A TURBULENCE MODEL FOR THE SOLAR WIND: APPLICATION OF THE TURBULENT MAGNETOHYDRODYNAMIC RESIDUAL-ENERGY EQUATION

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ABSTRACT

A turbulence model based on a statistical analysis of magnetohydrodynamic (MHD) turbulence with the flow-expansion effects incorporated is applied to the solar wind. In the model, the dynamics of the turbulent MHD residual energy (difference between the kinetic and magnetic energies) and the turbulent cross helicity (velocity–magnetic-field correlation) are simultaneously solved with those of the turbulent MHD energy and its dissipation rate. The observed radial behaviors of the solar-wind turbulence, i.e., the decay of the cross helicity and the stationariness of the Alfvén ratio ($r_{\rm A}$, the ratio of kinetic to magnetic-field energies) are reproduced by a numerical simulation of the model. The stationary value of $r_{\rm A} \simeq 0.5$ far from the Sun is elucidated as a stationary solution of the turbulence model.

INTRODUCTION

Spacecraft observations of solar wind have provided *in situ* information on the velocity and magnetic-field fluctuations in inhomogeneous plasmas. According to the observations, the solar-wind fluctuations show a high degree of Alfvénicity. Namely, (i) the cross-correlation between the velocity and magnetic-field fluctuations is very high, and (ii) the equipartition between the kinetic and magnetic energies is realized (Belcher and Davis, 1971). It is also known that the Alfvénicity decreases as the heliocentric distance increases: the scaled cross correlation $|W|/K (= 2|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle| / \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle) \simeq 1 \to 0.2$ and the Alfvén ratio $r_{\rm A} (\equiv \langle \mathbf{u}'^2 \rangle / \langle \mathbf{b}'^2 \rangle) \simeq 1 \to 0.5$ (Fig. 1). Not a few attempts have been done in elucidating this transitional behavior of solar-wind turbulence (see Roberts et al., 1990; Tu and Marsch, 1995 and references cited therein). However, it has been difficult to completely reproduce the radial evolution of the turbulent energy, cross helicity, and the energy difference, simultaneously. So, this transition is left as one of the unsolved problems in the current magnetohydrodynamic (MHD) turbulence theory for the solar wind.

The degree of Alvénicity in MHD turbulence can be described by one-point turbulence quantities such as the turbulent cross helicity $W (\equiv \langle \mathbf{u}' \cdot \mathbf{b}' \rangle)$ and turbulent MHD residual energy $K_{\rm R} (\equiv \langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle / 2)$. Recently, a model equation for $K_{\rm R}$ has been proposed on the basis of a statistical analysis of inhomogeneous MHD turbulence (Yokoi, 2006). As compared with the previous work relevant to $K_{\rm R}$ (Zhou and Matthaeus, 1990; Tu and Marsch, 1993; Matthaeus et al., 1994), its features may be summarized as (i) systematic incorporation of large-scale inhomogeneity through the nonlinear mixing terms and (ii) one-point modeling based on the Green's function formalism. These features make the model more appropriate than the previous models in describing turbulent behaviors in the outer heliosphere, where inhomogeneities of the large-scale velocity and magnetic field are considered to play an important role in the turbulence evolution.

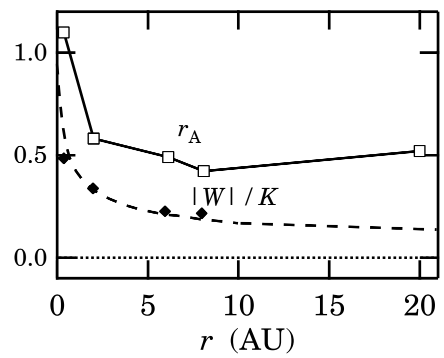

Figure 1: Observed radial evolution of the scaled cross helicity $|W|/K$ and the Alfvén ratio $r_{\rm A}$ against the heliocentric distance r (AU). Redrawn from Roberts et al. (1990).

EQUATION OF TURBULENT MHD RESIDUAL ENERGY

Fundamental equations

The density ρ, velocity $\mathbf{u}$, and magnetic field $\mathbf{b}_*$ of MHD fluids obey

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{j}_* \times \mathbf{b}_* + \nu \nabla^2 \mathbf{u} \tag{2}$$

$$\frac{\partial \mathbf{b}_*}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}_*) + \lambda \nabla^2 \mathbf{b}_* \tag{3}$$

$$\nabla \cdot \mathbf{b}_* = 0 \tag{4}$$

[ν: kinematic viscosity, λ: magnetic diffusivity]. Note that the magnetic field and electric-current density measured in the original units, $\mathbf{b}_*$ and $\mathbf{j}_*$, are related to the counterparts in the Alfvén-speed units as

$$\mathbf{b} = \frac{\mathbf{b}_*}{(\mu\rho)^{1/2}}, \quad \mathbf{j} = \frac{\mathbf{j}_*}{(\mu/\rho)^{1/2}} \tag{5}$$

(μ: magnetic permeability). As this consequence, Eqs. (1) and (4) are rewritten as

$$\nabla \cdot \mathbf{u} = -\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \ln \rho \tag{6}$$

$$\nabla \cdot \mathbf{b} = -\frac{1}{2} (\mathbf{b} \cdot \nabla) \ln \rho \tag{7}$$

