SCALING ANALYSIS AND SIMULATION OF STABLY STRATIFIED FLOWS

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ABSTRACT

Strongly stratified turbulent flows have been analysed and simulated. Scaling analysis has been applied to the governing equations and suggests the existence of two regimes with different dynamics. The determing parameter is $\mathcal{R} = ReF_h^2$, where Re and F_h are the Reynolds number and horizontal Froude number, respectively. If $\mathcal{R} \gg 1$, viscous effects are unimportant and three-dimensional but strongly anisotropic turbulence with a forward energy cascade can develop. The vertical length scale l_v scales in this case as $l_v \sim U/N$ (U is a horizontal velocity scale and N is the Brunt-Väisälä frequency). If $\mathcal{R} \ll 1$, the dynamics of the flow is determined by a balance between vertical viscous shearing and horizontal inertial forces. The vertical length scales behaves as $l_v \sim l_h Re^{-1/2}$ and is thus independent of the stratification strength. Direct numerical simulations with various values of Re and F_h have been carried and are consistent with the results of the scaling analysis. Horizontal one-dimensional spectra have been computed and show an approximate $k_h^{-5/3}$ range when $\mathcal{R} \gg 1$ and very steep slopes when $\mathcal{R} \ll 1$. A more comprehensive discussion of the scaling analysis and a large series of DNS of stratified turbulent flows are presented in Brethouwer et al. (2007).

INTRODUCTION

Measurements of the horizontal kinetic and potential energy spectra in the strongly stratified middle atmosphere show a $k_h^{-5/3}$ range, where k_h is the horizontal wave number. Since this range is observed for wave lengths between 1 and 500 km it is clear that these spectra are not the result of classical Kolmogorov turbulence. In the past decades two possible explanations for the spectra have been given and broadly discussed. One explanation is that the spectra are owing to two-dimensional turbulence producing an inverse energy cascade, the other explanation is that they are owing to internal waves producing a forward energy cascade. Recently, Lindborg (2006) presented an alternative hypothesis saying that the spectra are a result of a process called stratified turbulence. The basic idea is that large vertically oriented vortices split up into thin layers which in their turn split up into even thinner layers. In this process the vertical length scale of the structures become smaller and the vertical gradients steeper. As a result, instabilities appear which break down the large structures into smaller ones. This process causes a nonlinear and strongly anisotropic cascade of energy from large to small scales. This idea was supported by numerical simulations of strongly stratified fluids in elongated boxes (Lindborg, 2006). Moreover, the simulations produced horizontal $k_h^{-5/3}$-spectra. The conclusion was therefore that stratified turbulence might play a role in the middle atmosphere.

Direct numerical simulations of decaying stratified turbulent flows by Riley and deBruynKops (2003) were consistent with Lindborg's results. However, some other experiments and numerical simulations delivered different results, although in general they agree on the layer formation in stratified fluids. Smith and Waleffe (2002), Waite and Bartello (2004) and Praud et al. (2005) for example did not observe a $k_h^{-5/3}$-spectrum or a clear forward energy cascade. This leads to the question what the causes are of the differences. A possible answer is that they are due to different Reynolds and Froude numbers used in the experiments and simulations. The aim of our research is study the influence of the Reynolds number and Froude number on the dynamics of stratified flows.

SCALING ANALYSIS

We assume that the Boussinesq approximation is valid. To get some insight into the dynamics of strongly stratified fluids, we scale the governing equations. A more elaborate discussion of the scaling analysis is presented in Billant and Chomaz (2001) and Brethouwer et al. (2007). Here we restrict ourselves to the main results.

First, we introduce a characteristic horizontal velocity scale U and a horizontal and vertical length scale l_h and l_v respectively. The nondimensional parameters we obtain are the Reynolds number $Re = Ul_h/\nu$, the horizontal Froude number $F_h = U/Nl_h$ (N is the Brunt-Väisälä frequency), the Schmidt number $Sc = \nu/\kappa$ (ν and κ are the viscosity and diffusivity respectively) and the aspect ratio $\alpha = l_v/l_h$. Assuming a high Reynolds number and strong stratification, i.e. $Re \gg 1$ and $F_h \to 0$, we obtain the following nondimensional governing equations (Billant and Chomaz, 2001)

