{D}(\cdots) \end{aligned} \tag{64}$$

and where we use the operational definition, Germano (1992), of the generalized central moment of second order τ associated to two generic quantities a and b

$$\begin{aligned} \tau_f(a,b) &= \langle ab \rangle_f - \langle a \rangle_f \langle b \rangle_f \\ \tau_d(a,b) &= \langle ab \rangle_d - \langle a \rangle_d \langle b \rangle_d \end{aligned} \tag{65}$$

More explicitly we can write that

$$\begin{aligned} \langle u_i \rangle_p &= \langle \langle u_i \rangle_f \rangle_d = \\ &= 0.5 \langle u_i \rangle_f (\mathbf{x} + \frac{\mathbf{r}}{2}) + 0.5 \langle u_i \rangle_f (\mathbf{x} - \frac{\mathbf{r}}{2}) \end{aligned} \tag{66}$$

and if we assume that the subgrid model associated to the operator $\mathcal{F}$ is known

$$\tau_f(u_i, u_j) \sim M_f(\langle u_i \rangle_f, \langle u_j \rangle_f) \tag{67}$$

we can write

$$\begin{aligned} M_p(\langle u_i \rangle_f, \langle u_j \rangle_f) = 0.5 M_f(\mathbf{x} + \frac{\mathbf{r}}{2}) + 0.5 M_f(\mathbf{x} - \frac{\mathbf{r}}{2}) + \\ +0.25 \left(\langle u_i \rangle_f (\mathbf{x} + \frac{\mathbf{r}}{2}) - \langle u_i \rangle_f (\mathbf{x} - \frac{\mathbf{r}}{2}) \right) \cdot \\ \cdot \left(\langle u_j \rangle_f (\mathbf{x} + \frac{\mathbf{r}}{2}) - \langle u_j \rangle_f (\mathbf{x} - \frac{\mathbf{r}}{2}) \right) \end{aligned} \tag{68}$$

where

$$\begin{aligned} M_f(\mathbf{x} + \frac{\mathbf{r}}{2}) &= M_f \left(\langle u_i \rangle_f (\mathbf{x} + \frac{\mathbf{r}}{2}), \langle u_j \rangle_f (\mathbf{x} + \frac{\mathbf{r}}{2}) \right) \\ M_f(\mathbf{x} - \frac{\mathbf{r}}{2}) &= M_f \left(\langle u_i \rangle_f (\mathbf{x} - \frac{\mathbf{r}}{2}), \langle u_j \rangle_f (\mathbf{x} - \frac{\mathbf{r}}{2}) \right) \end{aligned}$$

We remark that the coupled subgrid stress consists of two parts, the first related to the dissipative functional operator that models the unrecoverable small scales, and the second related to the structural and partly recoverable numerical operator here simply expressed as a first approximation in terms of a two-point average. One problem obviously remains, due to the fact that the simulation produces the resolved values $\langle u_i \rangle_p$, so that we have to reconstruct $\langle u_i \rangle_f$ with an appropriate deconvolution applied to the computed velocity field $\langle u_i \rangle_p$.

CONCLUSIONS

We have examined the simplest filtering operator applied to the Navier Stokes equation : the two-point average in space, function both of the mean position $\mathbf{x}$ and the distance $\mathbf{r}$ between the two points selected. The associated subgrid stress depends on the velocity difference between the two points and its mean statistical value is given by the second order structure function. In order to better understand the interaction between the resolved field and the subgrid component we have written the balance equations for the subgrid turbulent energy k and the resolved turbulent energy K. The first one is the equation for the velocity difference, see Hill (2002), that is related both to the structure functions and to the statistical inertial and viscous range laws of Kolmogorov (1941), while the second is dual of the first and stands at the basis of a simple two-point large eddy approach. Then we have extended these results to a generic discrete filter and we have shown that the formulation of the subgrid stress in terms of velocity increments is not peculiar of the two-point average and the discrete filter but is a general property of the subgrid stress. As a consequence the statistical mean subgrid stress is a direct biconvolution of the second order structure function, and the subgrid force is related to its derivative in space. The product of a two-point average with a general dissipative filter has been finally examined in order to understand what happens in a real large eddy simulation where a numerical operator is usually coupled to an implicit dissipative filter generally expressed in terms of an eddy viscosity model. The coupled subgrid stress has been formally written and consists of two parts, the first related to the dissipative functional operator that models the unrecoverable small scales, and the second related to the structural and partly recoverable numerical operator here simply expressed as a first approximation in terms of a two-point average.