We decompose a field quantity f into the mean F and the fluctuation around it, f', as

$$f = F + f', \quad F = \langle f \rangle \tag{8}$$

with

$$f = (\rho,\ \mathbf{u},\ \mathbf{b},\ \mathbf{j},\ p) \tag{9}$$

$$F = (\bar{\rho},\ \mathbf{U},\ \mathbf{B},\ \mathbf{J},\ P) \tag{10}$$

$$f' = (\rho',\ \mathbf{u}',\ \mathbf{b}',\ \mathbf{j}',\ p') \tag{11}$$

($\langle \cdot \rangle$: ensemble average). In this work, we are interested in the effects of the large-scale stationary behavior of the solar-wind turbulence and consider the effects of the mean-density variation. Then we neglect the density fluctuation as

$$\rho = \bar{\rho}, \quad \rho' = 0 \tag{12}$$

throughout this work. This point does not deny the importance of the effects of density fluctuation, in particular at small spatial scales and short timescales.

Model of the turbulent MHD residual-energy equation

As far as a comparison with the solar-wind observation is concerned, the turbulent statistical quantities measured at a given heliocentric radius with an observed density bear a great deal of relevance to this work. So we will consider the total turbulent MHD energy $K (\equiv \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2)$, the cross helicity $W (\equiv \langle \mathbf{u}' \cdot \mathbf{b}' \rangle)$, and the turbulent MHD residual energy $K_{\rm R} (\equiv \langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle / 2)$ in this work, instead of the compressible counterparts such as $\langle \rho (\mathbf{u}'^2 + \mathbf{b}'^2) \rangle / 2$, $\langle \rho \mathbf{u}' \cdot \mathbf{b}' \rangle$, etc.

Applying the Reynolds decomposition [Eq. (8)] into Eqs. (2)-(4), we get equations for the velocity and magnetic-field fluctuations, $\mathbf{u}'$ and $\mathbf{b}'$. From these equations, we obtain the evolution equation of the turbulent MHD residual energy $K_{\rm R}$ as

$$\begin{aligned} \frac{DK_{\rm R}}{Dt} &\equiv \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) K_{\rm R} \\ &= -R_{\rm T}^{ab} \frac{\partial U^a}{\partial x^b} + W_{\rm T}^{ab} \frac{\partial B^a}{\partial x^b} - \mathbf{\Gamma} \cdot \mathbf{B} \\ &\quad - \varepsilon_{\rm R} + {\rm T}_{\rm R} + {\rm DRT} \end{aligned} \tag{13}$$

where $\varepsilon_{\rm R}$ is the dissipation rate of $K_{\rm R}$, ${\rm T}_{\rm R}$ is the transport rate of $K_{\rm R}$, and DRT is the density-variation-related terms. Here, $\mathbf{R}_{\rm T}$, $\mathbf{W}_{\rm T}$, and $\mathbf{\Gamma}$ in Eq. (13) are the self-correlation tensor, cross-correlation tensor, and torsional cross vector of MHD turbulence, respectively. They are defined by

$$R_{\rm T}^{\alpha\beta} \equiv \left\langle u'^\alpha u'^\beta + b'^\alpha b'^\beta \right\rangle \tag{14}$$

$$W_{\rm T}^{\alpha\beta} \equiv \left\langle u'^\alpha b'^\beta + b'^\alpha u'^\beta \right\rangle \tag{15}$$

$$\Gamma^\alpha \equiv \left\langle b'^a \frac{u'^a}{\partial x^\alpha} - u'^a \frac{\partial b'^a}{\partial x^\alpha} \right\rangle \tag{16}$$

These correlations represent nonlinear mixing due to turbulence and are expected to play a central role in the evolution of the turbulent MHD residual energy.

In Eq. (13), DRT represents the effects of mean-density variation. We see from Eqs. (6) and (7) that the mean-density variation can be related to the divergence of the mean velocity and magnetic fields as

$$\nabla \cdot \mathbf{U} = -\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) \ln \bar{\rho} = -(\mathbf{U} \cdot \nabla) \ln \bar{\rho} \tag{17}$$

$$\nabla \cdot \mathbf{B} = -\frac{1}{2} (\mathbf{B} \cdot \nabla) \ln \bar{\rho} \tag{18}$$

Since our interests in this work lie only in the steady state of the large-scale solar-wind structure, we dropped temporal derivative of the mean density in Eq. (17). The explicit expressions for the DRT are given later.

From a statistical theoretical analysis of the incompressible MHD turbulence, we obtained the expressions for $\mathbf{R}_{\rm T}$, $\mathbf{W}_{\rm T}$, and $\mathbf{\Gamma}$ in terms of the spectral functions (Yokoi, 2006) With the aid of these expressions, we model these quantities as

$$R_{\rm T}^{\alpha\beta} = \frac{2}{3} K \delta^{\alpha\beta} - \nu_{\rm R} S^{\alpha\beta} + {\rm HRT} \tag{19}$$

$$W_{\rm T}^{\alpha\beta} = \frac{2}{3} W \delta^{\alpha\beta} - \nu_{\rm R} M^{\alpha\beta} + {\rm HRT} \tag{20}$$

$$\mathbf{\Gamma} = C_{r1} \frac{\varepsilon}{K^2} K_{\rm R} \mathbf{B} + {\rm HRT} \tag{21}$$