$$\frac{\partial \boldsymbol{u}_h}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h \boldsymbol{u}_h + \frac{F_h^2}{\alpha^2} u_z \frac{\partial \boldsymbol{u}_h}{\partial z} = -\nabla_h p + \frac{1}{Re\,\alpha^2} \frac{\partial^2 \boldsymbol{u}_h}{\partial z^2} \quad (1)$$

$$0 = -\frac{\partial p}{\partial z} - \rho \quad (2)$$

$$\nabla_h \cdot \boldsymbol{u}_h + \frac{F_h^2}{\alpha^2} \frac{\partial u_z}{\partial z} = 0 \quad (3)$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h \rho + \frac{F_h^2}{\alpha^2} u_z \frac{\partial \rho}{\partial z} = u_z + \frac{1}{ReSc\,\alpha^2} \frac{\partial^2 \rho}{\partial z^2}, \quad (4)$$

where u_h and u_z are the horizontal and vertical velocity respectively, ρ is the density perturbation, z is the vertical coordinate and ∇_h is the horizontal gradient, all in nondimensional form.

The commonly made assumption is that the vertical Froude number $F_v = U/Nl_v \ll 1$, i.e. $\alpha \gg F_h$. In that case, the advection terms with vertical derivatives in (1) and (4) can be neglected as well as the last term on the left-hand-side of (3). The momentum equation for u_h contains then only the horizontal velocity which led to the hypothesis that a strongly stratified turbulent flow is dynamicly similar to two-dimensional turbulence.

In our study (see also Billant and Chomaz 2001, Lindborg 2006 and Brethouwer et al. 2007) we assume that l_v and thus α are not predetermined but determined by the dynamics of the flow. The ratio of the last terms on the left-hand-side and the right-hand-side of (1) is then given by the parameter $\mathcal{R} = ReF_h^2$. Depending on the value of $\mathcal{R}$, we can distinguish two regimes.

$\mathcal{R} \gg 1$: the stratified turbulence regime.

The influence of the viscous and diffusion term in (1) and (4) can be neglected if $\mathcal{R} \gg 1$. Billant and Chomaz (2001) suggested the scaling $l_v \sim U/N$ because then the governing are self-similar with respect to $z'N/U$, where z' is the dimensional vertical coordinate. Such a scaling has been observed in numerical simulations (Waite and Bartello, 2004). The scaling $l_v \sim U/N$ implies $\alpha \sim F_h$ and $F_v \sim 1$, i.e. the advection terms in (1) and in (4) are of the same order. Consequently, the vertical advection terms are of importance for the dynamics of the flow which leads to the hypothesis that stratified turbulence with $\mathcal{R} \gg 1$ is essentially governed by three-dimensional dynamics with a forward energy cascade, albeit strongly anisotropic. Lindborg (2006) argued that $l_h \sim U^3/\varepsilon$, where ε is the kinetic energy dissipation. Using this estimate, we obtain

$$\mathcal{R} = \frac{\varepsilon}{\nu N^2}. \quad (5)$$

If $\mathcal{R} \gg 1$ an inertial range can develop in stratified turbulence. Using scaling arguments, Lindborg showed that the horizontal kinetic and potential energy spectra have a $k_h^{-5/3}$ dependence in this range.

It is illustrative to write (5) in terms of the Ozmidov length scale $l_O = \varepsilon^{1/2}/N^{3/2}$ and the Kolmogorov length scale η which gives $\mathcal{R} = (l_O/\eta)^{4/3}$. Hence, stratified turbulence is to be expected in strongly stratified and high Reynolds number flows when $l_O \gg \eta$.

$\mathcal{R} \ll 1$: the viscosity affected stratified flow regime.

When $\mathcal{R} \ll 1$ we expect fundamentally different dynamics. In this case the vertical advection term in (1) is much smaller than the horizontal advection term and can be neglected. A balance between the horizontal advection term and the vertical viscous term in (1) is possible if

$$l_v \sim l_h Re^{-1/2}, \quad (6)$$

as argued by Godoy-Diana et al. (2004). We adopt this idea which implies that l_v is independent of the stratification strength. The dominant dynamics is then governed by a balance between horizontal inertial forces and vertical shearing between the layers. Hence, no clear forward energy cascade or inertial range will be seen.