APPENDIX I

Let us write here some useful relations that can help to derive the equations (39) and (40) in the text. First of all we remark that by using the Poisson's equation

$$\frac{\partial^2 p(\mathbf{y})}{\partial y_n \partial y_n} = -\frac{\partial u_i(\mathbf{y})}{\partial y_n} \frac{\partial u_n(\mathbf{y})}{\partial y_i} \tag{69}$$

and the definition of the dissipation ε

$$\varepsilon(\mathbf{y}) = \nu \frac{\partial u_i(\mathbf{y})}{\partial y_n} \frac{\partial u_i(\mathbf{y})}{\partial y_n} + \nu \frac{\partial u_i(\mathbf{y})}{\partial y_n} \frac{\partial u_n(\mathbf{y})}{\partial y_i} \tag{70}$$

we can write

$$\varepsilon(\mathbf{y}) = -\nu \frac{\partial^2 p(\mathbf{y})}{\partial y_n \partial y_n} + \nu \frac{\partial u_i(\mathbf{y})}{\partial y_n} \frac{\partial u_i(\mathbf{y})}{\partial y_n} \tag{71}$$

and similarly

$$\varepsilon(\mathbf{z}) = -\nu \frac{\partial^2 p(\mathbf{z})}{\partial z_n \partial z_n} + \nu \frac{\partial u_i(\mathbf{z})}{\partial z_n} \frac{\partial u_i(\mathbf{z})}{\partial z_n} \tag{72}$$

With reference now to the derivation of (39) and (40) we can write the following chain of relations

$$\bar{u}_i \frac{\partial^2 \bar{u}_i}{\partial y_n \partial y_n} + \bar{u}_i \frac{\partial^2 \bar{u}_i}{\partial z_n \partial z_n} =$$

$$= \frac{1}{2}\frac{\partial^2(\bar{u}_i\bar{u}_i)}{\partial y_n \partial y_n} + \frac{1}{2}\frac{\partial^2(\bar{u}_i\bar{u}_i)}{\partial z_n \partial z_n} - \frac{\partial \bar{u}_i}{\partial y_n}\frac{\partial \bar{u}_i}{\partial y_n} - \frac{\partial \bar{u}_i}{\partial z_n}\frac{\partial \bar{u}_i}{\partial z_n} \quad , \quad (73)$$

$$v_i \frac{\partial^2 v_i}{\partial y_n \partial y_n} + v_i \frac{\partial^2 v_i}{\partial z_n \partial z_n} =$$

$$= \frac{1}{2}\frac{\partial^2(v_i v_i)}{\partial y_n \partial y_n} + \frac{1}{2}\frac{\partial^2(v_i v_i)}{\partial z_n \partial z_n} - \frac{\partial v_i}{\partial y_n}\frac{\partial v_i}{\partial y_n} - \frac{\partial v_i}{\partial z_n}\frac{\partial v_i}{\partial z_n} \quad (74)$$