($S^{\alpha\beta}$: mean-velocity strain rate, $M^{\alpha\beta}$: mean-magnetic-field strain rate, HRT: helicity-related terms, C_{r1}: model constant). Here, ε is the dissipation rate of the turbulent MHD energy defined by

$$\varepsilon = \nu \left\langle \frac{\partial u'^a}{\partial x^b} \frac{\partial u'^a}{\partial x^b} \right\rangle + \lambda \left\langle \frac{\partial b'^a}{\partial x^b} \frac{\partial b'^a}{\partial x^b} \right\rangle \tag{22}$$

and $\nu_{\rm R}$ is the turbulent residual viscosity defined by

$$\nu_{\rm R} = \nu_{\rm K} \frac{K_{\rm R}}{K} \tag{23}$$

with $\nu_{\rm K}$ being the turbulent viscosity:

$$\nu_{\rm K} = C_{\nu{\rm K}} \frac{K^2}{\varepsilon} \tag{24}$$

($C_{\nu{\rm K}}$: model constant).

Using Eqs. (19)-(21), we model the equation for the turbulent residual energy ($K_{\rm R}$ equation) as

$$\begin{aligned} \frac{DK_{\rm R}}{Dt} &= \frac{1}{2} \nu_{\rm R} \left(\mathbf{S}^2 - \mathbf{M}^2\right) - C_{\varepsilon{\rm R}} \left(1 + \frac{C_{r1}}{C_{\varepsilon{\rm R}}} \frac{\mathbf{B}^2}{K}\right) \frac{\varepsilon}{K} K_{\rm R} \\ &\quad + \frac{1}{\bar{\rho}} \nabla \cdot \left(\frac{\nu_{\rm K}}{\sigma_{\rm R}} \bar{\rho} \nabla K_{\rm R}\right) + {\rm DRT} \end{aligned} \tag{25}$$

Here, the second ($C_{\varepsilon{\rm R}}$-related) and the third terms correspond to $\varepsilon_{\rm R}$ and ${\rm T}_{\rm R}$ in Eq. (13), respectively. As a first step of modeling the turbulent MHD residual-energy equation, we have dropped the helicity-related terms (HRT).

It is worth while to give a brief explanation on the structure of the $K_{\rm R}$ equation. Since the turbulent residual viscosity $\nu_{\rm R}$ is defined as Eq. (23), the right-hand side (RHS) of Eq. (25) with DRT dropped is linear in $K_{\rm R}$. This shows that no $K_{\rm R}$ can be generated unless a finite amount of $K_{\rm R}$ (seed of $K_{\rm R}$) already exists. With taking into account the compressible or flow-expansion effect, this situation must be changed. This point will be referred to in the following section. Here we should remark the following two points. Firstly, the first term of Eq. (25) represents $K_{\rm R}$ generation (or destruction) due to the relative magnitude of the mean velocity strain ($\mathbf{S}^2$) to the magnetic counterpart ($\mathbf{M}^2$). Secondly, the C_{r1}-related part of the second term of Eq. (25) shows that the presence of the mean magnetic field ($\mathbf{B}$) leads to returning turbulence to equipartition since it always destructs the existing $K_{\rm R}$. This corresponds to the Alfvén effect caused by the mean magnetic field. The second term of Eq. (25) suggests that the destruction of $K_{\rm R}$ is brought by a combination of two turbulence processes: the turn-over of eddies and the Alfvénic interaction.

APPLICATION TO THE SOLAR WIND TURBULENCE

System of model equations

As was referred to in Introduction, the solar wind is blown away from the Sun to the surrounding space. In order to quantitatively discuss the radial evolution of the solar-wind turbulence, we should take into account this expansion effect properly. For this purpose, we incorporate the effects of the mean-density stratification by way of Eqs. (17)-(18). In this formulation, the evolution equations for the turbulent MHD energy K, its dissipation rate ε, the cross helicity W, and the turbulent MHD residual energy $K_{\rm R}$ are given as