Table 1: Overview of the numerical and physical parameters used in the simulations. L_h and L_v are the horizontal and vertical dimension of the box respectively, and N_h, N_v are the number of nodes in the horizontal and vertical direction, respectively. $\mathcal{R}$ is computed using (5).

run	F_h ($\times 10^{-2}$)	$\mathcal{R}$	L_h/L_v	$N_h \times N_v$
A0.06	0.23	0.058	4	256×64
A0.2	0.53	0.21	4	256×64
A0.8	1.2	0.75	4	256×64
A1.8	1.5	1.75	4	256×80
A2.8	2.3	2.84	4	256×64
A9.3	4.2	9.3	2.9	256×96
B0.1	0.23	0.11	6	512×96
B0.4	0.45	0.40	6	512×96
B1.1	0.81	1.09	6	512×96
B3.0	1.5	2.97	5	512×128
B9.3	2.7	9.3	4	512×144

DIRECT NUMERICAL SIMULATIONS

To study strongly stratified fluids and to validate the ideas presented in the previous section, we carried out a series of direct numerical simulations (DNS) with different Re and F_h and $Sc = 0.7$. A standard pseudospectral method with periodic boundary conditions was used to solve the Boussinesq equations to simulate homogeneous turbulence with a uniform stratification. The horizontal dimension of the computational domain is larger than the vertical dimension because the flow is anisotropic. A constant rate of energy is injected at the large horizontal velocity scales to obtain a statistically stationary state. Only the vortical motions are thereby forced. For more details about the forcing technique we refer to Lindborg and Brethouwer (2007). Some of the numerical and physical parameters are listed in table 1. The physical parameters are extracted from the DNS after the flow reached statistical stationarity. The table shows that the DNS cover both regimes, $\mathcal{R} > 1$ and $\mathcal{R} < 1$. A more extensive set of DNS is presented in Brethouwer et al. (2007).

RESULTS

The simulations approach rather quickly a statistically stationary state whereby the constant energy input by the forcing is balanced on average by the dissipation of kinetic and potential energy, and the kinetic and potential energy stay approximately constant. This latter point is demonstrated in figure 1. The statistics and visualisations to be presented hereafter have been extracted from the simulations while the flow was statistically stationary.

Illustrative are visualisations of the flow field. Figure 2 presents snapshots of the fluctuating density field on a vertical plane extracted from two simulations, one with $\mathcal{R} < 1$ and one with $\mathcal{R} > 1$. It is clear that the structure of the

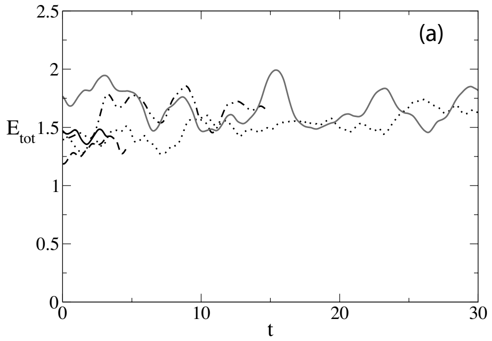

Figure 1: The time evolution of the total energy $E_{tot}/(PL_h)^{2/3}$, where P is the power input by the forcing. (———), B9.3; (− − −), B3.0; (· · · · ·), B1.1; (− · · − · · −), B0.4; (———, gray line), B0.1.

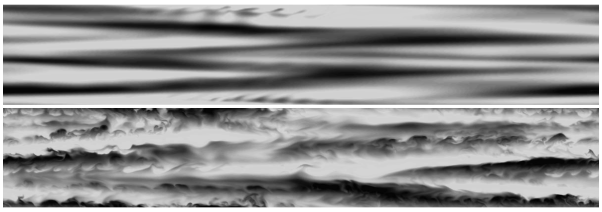

Figure 2: Snapshots of the density fluctuations in a vertical plane. $\mathcal{R} < 1$ (run B0.4) in the top figure and $\mathcal{R} > 1$ (run B9.3) in the bottom figure.

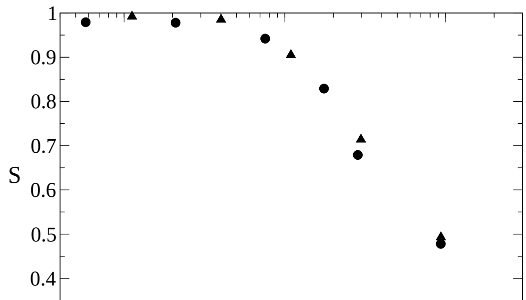

Figure 3: The anisotropy S of the dissipation defined by (7) as a function of $\mathcal{R}$. Circles, runs A; triangles, runs B.