$$\begin{aligned}
&\frac{\partial v_i}{\partial y_n}\frac{\partial v_i}{\partial y_n} + \frac{\partial v_i}{\partial z_n}\frac{\partial v_i}{\partial z_n} = \\
= \;& 4\left(\frac{\partial \bar{u}_i}{\partial y_n}\frac{\partial \bar{u}_i}{\partial y_n} + \frac{\partial \bar{u}_i}{\partial z_n}\frac{\partial \bar{u}_i}{\partial z_n}\right) = \\
= \;& \frac{\partial u_i(\mathbf{y})}{\partial y_n}\frac{\partial u_i(\mathbf{y})}{\partial y_n} + \frac{\partial u_i(\mathbf{z})}{\partial z_n}\frac{\partial u_i(\mathbf{z})}{\partial z_n} = \\
= \;& \frac{\varepsilon(\mathbf{y}) + \varepsilon(\mathbf{z})}{\nu} + \frac{\partial^2 p(\mathbf{y})}{\partial y_n \partial y_n} + \frac{\partial^2 p(\mathbf{z})}{\partial z_n \partial z_n} = \\
= \;& \frac{2\bar{\varepsilon}}{\nu} + 2\frac{\partial^2 \bar{p}}{\partial x_n \partial x_n} \qquad (75)
\end{aligned}$$

where $\bar{\varepsilon}$ represents the mean dissipation

$$\bar{\varepsilon} = \frac{\varepsilon(\mathbf{y}) + \varepsilon(\mathbf{z})}{2}$$

and where we remark that owing to the derivation rules (24) and (32) we have

$$\frac{\partial^2 \bar{p}}{\partial y_n \partial y_n} + \frac{\partial^2 \bar{p}}{\partial z_n \partial z_n} = \frac{\partial^2 \bar{p}}{\partial x_n \partial x_n} \quad (76)$$

APPENDIX II

Following Uberoi and Kovasznay (1953) we can relate the correlations of the original

$$\langle c_{ij}(\boldsymbol{x}, \boldsymbol{x}')\rangle = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}')\rangle \quad (77)$$

and filtered fields

$$\langle C_{ij}(\boldsymbol{x}, \boldsymbol{x}')\rangle = \langle \bar{u}_i(\boldsymbol{x}) \bar{u}_j(\boldsymbol{x}')\rangle \quad (78)$$

by the expression

$$\langle C_{ij}(\boldsymbol{x}, \boldsymbol{x}')\rangle = \iint G(\mathbf{x}-\boldsymbol{\xi})G(\mathbf{x}'-\boldsymbol{\xi}')\langle c_{ij}(\boldsymbol{\xi}, \boldsymbol{\xi}')\rangle d\boldsymbol{\xi} d\boldsymbol{\xi}' \quad (79)$$

and it is interesting to derive here a similar relation for the structure functions.

We remark that if we define a quantity analogous to d_{ij}

$$d_{ij}(\boldsymbol{x}, \boldsymbol{x}') = \big(u_i(\boldsymbol{x}) - u_i(\boldsymbol{x}')\big)\big(u_j(\boldsymbol{x}) - u_j(\boldsymbol{x}')\big) \quad (80)$$

composed now by the velocity increments of the filtered values $\bar{u}_i, \bar{u}_j$

$$D_{ij}(\boldsymbol{x}, \boldsymbol{x}') = \big(\bar{u}_i(\boldsymbol{x}) - \bar{u}_i(\boldsymbol{x}')\big)\big(\bar{u}_j(\boldsymbol{x}) - \bar{u}_j(\boldsymbol{x}')\big) \quad (81)$$

we have

$$\begin{aligned}
D_{ij}(\boldsymbol{x}, \boldsymbol{x}') \;=\; & \iint G(\mathbf{x}-\boldsymbol{\xi})G(\mathbf{x}'-\boldsymbol{\xi}') d_{ij}(\boldsymbol{\xi}, \boldsymbol{\xi}') d\boldsymbol{\xi} d\boldsymbol{\xi}' - \\
- \;& \tau_{ij}(\mathbf{x}) - \tau_{ij}(\mathbf{x}') \qquad (82)
\end{aligned}$$