$$\begin{aligned}\frac{\partial K}{\partial t} =& -(\mathbf{U}\cdot\nabla)K \\ &-\frac{1}{6}(3K+K_{\rm R})\nabla\cdot\mathbf{U} - 2W\nabla\cdot\mathbf{B} \\ &+\frac{1}{2}\nu_{\rm K}\mathbf{S}^2 - \frac{1}{2}\nu_{\rm M}\mathbf{S}:\mathbf{M} + \beta\mathbf{J}^2 - \gamma\mathbf{\Omega}\cdot\mathbf{J} \\ &-\varepsilon + \nabla\cdot(W\mathbf{B}) + \frac{1}{\bar{\rho}}\nabla\cdot(\nu_{\rm K}\bar{\rho}\nabla K)\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial \varepsilon}{\partial t} =& -(\mathbf{U}\cdot\nabla)\varepsilon + C_{\varepsilon 1}\frac{\varepsilon}{K}P_K - C_{\varepsilon 2}\frac{\varepsilon}{K}\varepsilon \\ &+\frac{1}{\bar{\rho}}\nabla\cdot\left(\frac{\nu_{\rm K}}{\sigma_\varepsilon}\bar{\rho}\nabla\varepsilon\right)\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial W}{\partial t} =& -(\mathbf{U}\cdot\nabla)W \\ &-\frac{1}{2}W\nabla\cdot\mathbf{U} - \left(2K - \frac{1}{3}K_{\rm R}\right)\nabla\cdot\mathbf{B} \\ &+\frac{1}{2}\nu_{\rm K}\mathbf{S}:\mathbf{M} - \frac{1}{2}\nu_{\rm M}\mathbf{M}^2 + \beta\mathbf{\Omega}\cdot\mathbf{J} - \gamma\mathbf{\Omega}^2 \\ &-C_{\varepsilon\rm W}\frac{\varepsilon}{K}W + \nabla\cdot(K\mathbf{B}) + \frac{1}{\bar{\rho}}\nabla\cdot\left(\frac{\nu_{\rm K}}{\sigma_{\rm W}}\bar{\rho}\nabla W\right)\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial K_{\rm R}}{\partial t} =& -(\mathbf{U}\cdot\nabla)K_{\rm R} - \frac{1}{6}(K+3K_{\rm R})\nabla\cdot\mathbf{U} \\ &-\frac{1}{3}W\nabla\cdot\mathbf{B} + \frac{1}{2}\nu_{\rm R}\mathbf{S}^2 - \frac{1}{2}\nu_{\rm R}\mathbf{M}^2 \\ &-C_{\varepsilon\rm R}\left(1 + \frac{C_{r1}}{C_{\varepsilon\rm R}}\frac{\mathbf{B}^2}{K}\right)\frac{\varepsilon}{K}K_{\rm R} + \frac{1}{\bar{\rho}}\nabla\cdot\left(\frac{\nu_{\rm K}}{\sigma_{\rm R}}\bar{\rho}\nabla K_{\rm R}\right)\end{aligned} \tag{29}$$

where

$$\beta = C_\beta\frac{K}{\varepsilon}K,\quad \gamma = C_\gamma\frac{K}{\varepsilon}W,\quad \nu_{\rm K} = \frac{7}{5}\beta,\quad \nu_{\rm M} = \frac{7}{5}\gamma \tag{30}$$

Here, P_K in the ε equation [Eq. (27)] denotes the production terms of K [the terms related to the inhomogeneity of $\mathbf{U}$ and $\mathbf{B}$ in Eq. (26)]. The dissipation rate of W, ε_W, defined by

$$\varepsilon_W = (\nu+\lambda)\left\langle\frac{\partial u'^a}{\partial x^b}\frac{\partial b'^a}{\partial x^b}\right\rangle \tag{31}$$

is modeled in the simplest possible way as the eighth or $C_{\varepsilon\rm W}$-related term in Eq. (28). A brief explanation on the derivation of this compressible modification for the K, W, and $K_{\rm R}$ equations [Eqs. (26), (28), and (29)] is given in Appendix. Note that here we included $\nabla\cdot\mathbf{B}$-related terms arising from the mean-density variation through Eq. (18) only but left the MHD-pressure-related terms not included. This is because the latter are regarded as being already included in the diffusion or transport-rate term of each equation.

This system of model equations [Eqs. (26)-(29)] is simultaneously solved in a numerical manner with appropriate boundary conditions. In the above turbulence model, we have formally 12 model constants. In this work, we adopt the following values:

eddy viscosity etc.

$$C_{\nu\rm K} = \frac{7}{5}C_\beta = 0.09,\ C_{\nu\rm M} = \frac{7}{5}C_\gamma = 0.09 \tag{32}$$

ε equation

$$C_{\varepsilon 1} = 1.4,\ C_{\varepsilon 2} = 1.9,\ \sigma_\varepsilon = 1.3 \tag{33}$$

W equation

$$C_{\varepsilon\rm W} = 1.8,\ \sigma_{\rm W} = 1.0 \tag{34}$$

$K_{\rm R}$ equation

$$C_{r1} = 0.01,\ C_{\varepsilon\rm R} = 1.0,\ \sigma_{\rm R} = 1.0 \tag{35}$$

It is important to stress that, in principle, these model constants are not adjustable parameters but universal constants in that, once optimized, they should be fixed throughout various applications of the model. Among them, the constants appearing in the K [Eq. (32)] and ε [Eq. (33)] equations should be identical with the counterparts in the hydrodynamic (HD) $k-\epsilon$ model since the MHD $K-\varepsilon$ model should be reduced to the HD $k-\epsilon$ model in the absence of the magnetic field, whose constants have been fully optimized through the various engineering applications. In contrast, the constants appearing in the W equation, $C_{\varepsilon\rm W}$ and $\sigma_{\rm W}$ [Eq. (34)], leave room for further optimization. Among the two, $C_{\varepsilon\rm W}$ is important for it is directly connected to the dissipation rate of W. The newly added constants are appearing in $K_{\rm R}$ equation. Among the $\varepsilon_{\rm R}$ (dissipation of $K_{\rm R}$)-related constants, C_{r1} and $C_{\varepsilon\rm R}$, $C_{\varepsilon\rm R}$ should be unity ($C_{\varepsilon\rm R} = 1$) since the equation for $K + K_{\rm R}$ should be reduced to the usual turbulent energy equation in the case of vanishing magnetic field. Then we adopt values as in Eq. (35).