flow field is quite different in the two cases. Layer formation can be seen in both visualisations but the density field is mostly smooth with only a few local disturbances when $\mathcal{R} < 1$ whereas small-scale turbulent-like disturbances are dominant when $\mathcal{R} > 1$. This suggests that energy dissipation by vertical shearing of the layers is important when $\mathcal{R} < 1$. On the other hand, when $\mathcal{R} > 1$ more isotropic dissipation as a consequence of the turbulent-like motions is expected. These expectations are confirmed by figure 3 showing the anisotropy as a function of $\mathcal{R}$. The dissipation

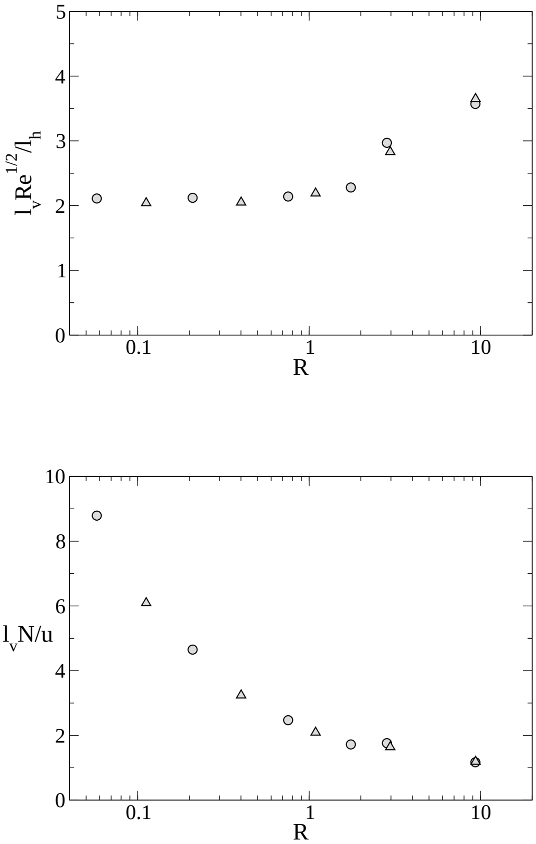

Figure 4: The scaled vertical length scale (top figure) $l_v Re^{1/2}/l_h$ and (bottom figure) $l_v N/U$ as a function of $\mathcal{R}$. Circles, runs A; triangles, runs B.

anisotropy is defined as

$$S = \frac{\nu\langle\left(\frac{\partial u'_x}{\partial z'}\right)^2 + \left(\frac{\partial u'_y}{\partial z'}\right)^2\rangle}{\varepsilon} \tag{7}$$

and is nearly equal to one implying almost purely vertical shearing if $\mathcal{R} < 1$ whereas the dissipation becomes more isotropic for $\mathcal{R} > 1$.

The scaling behaviour of l_v constitutes an important test of the scaling analysis presented here before. Figure 4 shows l_v scaled by $l_h Re^{-1/2}$ and scaled by U/N as a function of $\mathcal{R}$. The computation of l_v is explained in Brethouwer et al. (2007). The figure shows the results of several DNS with a fairly wide range of F_h and Re. The results of the DNS support the scaling $l_v \sim l_h Re^{-1/2}$ for $\mathcal{R} < 1$ and the scaling $l_v \sim U/N$ for $\mathcal{R} > 1$, which is in agreement with the scaling analysis.

Figure 5 shows compensated horizontal one-dimensional kinetic and potential energy spectra for several values of $\mathcal{R}$. Lindborg (2006) and Lindborg and Brethouwer (2007) observed in their simulations with hyperviscosity that the computed compensated kinetic and potential energy spectra have a clear inertial range and a constant value of approximately 0.5. This line is also plotted in figure 5. In none of the present DNS we see such a clear inertial range because viscous effects, i.e. $\mathcal{R}$, are not small enough in the inertial range. The spectra obtained from the runs B3.0 and B9.3 with $\mathcal{R} > 1$ have, nevertheless, a wave number