and this relation deserves some interest. First of all if we put in the relation (82) $\mathbf{x} = \mathbf{x}'$, we recover the relation (51) of the paper. Secondly it shows, by averaging statistically the terms, that the second order structure function of the filtered velocity is equal to a double convolution of the second order structure function of the original field minus the difference of the mean values of the subgrid stress

$$\begin{aligned}
\langle D_{ij}(\boldsymbol{x}, \boldsymbol{x}')\rangle \;=\; & \iint G(\mathbf{x}-\boldsymbol{\xi})G(\mathbf{x}'-\boldsymbol{\xi}')\langle d_{ij}(\boldsymbol{\xi}, \boldsymbol{\xi}')\rangle d\boldsymbol{\xi} d\boldsymbol{\xi}' - \\
- \;& \langle \tau_{ij}(\mathbf{x})\rangle - \langle \tau_{ij}(\mathbf{x}')\rangle \qquad (83)
\end{aligned}$$

REFERENCES

Brun, C., and Friedrich, R., 2001, "The spatial velocity increment as a tool for SGS modeling", *Modern Simulation Strategies for Turbulent Flow*, pp. 57-84, Geurts, B. J., (ed.), Edwards Publishing House, pp. 57-84

Brun, C., Friedrich, R., and da Silva, C. B., 2006, "A non-linear SGS model based on the spatial velocity increment", *Theor. Comp. Fluid Dyn.*, Vol. 20, pp. 1-21

Carati, D., Winckelmans G. S., and Jeanmart H., 2001, "On the modelling of the subgrid scale filtered scale stress tensors in large eddy simulation", *J. Fluid Mech.*, Vol. 441, pp. 119-138

Germano, M., 1992, "Turbulence: the Filtering Approach", *Journal of Fluid Mechanics*, Vol. 238, pp. 325-336.

Germano, M., 2004, "Properties of the hybrid RANS/LES filter", *Theoret. Comput. Fluid Dynamics*, Vol. 17, pp. 225-231.

Germano, M., 2007, "A direct relation between the filtered subgrid stress and the second order structure function", *Phys. Fluids*, Vol. 19, pp. 038102/2

Hill, R. J., 2001, "Equations relating structure functions of all orders", *J. Fluid Mech.*, Vol. 434, pp. 379-388

Hill, R. J., 2002, "Exact second order structure function relationships", *J. Fluid Mech.*, Vol. 468, pp. 317-326

Kolmogorov, A. N., 1941, "Dissipation of energy in locally isotropic turbulence", *Dokl. Akad. Nauk SSSR*, Vol. 32, pp. 16-18

Leonard, A., 1974, "Energy Cascade in Large-Eddy Simulations of Turbulent Fluid Flows", *Adv. in Geophysics*, Vol. 18, pp. 237-248.

Najjar, F. M., and Tafti, D. K., 1996, "Study of discrete test filters and finite difference approximations for the dynamic subgrid scale stress model", *Phys. Fluids*, Vol. 8, pp. 1076-1088

Sagaut, P., and Grohens, R., 1999, "Discrete filters for large eddy simulation", *Int. J. Numer. Meth. Fluids*, Vol. 31, pp. 1195-1220

Sagaut, P., 2005, "Large Eddy Simulation for Incompressible Flows", Third edition, Springer.

Schumann, U., 1975 "Subgrid Scale Model for Finite Difference Simulations of Turbulent Flows in Plane Channels and Annuli", *J. Comput. Phys.*, Vol. 18, pp. 376-404

Stolz, S., and Adams, N. A., 1999, "An approximate deconvolution procedure for large eddy simulation", *Phys. Fluids*, Vol.11, pp. 1699-1701.

Stolz, S., Adams, N. A., and Kleiser, L., 2001, "An approximate deconvolution model for large eddy simulation with application to incompressible wall bounded flows", *Phys. Fluids*, Vol. 13, pp. 997-1015

Uberoi, M. S., and Kovasznay, L. S. G., 1953, "On mapping and measurement of random fields", *Quart. Appl. Math.*, Vol. 10, pp. 375-393