Numerical simulations

We assume time independent mean fields of a solar wind. As a first step of simulation, we fix the mean velocity and magnetic-field throughout the simulation. We adopt spatial distributions that are given by the current theories of the solar wind since they agree with observations well (Weber and Davis, 1967). In the theoretical model, the mean fields in the equatorial plane are considered under the assumptions of (i) time independence; (ii) axsymmetry around the rotation axis; (iii) plasmas constrained in the equatorial plane; and symmetry with respect to the equatorial plane. In the spherical polar coordinate (r, θ, ϕ), the mean velocity and magnetic fields are expressed as

$$\mathbf{U} = \left(U^r(r),\ U^\theta(r),\ U^\phi(r)\right)$$
$$= \left(U^r,\ 0,\ \frac{\Omega_\odot r}{M_\mathrm{a}^2 - 1}\left[\frac{M_\mathrm{a}^2}{(r/r_\mathrm{a})^2} - 1\right]\right) \tag{36}$$
$$\mathbf{B} = \left(B^r(r),\ B^\theta(r),\ B^\phi(r)\right)$$
$$= \left(\frac{r_\mathrm{a}}{r}\sqrt{V_\mathrm{a}}\sqrt{U^r},\ 0,\ \frac{U^\phi - \Omega_\odot r}{U^r}B^r\right) \tag{37}$$

where $\Omega_\odot$ is the angular velocity of the Sun and r_a is the Alfvén point defined by $U^r(r_\mathrm{a}) = B^r(r_\mathrm{a}) (\equiv V_\mathrm{a})$. Here, M_a is the Alfvén Mach number defined by

$$M_\mathrm{a}(r) = \frac{U^r(r)}{B^r(r)} \tag{38}$$

and we have $M_\mathrm{a} = 1$ at the Alfvén point r_a. As can be seen in Eqs. (36) and (37), we dropped the θ or polar-angle dependence of the mean fields. This is because we suppose a solar-wind region near the equatorial plane within a magnetic sector ($B^r > 0$) where the θ dependence can be neglected as a first-order approximation. This treatment does not deny the importance of the θ dependence in case we consider a phenomenon such as a flow across the current sheet, where the up-down asymmetry with respect to the current sheet near the equatorial plane is of essential importance.

As far as the mean radial velocity U^r is concerned, the prediction given by the non-rotating unmagnetized model is almost identical with the one by the magnetic rotator model (Parker, 1965). So we use in Eqs. (36) and (37) the solar-wind speed given by the former model, which is a solution of

$$\frac{U^{r\,2}}{c_\mathrm{s}^2} - \ln\frac{U^{r\,2}}{c_\mathrm{s}^2} = -3 + \ln\frac{r^4}{r_\mathrm{s}^4} + 4\frac{r_\mathrm{s}}{r} \tag{39}$$

Here r_s is the sonic point where the radial speed of the solar wind, U^r, is equal to the speed of sound c_s. The sonic point is calculated as $r_\mathrm{s} = GM_\odot/2c_\mathrm{s}^2$ (G: gravity constant, $M_\odot$: solar mass). In the region treated in this work ($r > r_\mathrm{a} > r_\mathrm{s}$), U^r can be approximated by the expression in the region far from the sonic point ($r \gg r_\mathrm{s}$) as

$$U^r(r) = 2c_\mathrm{s}[\ln(r/r_\mathrm{s})]^{1/2} \tag{40}$$

The mean velocity and magnetic fields [Eqs. (36) and (37)] with Eq. (40) are plotted in Fig. 2. Here, the fields are scaled by the Alfvén speed at the Alfvén point, V_a, as $\hat{U}^r = U^r/V_\mathrm{a}, \hat{B}^r = B^r/V_\mathrm{a}$, etc. Note that $\hat{U}^r = \hat{B}^r = 1$ at the Alfvén point ($\hat{r} = r/r_\mathrm{a} = 1$).

As for the boundary conditions for K, ε, W and K_R, we fix the values of them at the inner boundary:

$$K = K_0 = 1.0 \times 10^{-2},\ \varepsilon = \varepsilon_0 = 8.0 \times 10^{-3},$$
$$W = W_0 = -7.5 \times 10^{-3},\ K_\mathrm{R} = K_{\mathrm{R}0} = 0 \quad \text{at}\ \hat{r} = 1 \tag{41}$$

and assign the vanishing of the second derivatives at the outer boundary:

$$\frac{\partial^2 K}{\partial r^2} = \frac{\partial^2 \varepsilon}{\partial r^2} = \frac{\partial^2 W}{\partial r^2} = \frac{\partial^2 K_\mathrm{R}}{\partial r^2} = 0 \quad \text{at}\ \hat{r} = 100 \tag{42}$$

The system of the inner boundary values [Eq. (41)] may be set differently. Actually, the simulation results have not changed drastically if we varied these values.

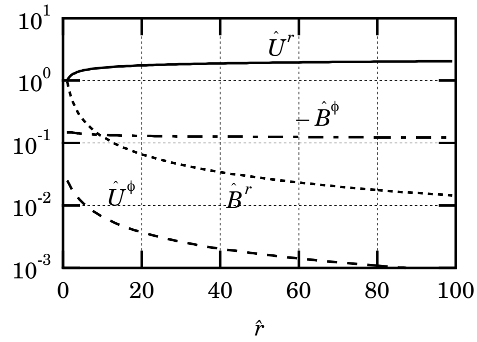

Figure 2: Radial distribution of the mean velocity and magnetic fields. The mean fields scaled by V_a are denoted as $\hat{U}^r$, ——; $\hat{U}^\phi$, – – –; $\hat{B}^r$, ······; $\hat{B}^\phi$, – - –. The heliocentric distance $\hat{r}$ is scaled by the Alfvén radius r_a, on which we adopt $r_\mathrm{a} = 21.5 r_\odot$ in this work ($r_\odot$: solar radius). The orbit of the Earth corresponds to $\hat{r} = 10$.