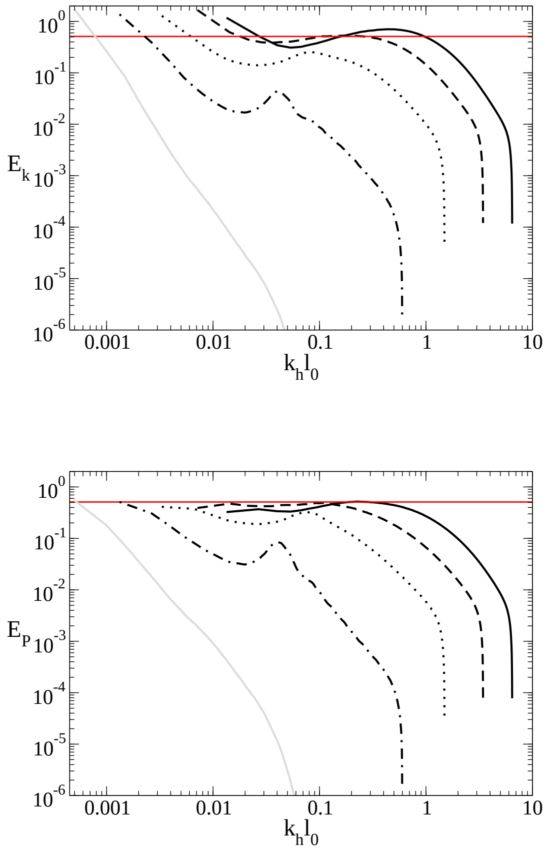

Figure 5: The compensated horizontal one-dimensional kinetic energy spectra $E_K(k_h)k_h^{5/3}/\varepsilon^{2/3}$ (top figure) and potential energy spectra $E_P(k_h)k_h^{5/3}\varepsilon^{1/3}/\varepsilon_P$ (bottom figure), where ε_P is the potential energy dissipation.. The horizontal wavenumber k_h is scaled by the Ozmidov length scale l_O. (———), B9.3; (− − −), B3.0; (· · · · ·), B1.1; (− · · − · · −), B0.4; (———, gray line), B0.1.

range that approaches the straight line and thus approaches the $k_h^{-5/3}$-dependence. The spectra obtained from the other runs with $\mathcal{R} < 1$ do not reveal such an approximate inertial range but are instead very steep. This implies that the small scales contain very little energy. The study by Brethouwer et al. (2007) shows that energy flux from large to small scales is very weak or virtually nonexistent when $\mathcal{R} < 1$ whereas there is a clear energy flux when $\mathcal{R} > 1$.

CONCLUSIONS

Figure 6 summarises some of the results of our study in a diagram. The different regimes and how these are related to Re and F_h according to the scaling analysis and the DNS are displayed in the figure. Stratified turbulence with a forward energy cascade and an approximate inertial range with a $k_h^{-5/3}$-dependence is observed when $\mathcal{R} = ReF_h^2 > 1$ and for $F_h < 0.02$, see Brethouwer et al. (2007). For larger values of F_h the flow is weakly stratified. If $\mathcal{R} = ReF_h^2 < 1$ the flow is strongly affected by vertical viscous shearing between the layers as suggested by the scaling analysis and confirmed by the DNS. The spectra are very steep in this case.

In the middle atmosphere and the ocean in general $\mathcal{R} \gg 1$ and therefore stratified turbulence seems to be most

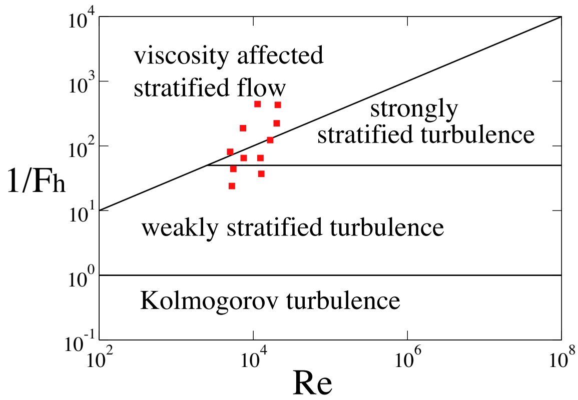

Figure 6: The different regimes in stably stratified fluids. The conditions under which the our DNS are carried out are represented by the symbols.

relevant. However, in laboratory experiments and numerical simulations it is difficult to create the condition $\mathcal{R} \gg 1$ since it requires very high Re when the stratification is strong. This is illustrated by figure 6 where the present DNS are represented by symbols. It shows that some of these DNS have just met the conditions $\mathcal{R} > 1$ and $F_h < 0.02$. Much larger computations are required to really satisfy the condition $\mathcal{R} \gg 1$ and reveal a clear inertial range without viscous effects.

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