Results

The simulated radial distributions of the turbulent MHD energy K, its dissipation rate ε, the turbulent cross helicity W, and the turbulent MHD residual energy K_R are plotted against the heliocentric distance in Fig. 3. With the heliocentric distance $\hat{r}$, the absolute values of K, ε, and W decrease from their inner boundary values at $\hat{r} = 1$. In contrast, the absolute value of K_R increases near the Sun then decreases with $\hat{r}$. The value of K_R decreases from its inner boundary value of $K_{\mathrm{R}0} = 0$ to its minimum $K_\mathrm{R} \simeq -0.87 \times 10^{-3}$ at $\hat{r} \simeq 3$.

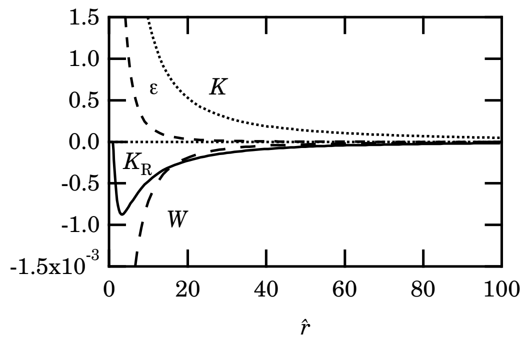

Figure 3: Simulated radial distributions of the turbulent statistical quantities. K, ······; ε, - - -; W, – – –; K_R, ——.

The energy dissipation rate ε, the cross helicity W, and the residual energy K_R scaled by the turbulent energy K are plotted against the heliocentric distance in Fig. 4. We see from this figure that the characteristics of decays in W/K and in K_R/K are well reproduced.

The radial evolutions of the magnitude of the scaled cross helicity $|W|/K$ and the Alfvén ratio r_A are given in Fig. 5. This figure should be compared with Fig. 1. Note that the range of $\hat{r} = 1 - 100$ in the present simulation corresponds to the heliocentric distance of $r = 0.1 - 10$ AU. The general tendency of the observation is reproduced by the simulation.

DISCUSSIONS

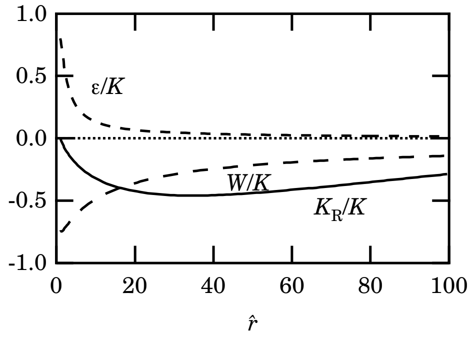

Figure 4: Simulated radial evolution of the energy dissipation rate ε, the cross helicity W, and the residual energy K_{R}, scaled by the turbulence energy K. ε/K, - - -; W/K, – – –; K_{R}/K, ——.

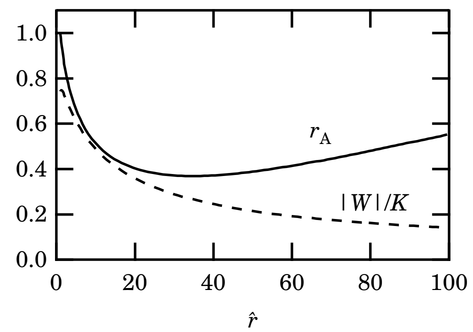

Figure 5: Simulated radial evolution of the cross helicity scaled by the turbulence energy, $|W|/K$, and the Alfvén ratio $r_{\mathrm{A}}[= (K+K_{\mathrm{R}})/(K-K_{\mathrm{R}})]$. $|W|/K$, – – –; r_{A}, ——.

Dominant balances in energetics

In order to increase our understanding of the dynamics of the solar-wind turbulence, in this subsection, we examine the dominant balances of the terms in the turbulent MHD energy (K) [Eq. (26)], cross-helicity (W) [Eq. (28)], and residual-energy (K_{R}) [Eq. (29)] equations. For the purpose of seeing the general tendency, we list the representative values of each term in both regions near and far from the Sun in Tab. 1. Note that each value is scaled by the value of the convection terms.

K_{R} equation. Near the Sun, a negative K_{R} is dominantly generated by the flow-expansion effect coupled with the turbulent MHD energy, $-(K/6)\nabla\cdot\mathbf{U}$. This effect can be interpreted as follows: Both the turbulent kinetic energy $\langle \mathbf{u}'^2\rangle/2$ and the magnetic one $\langle \mathbf{b}'^2\rangle/2$ are reduced by a flow expansion represented by $\nabla\cdot\mathbf{U}$. We should remember that $-(K/2)\nabla\cdot\mathbf{U}$ in the K equation reduces the K production for $\nabla\cdot\mathbf{U} > 0$ (Tab. 1). The degree of reduction is larger in $\langle \mathbf{u}'^2\rangle/2$ than in $\langle \mathbf{b}'^2\rangle/2$, then a negative $K_{\mathrm{R}}(= \langle \mathbf{u}'^2 - \mathbf{b}'^2\rangle/2)$ is associated with the flow expansion. These effects associated with the flow expansion of a solar wind, are of great importance in particular in the vicinity of the Sun, where equipartition between the kinetic and magnetic energies is realized. In other words, only these effects can work for the production of $|K_{\mathrm{R}}|$ even in the absence of the seed of K_{R}.

The destruction of K_{R}, ε_{R}, is represented by the sixth or $C_{\varepsilon\mathrm{R}}$ and C_{r1}-related term in Eq. (29):

$$\varepsilon_{\mathrm{R}} = C_{\varepsilon\mathrm{R}}\frac{\varepsilon}{K}K_{\mathrm{R}} + C_{r1}\frac{\mathbf{B}^2}{K}\frac{\varepsilon}{K}K_{\mathrm{R}} \qquad (43)$$

This is a combination of the eddy-distortion [the first term in Eq. (43)] and the Alfvén (the second term) effects. The appearance of the second or $\mathbf{B}$-related term should be remarked. Deviation from equipartition between the kinetic and magnetic energies or a finite value of K_{R} is suppressed by the Alfvén effects associated with a strong magnetic field. In the solar-wind turbulence, these effects may play a certain role both in the regions near and far from the Sun, since $\mathbf{B}$ itself remains to be fininte even in the region where the shear of $\mathbf{B}$ becomes negligibly small. Quantitatively, this situation directly depends on the magnitude of the Alfvén effects through our choice of model constant C_{r1}. The value of C_{r1} is closely connected with the timescales of MHD turbulence.

In the region near the Sun, the production of K_{R} is dominantly attributed to $(K/6)\nabla\cdot\mathbf{U}$. This production, with the convection of K_{R} by $\mathbf{U}$, is mainly balanced by the flow expansion effects coupled with W and K_{R}, $(W/3)\nabla\cdot\mathbf{B}$ and $(K_{\mathrm{R}}/2)\nabla\cdot\mathbf{U}$, and ε_{R} [Eq. (43)] as

$$-(\mathbf{U}\cdot\nabla)K_{\mathrm{R}} - \frac{1}{6}K\nabla\cdot\mathbf{U} - \frac{1}{2}K_{\mathrm{R}}\nabla\cdot\mathbf{U} - \frac{1}{3}W\nabla\cdot\mathbf{B} - \varepsilon_{\mathrm{R}} \simeq 0 \qquad (44)$$

As the heliocentric distance increases, the relative importance of the convection by $\mathbf{U}$ in the local $|K_{\mathrm{R}}|$ generation increases. Noting that $|-(W/3)\nabla\cdot\mathbf{B}|$ rapidly attenuates with $\hat{r}$, we have

$$-(\mathbf{U}\cdot\nabla)K_{\mathrm{R}} - \frac{1}{6}K\nabla\cdot\mathbf{U} - \frac{1}{2}K_{\mathrm{R}}\nabla\cdot\mathbf{U} - \varepsilon_{\mathrm{R}} \simeq 0 \qquad (45)$$

in the region far from the Sun (approximately corresponding to the outer heliosphere).

The order of magnitude of the terms in the K_{R} equation [Eq. (29)] is depicted in Fig. 6. The dent of the convection term $|(\mathbf{U}\cdot\nabla)K_{\mathrm{R}}|$ at $\hat{r}\simeq 3$ shows the value of $(\mathbf{U}\cdot\nabla)K_{\mathrm{R}}$ reverses its sign there.

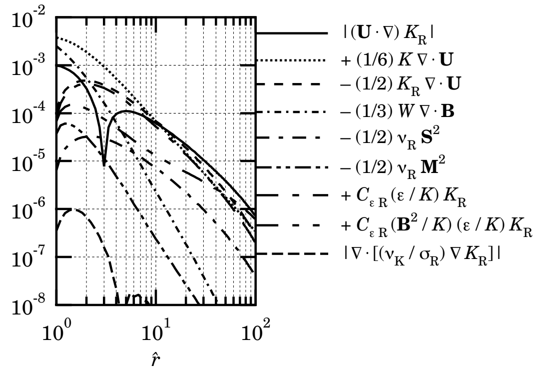

Figure 6: The order of magnitude of terms in the turbulent MHD residual-energy (K_{R}) equation [Eq. (29)] against the scaled heliocentric distance $\hat{r}$.

Alfvén ratio stationarity in space

With understanding the above energetics of the turbulent statistical quantities, we examine here the Alfvén-ratio stationarity in space ($r_{\mathrm{A}}\simeq 0.5$) observed in the region far from

Table 1: The balances between the major terms in the turbulent MHD energy (K) [Eq. (26)], cross-helicity (W) [Eq. (28)], and residual energy (K_{R}) [Eq. (29)] equations, respectively. Note that the values are scaled by that of the convection terms. So the signs of the entries change depending on the signs of the convected quantities.

$\hat{r}$	$\partial K/\partial t$	$-(\mathbf{U}\cdot\nabla)K$	$-\frac{1}{2}K\nabla\cdot\mathbf{U}$	$-\frac{1}{6}K_{\mathrm{R}}\nabla\cdot\mathbf{U}$	$-2W\nabla\cdot\mathbf{B}$	$+\frac{1}{2}\nu_{\mathrm{K}}\mathbf{S}^2$	$\nabla\cdot(W\mathbf{B})$	$-\varepsilon$
7.4	0	1	−0.75	0.068	0.090	0.072	0.11	−0.60
90.0	0	1	−0.70	0.074	1.6×10^{-3}	0.094	2.5×10^{-3}	−0.48
$\hat{r}$	$\partial W/\partial t$	$-(\mathbf{U}\cdot\nabla)W$	$-\frac{1}{2}W\nabla\cdot\mathbf{U}$	$-2K\nabla\cdot\mathbf{B}$	$\frac{1}{3}K_{\mathrm{R}}\nabla\cdot\mathbf{B}$	$+\frac{1}{2}\nu_{\mathrm{K}}\mathbf{S}:\mathbf{M}$	$\nabla\cdot(K\mathbf{B})$	$-C_{\varepsilon\mathrm{W}}\frac{\varepsilon}{K}W$
7.4	0	1	−0.61	0.24	0.011	−0.040	0.28	−0.88
90.0	0	1	−0.50	0.051	2.7×10^{-3}	−0.011	0.063	−0.61
$\hat{r}$	$\partial K_{\mathrm{R}}/\partial t$	$-(\mathbf{U}\cdot\nabla)K_{\mathrm{R}}$	$-\frac{1}{6}K\nabla\cdot\mathbf{U}$	$-\frac{1}{2}K_{\mathrm{R}}\nabla\cdot\mathbf{U}$	$-\frac{1}{3}W\nabla\cdot\mathbf{B}$	$+\frac{1}{2}\nu_{\mathrm{R}}\mathbf{S}^2$	$-C_{r1}\frac{\mathbf{B}^2}{K}\frac{\varepsilon}{K}K_{\mathrm{R}}$	$-C_{\varepsilon\mathrm{R}}\frac{\varepsilon}{K}K_{\mathrm{R}}$
7.4	0	1	1.74	−1.41	−0.10	0.13	−0.24	−1.12
90.0	0	1	0.45	−0.43	-5.3×10^{-4}	0.059	−0.78	−0.30

the Sun ($r >\sim 3$ AU) from the viewpoint of the stationary solution of the turbulent residual-energy model equation.

If we remark that the diffusion effects in the present simulation are negligible in the whole regions, the balance in the energetics of statistical quantities will be generally kept among the production, convection, and dissipation terms. In the framework of purely incompressible turbulence model, an increase of the residual energy K_{R} due to the convection by the mean flow $\mathbf{U}$ and the production by the shears $\mathbf{S}$ and $\mathbf{J}$ must be balanced by the dissipation of K_{R}, ε_{R}, due to the eddy distortion and to the Alfvén effects:

$$-(\mathbf{U}\cdot\nabla)K_{\mathrm{R}} + \frac{1}{2}\nu_{\mathrm{R}}\mathbf{S}^2 + \beta\mathbf{J}^2 - \varepsilon_{\mathrm{R}} \simeq 0 \qquad (46)$$

In this simulation, the second and third or $\mathbf{S}^2$- and $\mathbf{J}^2$-related terms in Eq. (46) are much smaller than the rest terms, so they may be neglected. Then we have

$$-(\mathbf{U}\cdot\nabla)K_{\mathrm{R}} - \varepsilon_{\mathrm{R}} \simeq 0 \qquad (47)$$

In this work, we have extra terms arising from the flow-expansion effects, the $\nabla\cdot\mathbf{U}$- and $\nabla\cdot\mathbf{B}$-related terms. They are originated from the compressibility or the mean-density variation in the solar wind, and in this sense, have origins different from the terms appearing in Eq. (46) which are inherently incompressible. With this point in mind, we examine Tab. 1 and Fig. 6. Then we find the balance relation (47) [or (46)] roughly applies. This seems to be the case irrespective of the region near or far from the Sun. In particular, in the region far from the Sun, the incompressible balance (47) holds with reasonable accuracy.

Provided that the incompressible balance relation of Eq. (46) is realized, we see from Eq. (44) that in the region near the Sun we have

$$-\frac{1}{2}K_{\mathrm{R}}\nabla\cdot\mathbf{U} - \frac{1}{6}K\nabla\cdot\mathbf{U} - \frac{1}{3}W\nabla\cdot\mathbf{B} \simeq 0 \qquad (48)$$

In the region far from the Sun, Eq. (45) with Eq. (47) is reduced to

$$-\frac{1}{2}K_{\mathrm{R}}\nabla\cdot\mathbf{U} - \frac{1}{6}K\nabla\cdot\mathbf{U} \simeq 0 \qquad (49)$$

which suggests that the relation

$$\frac{K_{\mathrm{R}}}{K} \simeq -\frac{1}{3} \qquad (50)$$

would be realized there. This may explain, from the viewpoint of the steady solution of a turbulence model, why the stationary value of $r_{\mathrm{A}} \simeq 0.5$ is observed in the region far from the Sun.

CONCLUSIONS

A turbulence model, constituted of the turbulent MHD residual energy (K_{R}) and the turbulent cross helicity (W) as well as the turbulent MHD energy (K) and its dissipation rate ε, was applied to the solar wind. With the aid of the numerical simulation, the observed radial evolution of the solar-wind turbulence was shown to be reproduced by the model. The stationary value of the Alfvén ratio ($r_{\mathrm{A}} \simeq 0.5$) in the region far from the Sun was elucidated from the viewpoint of a stationary solution of the turbulence model.

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